(These concepts from general topology are assumed to be familiar, at least from $\mathbb{R}^n$, and their elementary properties are used without further comment. In particular: a continuous image of a compact (resp. connected) set is compact (resp. connected). This follows directly from the definitions.

8. If a linear group $G$ has a connected dense subset $A$, then $G$ itself is connected.

9. Show that a subgroup $H$ of a connected linear group $G$ is dense in $G$ if and only if the closure of $H$ contains a neighborhood of 1 in $G$. Give an example with $H$ connected, $H \neq G$.

10. If a linear group $G$ has an Abelian dense subgroup $H$, then $G$ itself is Abelian. Give an example with $H \neq G$.

11. Show that a subgroup $H$ of a linear group $G$ is discrete in $G$ if and only if there is a neighborhood of 1 in $G$ that contains no other elements of $H$.

12. Show that a discrete normal subgroup of a connected linear group $G$ is contained in the center of $G$. [Suggestion: for a continuous path $a(\tau)$ in $G$ and an element $c$ in a discrete normal subgroup, consider $a(\tau)c a(\tau)^{-1}$ as a function of $\tau$.]

13. Show that a linear group $G$ is discrete (in its group topology, as defined above) if and only if its Lie algebra $\mathfrak{g}$ is zero. [Careful: $GL(n, \mathbb{Q})$ is discrete in its group topology, but is not discrete in $GL(n, \mathbb{R})$; it is dense in $GL(n, \mathbb{R})$.]

14. Prove Remark 5. [First prove Lemma 3A for the form $\langle x, y \rangle$ on $\mathbb{R}^{p+1}$. Check that the remaining argument carries over from $n = 3$ to $n = p + 1$.]

2.5 The Lie correspondence

We shall prove in this section that the passage from a linear group to its Lie algebra sets up a one-to-one correspondence between connected linear groups and linear Lie algebras, the inverse being the passage from a linear Lie algebra to the group generated by its exponentials. This is the essence of Lie theory, as Lie must have understood it, even though Lie's conception was on the one hand broader, in that he considered transformations which were not necessarily linear, but on the other hand less complete, in that he took a local point of view. The global Lie correspondence is a refinement that is rather routine once the appropriate topological notions are available. The version of the Lie correspondence stated above follows from Satz 1 of Freudenthal (1941) and can be found in Bourbaki (1960) in the context of general Lie groups.

The essence of the Lie correspondence, in turn, is the Campbell–Baker–Hausdorff formula in its qualitative form, saying that in exponential coordinates the group multiplication is given by a bracket series and therefore completely determined by the Lie algebra, at least in a neighborhood of the identity. (Actually, Lie himself might object, if he could: he was not fond of any such algebraic formulation of his theory, which he conceived of as being essentially geometric and analytic. Even today the Lie correspondence is often established without Campbell–Baker–Hausdorff; but the principle that 'the Lie algebra determines the group' is certainly most simply and forcefully expressed by this formula.)

To succinctly state the Lie correspondence we use the following notation. The Lie algebra of a linear group $G$ will be denoted $L(G)$ rather than $\mathfrak{g}$ when it is necessary to bring out its dependence on $G$. Furthermore, we shall use the characterization of $L(G)$ in terms of exp:

$$L(G) = \{ x \in M \mid \exp x \in G \text{ for all } x \in \mathbb{R} \}.$$ 

On the other hand, given a linear Lie algebra $\mathfrak{g}$, we denote by $\Gamma(\mathfrak{g})$ the linear group generated by exp:

$$\Gamma(\mathfrak{g}) = \{ \exp x_1 \cdots \exp x_k \mid x_1, x_2, \ldots, x_k \in \mathfrak{g} \}.$$ 

$\Gamma(\mathfrak{g})$ is simply called the linear group generated by $\mathfrak{g}$.

**Theorem 1 (The Lie Correspondence).** There is a one-to-one correspondence between connected linear groups $G$ and linear Lie algebras $\mathfrak{g}$ given by

$$G \leftrightarrow \mathfrak{g}$$

if

$$\mathfrak{g} = L(G) \quad \text{or equivalently} \quad G = \Gamma(\mathfrak{g}).$$

**Proof** $\Gamma(L(G)) = G$. This says that $G$ is generated by $\exp L(G)$, which is (d) of Proposition 1, §2.4.

$\mathfrak{g} = L(\Gamma(\mathfrak{g}))$. Let $\mathfrak{g}$ be a linear Lie algebra. Then $\Gamma(\mathfrak{g})$ is connected because any element $\exp x_1 \exp x_2 \cdots \exp x_k$ of $\Gamma(\mathfrak{g})$ can be joined to 1 by the continuous path $\exp \tau x_1 \exp \tau x_2 \cdots \exp \tau x_k$, $\tau \in \mathbb{R}$.

