Math 212b, Spring 2009, Homework # 5

Invariant Measures

1. Suppose that a compact group G acts continuously on a compact metric space X. Show that there is a metric d on X defining the same topology that is G-invariant, i.e. with $d(gx, gy) = d(x, y)$ for all $x, y \in X$, $g \in G$. [Hint: Zimmer, Problem 2.12]

2. Let $X = \mathbb{R}/\mathbb{Z}$ (equivalently, $[0, 1]$ with endpoints identified).

(a) Find all measures on X invariant under $T : x \mapsto x^2$.

(b) Construct a self-map of X discontinuous in just one point which has no invariant measures.

(c) Construct two continuous self-maps of X such that there are no measures invariant under both of them.

3. Let $X = \mathbb{R}^2/\mathbb{Z}^2$, $\alpha \notin \mathbb{Q}/\mathbb{Z}$, and let $T : X \to X$ be given by $(x, y) \mapsto (x+\alpha, x+y)$. Prove that T preserves Lebesque measure and is ergodic.

In the remaining problems, $(X, \mathcal{A}, \mu)$ is a measure space, $\mu(X) = 1$, and T is a measurable self-map of X.

4. Prove that T preserves μ (that is, $\mu\big(T^{-1}(A)\big) = \mu(A)$ for all $A \in \mathcal{A}$) if and only if $\mu\big(T^{-1}(A)\big) \le \mu(A)$ for all $A \in \mathcal{A}$.

In the remaining problems, assume that T preserves μ.

5. Let $f \in L^1(X, \mathcal{A}, \mu)$ be such that $f\big(T(x)\big) \le f(x)$ for μ-a.e. $x \in X$. Prove that $f\big(T(x)\big) = f(x)$ for μ-a.e. x (that is, f is essentially T-invariant).

6. Suppose that T is ergodic, and let U be the unitary operator on $L^2(X, \mathcal{A}, \mu)$ associated to T.

(a) Prove that the eigenvalues of U form a subgroup.

(b) Assume in addition that μ is non-atomic (that is, $\mu\big(\{x\}\big) = 0$ for all $x \in X$). Then every point on the unit circle in $\mathbb{C}$ is an *approximate eigenvalue* of U, that is, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ there exists a sequence $\{f_n\} \subset L^2$ with $\|f_n\|_2 = 1$ for all n and $\|Uf_n - \lambda f_n\|_2 \to 0$ as $n \to \infty$. Consequently, the spectrum of U is the entire unit circle.