1. Suppose that a compact group $G$ acts continuously on a compact metric space $X$. Show that there is a metric $d$ on $X$ defining the same topology that is $G$-invariant, i.e. with $d(gx, gy) = d(x, y)$ for all $x, y \in X$, $g \in G$. [Hint: Zimmer, Problem 2.12]

2. Let $X = \mathbb{R}/\mathbb{Z}$ (equivalently, $[0, 1]$ with endpoints identified).
   (a) Find all measures on $X$ invariant under $T : x \mapsto x^2$.
   (b) Construct a self-map of $X$ discontinuous in just one point which has no invariant measures.
   (c) Construct two continuous self-maps of $X$ such that there are no measures invariant under both of them.

3. Let $X = \mathbb{R}^2/\mathbb{Z}^2$, $\alpha \notin \mathbb{Q}/\mathbb{Z}$, and let $T : X \to X$ be given by $(x, y) \mapsto (x + \alpha, x + y)$. Prove that $T$ preserves Lebesgue measure and is ergodic.

In the remaining problems, $(X, \mathcal{A}, \mu)$ is a measure space, $\mu(X) = 1$, and $T$ is a measurable self-map of $X$.

4. Prove that $T$ preserves $\mu$ (that is, $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$) if and only if $\mu(T^{-1}(A)) \leq \mu(A)$ for all $A \in \mathcal{A}$.

In the remaining problems, assume that $T$ preserves $\mu$.

5. Let $f \in L^1(X, \mathcal{A}, \mu)$ be such that $f(T(x)) \leq f(x)$ for $\mu$-a.e. $x \in X$. Prove that $f(T(x)) = f(x)$ for $\mu$-a.e. $x$ (that is, $f$ is essentially $T$-invariant).

6. Suppose that $T$ is ergodic, and let $U$ be the unitary operator on $L^2(X, \mathcal{A}, \mu)$ associated to $T$.
   (a) Prove that the eigenvalues of $U$ form a subgroup.
   (b) Assume in addition that $\mu$ is non-atomic (that is, $\mu(\{x\}) = 0$ for all $x \in X$). Then every point on the unit circle in $\mathbb{C}$ is an approximate eigenvalue of $U$, that is, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ there exists a sequence $\{f_n\} \subset L^2$ with $\|f_n\|_2 = 1$ for all $n$ and $\|Uf_n - \lambda f_n\|_2 \to 0$ as $n \to \infty$. Consequently, the spectrum of $U$ is the entire unit circle.