$\mathfrak{g} \subset L(\Gamma(\mathfrak{g}))$ is clear since $\exp (\tau X) \in \Gamma(\mathfrak{g})$ for all $X \in \mathfrak{g}$. The point is to show that $L(\Gamma(\mathfrak{g})) \subset \mathfrak{g}$. Let

$$U = \{ x \in \mathfrak{g} \mid \| x \| < \epsilon \} \quad \text{and} \quad \bar{U} = \{ x \in \mathfrak{g} \mid \| x \| \leq \epsilon \}$$

for small $\epsilon > 0$. From Campbell–Baker–Hausdorff we know that the equation

$$\exp Z = \exp x \exp Y$$

defines a map $Z = C(X, Y)$ from $\bar{U} \times \bar{U}$ to a neighborhood of 0 in $\mathfrak{g}$; $C(X, Y)$ is the Campbell–Baker–Hausdorff series. We set $V = C(U, U)$, $\bar{V} = C(\bar{U}, \bar{U})$. Thus $\exp (V) = \exp (U) \exp (U)$. Since $C(X, Y)$ reduces to $C(0, Y) = Y$ for $X = 0$, the map $U \rightarrow \mathfrak{g}$, $Y \rightarrow C(X, Y)$ (X fixed) has a differential of rank $\dim \mathfrak{g}$ at $Y = 0$, as is obvious if $X = 0$ and remains true for $X$ in a neighborhood of 0 by continuity. The Inverse Function Theorem implies that $C(X, U)$ is an open neighborhood of $X$ in $\mathfrak{g}$ provided $X$ is sufficiently close to 0 in $\mathfrak{g}$ (which we may
assume to be the case for \( X \in \bar{V} \) and provided the \( \epsilon \) defining \( U \) is sufficiently small. This we assume to be so.

The set \( \bar{V} = C(\bar{U}, \bar{U}) \) is covered by the open sets \( C(X, U), X \in \bar{V} \) (because certainly \( X \in C(X, U) \)). Since \( V \) is a compact subset of \( g \) (being a continuous image of the compact set \( \bar{U} \times \bar{U} \)) already finitely many \( C(X, U) \) cover \( \bar{V} \), say \( C(X_j, U), j = 1, \ldots, N, X_j \in V \). Write \( \exp X_j = a_j^j a_j^{j'} \) with \( a_j^j, a_j^{j'} \in \exp \bar{U} \) and apply \( \exp \) to \( V \subset \bigcup_{j} C(X_j, U) \) to find that

\[
\exp \bar{V} \exp \bar{U} \subset \bigcup_j a_j^j a_j^{j'} \exp \bar{U},
\]
even with \( \bar{U} \) replaced by \( U \) on the right. Let \( \{b_j : j = 1, 2, \ldots \} \) be the (countable) set of all products of finite sequences from \( \{a_j^j, a_j^{j'} : j = 1, \ldots, N \} \) and write \( \exp \bar{U} \) for the set of \( k \)-fold products of elements of \( \exp \bar{U} \). From the above inclusion one gets inductively that

\[
(\exp \bar{V})^k \subset \bigcup_{j=1}^{\infty} b_j (\exp \bar{U})^k
\]
for all \( k \geq 1 \). Hence \( \Gamma(g) = \bigcup_{k=1}^{\infty} (\exp \bar{U})^k \) is expressible as a countable union

\[
\Gamma(g) = \bigcup_{j=1}^{\infty} b_j (\exp \bar{U})
\]
for appropriate \( b_j \in \Gamma(g) \). By Baire’s Covering Lemma (proved below; see also the comment (b) thereafter):

**some \( b_j \exp U \) contains a neighborhood of some point \( a_o \) in \( \Gamma(g) \).**

(2)

