Math 212b, Spring 2009, Homework # 4

Compact and Fredholm Operators

1. [Lang, Problem 11 on p. 437] Let $X = \mathbb{R}/\mathbb{Z}$ be the unit circle and consider $\Phi : X \times X \to \mathbb{C}$ given by $\Phi(x,t) = e^{2\pi i(x-t)}$. Describe the spectrum of T_Φ, the integral operator on $L^2(X)$ corresponding to Φ.

2. Let $E = C([0,1])$ and define $T : E \mapsto E$ by

$$(Tf)(x) = \int_0^x f(t)\,dt\,.$$

(a) Prove that T is compact (note that $T = T_\Phi$ with non-continuous Φ).

(b) Prove that $T(E_1)$ is not compact.

(c) Describe the spectrum of T.

3. Let $M_\varphi : C([0,1]) \mapsto C([0,1])$ be the operator of multiplication by $\varphi \in C([0,1])$. State necessary and sufficient conditions (in terms of φ) for

(a) M_φ to be compact; (b) M_φ to be Fredholm;

(c) $\lambda \in \mathbb{C}$ to be an eigenvalue of M_φ;

(d) $\lambda \in \mathbb{C}$ to be in the spectrum of M_φ.

4. Let X be a separable metric space, μ a finite measure on X and $\varphi \in L^\infty(X,\mu)$. State necessary and sufficient conditions (in terms of φ) for M_φ to be compact. [Hint: Zimmer, Problem 3.13]

5. Let $T : E \to E$ be compact and let λ be a nonzero eigenvalue of T. Show that the sequence of subspaces

$$\mathrm{Ker}(T-\lambda I) \subset \mathrm{Ker}(T-\lambda I)^2 \subset \cdots \subset \mathrm{Ker}(T-\lambda I)^n \subset \ldots$$

eventually stabilizes.

6. Suppose G is a compact group, π is its regular representation, and $V \subset L^2(G)$ is finite-dimensional and $\pi(G)$-invariant. Show that $V \subset C(G)$. [Hint: Zimmer, Problem 3.18]