Say

\[
b_j \exp U \supset a_o \exp \bar{U},
\]
where \( \bar{U} = \{X \in L(\Gamma(g)) \mid \|X\| < \epsilon \} \) for some \( \epsilon > 0 \). Then

\[
\exp \bar{U} \subset c \exp U,
\]
where \( c = a_o^{-1} b_j \). This implies that for all \( X \in \bar{U} \)

\[
\exp \bar{X} = c \exp X
\]
with \( X \in U \). Furthermore, for \( \epsilon \) and \( \bar{\epsilon} \) sufficiently small, \( X \in U \) and \( \bar{X} \in \bar{U} \) will be arbitrarily close to 0, hence \( c = \exp(-X) \exp(\bar{X}) \) will be close to 1 and the unique solution of \( (X) \) for \( X \in U \) is

\[
X = \log \left( c^{-1} \exp \bar{X} \right).
\]
Replacing \( \bar{X} \) by \( \tau \bar{X} \) with \( \tau \) close to 0 in \( \mathbb{R} \) we see that

\[
\exp \tau \bar{X} = c \exp X(\tau)
\]
with \( X(\tau) \in U \) depending differentiably on \( \tau \). Setting \( \tau = 0 \) we find \( c = \exp(-X(0)) \) and therefore

\[
\exp \tau \bar{X} = \exp(-X(0)) \exp X(\tau).
\]
Differentiating this equation at \( \tau = 0 \) we obtain

\[
\bar{X} = -\exp(\text{ad}(X(0))) X'(0),
\]
which lies in \( g \) since \( X(0), X'(0) \in g \) and \( g \) is a Lie algebra. Thus the neighborhood \( \bar{U} \) of 0 in \( L(\Gamma(g)) \) is contained in \( g \), hence \( L(\Gamma(g)) \subset g \) as required.

It remains to show that (1) implies (2). This will follow from the following general lemma, which will be useful more than once.

**Lemma 2 (Baire’s covering lemma).** Let \( \{A_j\} \) be a countable family of subsets of \( G \) that cover \( G \):

\[
G = \bigcup_{j=1}^{\infty} A_j.
\]

Then the closure \( \bar{A}_j \) of some \( A_j \) contains an open subset of \( G \).

**Comments.**

(a) One could as well take the \( A_j \) to be closed in the first place.

(b) If \( A_j \) contains an open, dense subset of its closure \( \bar{A}_j \), as does a ball, for example, then some \( A_j \) itself will have to contain an open subset of \( G \).

(c) The lemma (and its proof) hold in more general spaces, in particular in manifolds, to be defined later.

**Proof.** (By contradiction.) Assume no \( \bar{A}_j \) contains an open subset of \( G \). The part of \( G \) outside of \( \bar{A}_1 \) is then certainly non-empty, hence (being open) contains
contains no other element of $\Gamma$: this follows from the fact that $\exp: a \to A$ is invertible in a neighborhood of 0.

We need a lemma, which holds in a more general context:

**Lemma 5.** For any discrete subgroup $\Gamma$ of an $n$-dimensional real vector space $V$ there is a basis $u_1, u_2, \ldots, u_n$ of $V$ so that $\Gamma$ consists of exactly the elements of $V$ of the form $\sum u_1 + \cdots + \sum u_p$ with $n_1, \ldots, n_p \in \mathbb{Z}$ (some $p \leq n$).

Assuming this, for the moment, we choose such a basis for $V = a$ and $\Gamma$ as above. The map $\mathbb{R}^n \to A$, $(\xi_1, \ldots, \xi_n) \to \exp(\sum \xi_1 u_1 + \cdots + \sum \xi_n u_n)$ induces the required isomorphism $\mathbb{T}^p \times \mathbb{R}^q \to A$ when we identify $\mathbb{T}^p$ with $\mathbb{R}^p/\mathbb{Z}^p$ ($n = p + q$).

This completes the proof of proposition. QED

**Proof of Lemma 5.** The proof will provide an inductive construction of such a basis $\{u_k\}$. Let $W$ be a subspace of $V$ so that $\Gamma \cap W = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$ for some basis $u_1, \ldots, u_m$ of $W$. (We could start with $W = \{0\}$. Suppose there is an element $u$ of $\Gamma$ that does not lie in $U$. Consider the set of points of $V$ in the form (Figure 2)

$$\sum \xi_1 u_1 + \cdots + \sum \xi_m u_m + \sum \xi u, \quad 0 \leq \xi_k \leq 1, \quad 0 \leq \xi \leq 1.$$  

Since this set is bounded (in a Euclidean norm on $V$), it can contain only finitely many points from the discrete set $\Gamma$. Hence there is a point $v$ of the form (7) in $\Gamma$ with minimal non-zero coefficient $\xi$, say $\xi = \mu > 0$. Any other element $w$ of $\Gamma$ of the form $\sum \xi_1 u_1 + \cdots + \sum \xi_m u_m + \sum \xi w$, will then have as coefficient $\xi$ of $w$ a multiple of the (minimal) coefficient $\mu$. (Otherwise one could write $\xi = k\mu + \nu$ with $k \in \mathbb{Z}$ and $0 < \nu < \mu$ to obtain an element $w = k\nu$ of $\Gamma$ of the form (8) whose coefficient $\xi = \nu$ is less than that of $v$. The coefficients $\xi_k$ of this element may be made to satisfy $0 \leq \xi_k \leq 1$ after subtraction of suitable multiples of the $u_k$.) This means that

$$\Gamma \cap (W + \mathbb{R}u) = (\Gamma \cap W) + \mathbb{Z}u = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m + \mathbb{Z}v.$$

$\{u_1, \ldots, u_m, v\}$ forms a basis for $W + \mathbb{R}v$. Replacing $W$ by $W + \mathbb{R}v$ and repeating the argument one finds that $\Gamma = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ for linearly independent vectors $u_1, \ldots, u_p$, which may be completed to a basis for $V$.

QED

We continue with the 'group $\to$ Lie algebra dictionary'.

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3A coordinate-ball consists of the elements with coordinates (relative to some coordinate system) in some ball in $\mathbb{R}^n$. 

---

2. Lie theory

2.5. The Lie correspondence

Fig. 2
Proposition 6. Let $G$ be a connected linear group, $\mathfrak{g}$ its Lie algebra. A connected subgroup of $G$ is normal if and only if its Lie algebra is an ideal of $\mathfrak{g}$.

Explanation. A subgroup $H$ of $G$ is normal subgroup if $aHa^{-1} = H$ for all $a \in G$; a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is an ideal if $[X,h] \subset h$ for all $X \in \mathfrak{g}$ (briefly: $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$).

Proof. Assume $H$ is normal in $G$. Then $a\exp(\tau Y)a^{-1} \in H$ for all $Y \in \mathfrak{h}$, $a \in G$. Differentiating at $\tau = 0$ we find that $\text{Ad}(a)Y \in \mathfrak{h}$ for all $a \in G$. Take $a = \exp \sigma X$ with $X \in \mathfrak{g}$, $\sigma \in \mathbb{R}$ and differentiate with respect to $\sigma$ at $\sigma = 0$ to see that $[X,Y] \subset \mathfrak{h}$.

Conversely, assume $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Then for $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, $\exp X \exp Y \exp -X = \text{Ad}(\exp X) Y = \exp (\exp(\text{ad} X)Y) \in H$. Since $G$ and $H$ are generated by exponentials, one sees that $aba^{-1} \in H$ for all $a \in G$ and $b \in H$.

QED

Proposition 7. Let $G$ be a connected linear group, $\mathfrak{g}$ its Lie algebra. The commutator subgroup $(G,G)$ of $G$ is connected and its Lie algebra is the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

Explanation. The commutator subgroup of $G$, denoted $(G,G)$, is the subgroup generated by all $aba^{-1}b^{-1}$, $a,b \in G$. The commutator subalgebra of $\mathfrak{g}$ is the subalgebra generated (namely spanned, in view of Jacobi) by all $[X,Y], X,Y \in \mathfrak{g}$.

Proof. Elements of $(G,G)$ are products of commutators $aba^{-1}b^{-1}$ with $a,b \in G$. Since $G$ is connected, $a$ and $b$ can be joined to 1 by continuous curves $a(\tau)$ and $b(\tau)$. Elements of $(G,G)$ can be joined to 1 by the corresponding products of commutators $a(\tau)b(\tau)a(\tau)^{-1}b(\tau)^{-1}$. So $(G,G)$ is connected.

$[\mathfrak{g}, \mathfrak{g}] \subset L((G,G))$. Let $X,Y \in \mathfrak{g}$. Then $\exp X \exp Y \exp -X \exp -Y$ lies in $(G,G)$, and differentiation at $\sigma = \tau = 0$ shows that $[X,Y] \in L((G,G))$.

$L((G,G)) \subset [\mathfrak{g}, \mathfrak{g}]$. We shall show the equivalent statement $(G,G) \subset \Gamma([\mathfrak{g}, \mathfrak{g}])$.

$(G,G)$ is generated by elements $\exp X \exp Y \exp -X \exp -Y$ with $X,Y \in \mathfrak{g}$, so it suffices to show that these lie in $\Gamma([\mathfrak{g}, \mathfrak{g}])$. For this purpose, consider

$$a(\tau) = \exp -X \exp \tau Y \exp X \exp -\tau Y$$

with $\tau \in \mathbb{R}$. One can write

$$a(\tau) = \exp -X \exp (\text{Ad}(\exp \tau Y)X)$$

$$= \exp -X \exp (\exp(\tau \text{ad} Y)X)$$

$$= \exp -X \exp X(\tau),$$

where we have put $X(\tau) = (\exp \tau \text{ad} Y)X$. Differentiating with respect to $\tau$ we get

$$a'(\tau) = a(\tau) \frac{1 - \exp(-\text{ad}(X(\tau)))}{\text{ad}(X(\tau))} X'(\tau).$$

From the definition of $X(\tau)$ it is clear that $X'(\tau) \in [\mathfrak{g}, \mathfrak{g}]$, so

$$a'(\tau) \in a(\tau)[\mathfrak{g}, \mathfrak{g}].$$

QED

2.5. The Lie correspondence

By Corollary 4 of §2.2, applied to the group $\Gamma([\mathfrak{g}, \mathfrak{g}])$, this implies that $a(\tau) \in \Gamma([\mathfrak{g}, \mathfrak{g}])$ for all $\tau$. In particular $a(1) = \exp -X \exp Y \exp X \exp -Y \in \Gamma([\mathfrak{g}, \mathfrak{g}])$.

The centralizer (in $G$) of a subset $A$ of a linear group $G$ consists of the elements of $G$ that commute with those of $A$:

$$Z_G(A) = \{ c \in G \mid ca = ac \quad \text{for all} \quad a \in A \};$$

the centralizer (in $\mathfrak{g}$) of a subset $a$ of its Lie algebra $\mathfrak{g}$ consists similarly of the elements of $\mathfrak{g}$ which commute with those of $a$:

$$Z_\mathfrak{g}(a) = \{ c \in \mathfrak{g} \mid \text{Ad}(c)X = X \quad \text{for all} \quad X \in \mathfrak{a} \}.$$

In the Lie algebra of $G$ one defines analogously:

$$z_\mathfrak{g}(A) = \{ Y \in \mathfrak{g} \mid \text{Ad}(a)Y = Y \quad \text{for all} \quad a \in A \},$$

$$z_\mathfrak{g}(a) = \{ Y \in \mathfrak{g} \mid [X,Y] = 0 \quad \text{for all} \quad X \in \mathfrak{a} \},$$

called the centralizer (in $\mathfrak{g}$) of $A$ and of $a$, respectively. (There is little lost if one considers only subgroups for $A$ and subalgebras for $a$, since centralizers do not change by passing to the subgroup generated by $A$ or to the subalgebra generated by $a$. But it is often convenient to use centralizers of single elements, for example.)

Proposition 8. Let $G$ be a linear group, $\mathfrak{g}$ its Lie algebra, $A$ a subset of $G$, and $a$ a subset of $\mathfrak{g}$.

(a) $Z_G(A)$ is a group with Lie algebra $z_\mathfrak{g}(A)$.

(b) $Z_\mathfrak{g}(a)$ is a group with Lie algebra $z_\mathfrak{g}(a)$.

(c) If $A$ is a connected subgroup of $G$ and $a$ its Lie algebra, then $Z_G(A) = Z_\mathfrak{g}(a)$ and has Lie algebra $z_\mathfrak{g}(A) = z_\mathfrak{g}(a)$.

QED

We omit the proof, which follows the by now familiar differentiation-exponentiation pattern. We record in particular the special case when $A = G$ and $a = \mathfrak{g} : Z_G(G)$, the subgroup whose elements commute with all of $G$, is called the center of $G$, denoted $Z(G)$; similarly, $z_\mathfrak{g}(\mathfrak{g})$, the subalgebra whose elements commute with all of $\mathfrak{g}$, is called the center of $\mathfrak{g}$, denoted $z(\mathfrak{g})$.

Upon taking $A = G$, $a = \mathfrak{g}$ in part (c) one finds:

Corollary 9. Assume $G$ is a connected linear group. The Lie algebra of the center $Z(G)$ of $G$ is the center $z(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$.

In analogy with centralizers one can define normalizers. We shall only consider two cases. The normalizer (in $G$) of a subgroup $A$ of a linear group $G$ is:

$$N_G(A) = \{ c \in G \mid cAc^{-1} = A \}.$$
The normalizer (in $g$) of a subalgebra $a$ of $g$ is:

$$n_g(a) = \{Y \in g \mid [Y, a] \subset a\}.$$ 

**Proposition 10.** Let $G$ be a linear group, $g$ its Lie algebra, $a$ a connected subgroup of $G$, and $a$ its Lie algebra. Then $N_G(a) = N_g(a)$ and has Lie algebra $n_G(A) = n_g(a)$. (QED)

Again we omit the proof.

**Problems for §2.5**

1. (a) Prove Proposition 8, (b) prove Proposition 10.

2. Let $G \subset GL(E)$ be a connected linear group, $F$ a subspace of $E$. Show: $F$ is $G$-stable [i.e. $aF \subset F$ for all $a \in G$] if and only if $F$ is $g$-stable [i.e. $XF \subset F$ for all $X \in g$].

3. *Lie algebra cocycles.* Let $g$ be a real Lie algebra. A **cocycle** on $g$ is a bilinear function $\omega: g \times g \to \mathbb{R}$ that satisfies

$$\omega(X, Y) = -\omega(Y, X),$$

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0.$$

A **coboundary** is a cocycle of the form $\omega(X, Y) = \varphi([X, Y])$ where $\varphi \in g^*$ is a linear functional on $g$.

Let $G$ be a linear group, $g$ its Lie algebra, $\omega$ a cocycle on $g$. For any $a \in G$ define another cocycle $\omega^a$ on $g$ by the formula

$$\omega^a(X, Y) = \omega(\text{Ad}(a)X, \text{Ad}(a)Y).$$

Show that for a **connected** linear group $G$, $\omega^a$ differs from $\omega$ by a coboundary for any $a \in G$. [Suggestion: consider a $C^1$ path $a = a(\tau)$ and differentiable $\omega^a(X, Y)$ with respect to $\tau$; write $a'(\tau)$ as $a'(\tau) = a(\tau)Z(\tau)$ with $Z(\tau) \in g$.]

4. If a linear group of dimension $\geq 2$ has a dense one-parameter subgroup, then it is isomorphic with a torus $T^n$.

**Complex linear groups.** A linear group $G \subset GL(n, \mathbb{C})$ is complex if the Lie algebra $g$ of $G$ is a complex subspace of $gl(n, \mathbb{C})$.

5. List all complex groups among the groups mentioned in §2.1. [Make sure your list is complete.]

6. The only connected complex abelian group which is a compact subset of the matrix space is the trivial group.

**Semidirect products.** Let $G$ be a group, $M$ and $N$ subgroups of $G$ with $N$ normal. We say $G$ is the semidirect product of $M$ and $N$ if every $a \in G$ can be uniquely written in the form $a = mn$ with $m \in M$ and $n \in N$. (Equivalently: $G = MN$, $M \cap N = \{1\}$.) We then write $G = MN$ (semidirect), or $G = M \rtimes N$. One could also write $G = NM$ (semidirect) or $G = N \rtimes M$. If not clear from the context, it must be specified which of the two groups $M, N$ is normal. If they are both normal, the product is direct. When $G$ is a linear group we also require that the map $G \to M \times N, mn \to (m, n)$ be analytic. This requirement is superfluous if $M$ and $N$ have countably many connected components, see problem 9.

7. Suppose the linear group $G$ is a semidirect product $G = MN$.

(a) Show that its Lie algebra is the direct sum $g = m \oplus n$ of the Lie algebras $m$ and $n$, $m$ being a subalgebra, $n$ an ideal of $g$. [One says that the Lie algebra $g$ is the semidirect product of $m$ and $n$.]

(b) Show that the exponential map $\exp: g = m + n \to G = MN$ of $G$ takes the form

$$\exp(X + Y) = \exp X \exp A(X) Y,$$

where $A(X)$ is a (generally non-linear) transformation of $n$ depending on $X \in m$. [It is understood that $X \in m, Y \in n$, both sufficiently close to 0.]

(c) Show that when $N$ is abelian, $A(X) : n \to n$ is given by

$$A(X) = \frac{\exp(-\text{ad } X) - 1}{\text{ad } X}.$$

(d) Show that the group of affine transformations $x \to ax + b, a \in GL(E), b \in E,$ is the semidirect product of the subgroup $GL(E)$ of linear transformations $x \to ax$ and the subgroup $E$ of translations $x \to x + b$. [See Example 6, §2.1.]

8. Let $G$ be the group of block-triangular matrices of the form

$$
\begin{bmatrix}
a_1 & * & * & \cdots & * \\
0 & a_2 & * & \cdots & * \\
& & \ddots & \ddots & \\
& & & \ddots & \ddots \\
0 & 0 & 0 & \cdots & a_m
\end{bmatrix}
$$
Each \( a_k \) is an invertible block of some fixed size. Let \( M \) be the subgroup of block-diagonal matrices

\[
\begin{bmatrix}
a_1 & 0 & 0 & \cdots & 0 \\
0 & a_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_m
\end{bmatrix}
\]

\( N \) the subgroup of unipotent block-triangular matrices

\[
\begin{bmatrix}
1 & * & * & \cdots & * \\
0 & 1 & * & \cdots & * \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

Show that \( G \) is the semidirect product \( G = MN \).

9. Show that the analyticity requirement in the definition of ‘semidirect product’ is superfluous if \( M \) and \( N \) have countably many connected components. [Suggestion: use Lemma 2 to show that \( g = m + n \).]

10. Let \( G \) be a linear group, \( A, B \) two subgroups of \( G \), and \( g, a, b \) their algebras. Show:

(a) Assume \( g = a + b \), \( G \) is connected, and \( AB \) is closed in \( G \). Then \( G = AB \).

(b) Give an example with \( g = a + b \), \( G \) connected, but \( G \neq AB \) (even with \( A \) and \( B \) closed).

(c) Assume \( G = AB \) and \( A, B \) have countably many components. Then \( g = a + b \). [Suggestion for (b). Take \( G = \text{SL}(2, R) \), \( A = \{ \text{upper triangular} \} \), and find a suitable \( B \).]

11. Let \( G \) be a connected linear group with the property that \( G^* = G \). (\( a^* \) is the adjoint of \( a \) with respect to a positive definite form.) Fix a self-adjoint element \( Z \in g \); \( Z^* = Z \). For \( \lambda \in R \), let \( g_\lambda = \{ X \in g \mid \text{ad}(Z)X = \lambda X \} \). Show:

(a) \( g = \sum_\lambda g_\lambda \) (direct sum).

(b) \( [g_\lambda, g_\mu] \subseteq g_{\lambda + \mu} \).

(c) \( (g_\lambda)^* = g_{-\lambda} \).

Problems for §2.5

12. In problem 11 take \( G = \text{SL}(2, R) \),

\[
Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Find \( K \) and \( Q \).

13. In problem 11, take \( G = \text{GL}(n, R) \), and \( Z = (1, 0, \ldots, 0) \) (diagonal matrix). Find \( K \) and \( Q \).

14. In problem 11, take \( G = \text{GL}(n, R) \), and \( Z = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) with \( \lambda_1 > \lambda_2 > \cdots > \lambda_n \) (diagonal matrix). Find \( K \) and \( Q \). [Compare your answer with exercise 9, §2.1. Suggestion: to find the \( g_\lambda \), review the proof of Lemma 8, §1.2.]

15. Let \( J \) be the \( n \times n \) Jordan block

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

(a) Show that the centralizer of \( J \) in \( \text{gl}(n, R) \) or \( \text{gl}(n, C) \) consists of all matrices of the form

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_{n-1} & a_n \\
0 & a_1 & a_2 & \cdots & a_{n-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & a_1 & a_0
\end{bmatrix}
\]

(b) Describe the one-parameter group generated by \( J \) and its centralizer in \( \text{GL}(n, R) \) or \( \text{GL}(n, C) \).
2.6 Homomorphisms and coverings of linear groups

The Lie correspondence 'groups $\rightarrow$ Lie algebras' works not only on the level of sets, but also on the level of maps. The main point is:

**Theorem 1.** Let $f : G \rightarrow H$ be a differentiable homomorphism of linear groups, $\varphi : g \rightarrow h$ its differential at 1. Then $\varphi$ is a homomorphism of Lie algebras and

$$f(\exp X) = \exp \varphi(X)$$

for all $X \in g$.

**Proof.** The proof uses the familiar 'differentiation-exponentiation' method. We first prove that $f(\exp X) = \exp \varphi(X)$. We have:

$$\varphi(X) = \left. \frac{d}{d\tau} f(\exp \tau X) \right|_{\tau = 0}$$

and generally

$$\left. \frac{d}{d\sigma} f(\exp \sigma X) \right|_{\tau = 0} = \left. \frac{d}{d\tau} f(\exp(\sigma + \tau)X) \right|_{\tau = 0}$$

$$= \left. \frac{d}{d\tau} f(\exp \sigma X \exp \tau X) \right|_{\tau = 0}$$

$$= f(\exp \sigma X) \left. \frac{d}{d\tau} f(\exp \tau X) \right|_{\tau = 0}$$

$$= f(\exp \sigma X) \varphi(X).$$

This shows that $\alpha(\sigma) = f(\exp \sigma X)$ satisfies the differential equation $\alpha'(\sigma) = \alpha(\alpha) \varphi(X)$ and of course also the initial condition $\alpha(0) = 1$. Therefore $\alpha(\sigma) = \exp \sigma \varphi(X)$ and in particular $f(\exp X) = \exp \varphi(X)$ as required.

To show that $\varphi : g \rightarrow h$ is a Lie algebra homomorphism we must show that

$$\varphi[X, Y] = [\varphi X, \varphi Y]$$

for all $X, Y \in g$. To prove this, start with the equation

$$f(\exp \tau X \exp \sigma Y \exp -\tau X) = \exp(\tau \varphi X) \exp(\sigma \varphi Y) \exp(-\tau \varphi X),$$

valid for all $\sigma, \tau \in \mathbb{R}$, by what has just been proved. Differentiate with respect to $\sigma$ at $\sigma = 0$:

$$\varphi(\exp(\tau X) Y \exp(-\tau X)) = \exp(\tau \varphi X) \varphi(Y) \exp(-\tau \varphi X).$$

Now differentiate with respect to $\tau$ at $\tau = 0$:

$$\varphi[X, Y] = [\varphi X, \varphi Y].$$

QED

**Terminology and notation.** A homomorphism of linear groups $f : G \rightarrow H$ is required to be differentiable, by definition. Its differential at $1$, $\varphi : g \rightarrow h$ is called its Lie map, denoted $L(f)$ or $L f$.

**Remark 2.** One sees from the theorem that a differentiable homomorphism between matrix groups is in fact analytic. It can be shown that the same is true even for a continuous homomorphism. While the proof is not difficult, we shall not give it here, as its only purpose would be a justification for restricting oneself to differentiable homomorphisms in the first place. The proof may be found in §11 of Freudenthal and de Vries (1959), for example.

**Example 3 (Aut, Der, Ad, and ad).** Let $G$ be a linear group. For $a \in G$, define the conjugation by $a$ to be the transformation $c(a) : G \rightarrow G, b \rightarrow aba^{-1}$. $c(a)$ is an automorphism of $G$, and is also called the inner automorphism determined by $a$. Since

$$\left. \frac{d}{d\tau} (a \exp(\tau X) a^{-1}) \right|_{\tau = 0} = a X a^{-1},$$

the Lie map of $c(a) : G \rightarrow G, b \rightarrow aba^{-1}$ is the automorphism $a : g \rightarrow g, X \rightarrow a X a^{-1}$. The theorem says that $\text{Ad} a$ is a Lie algebra homomorphism and that

$$a \exp(X) a^{-1} = \exp(a X a^{-1}),$$

which is clear.

Note that $\text{Ad} a$ is invertible (with inverse $\text{Ad}(a^{-1})$) and

$$\text{Ad}(ab) = \text{Ad}(a) \text{Ad}(b),$$

so that $a \rightarrow \text{Ad} a$ is a homomorphism from $G$ to the group $\text{Aut}(g)$ of automorphisms of the Lie algebra $g$:

$$\text{Ad} : G \rightarrow \text{Aut}(g).$$

The Lie algebra of $\text{Aut}(g)$ is $\text{Der}(g) = \{\text{derivations of } g\}$ (Proposition 9, §2.2). The Lie map of $\text{Ad} : G \rightarrow \text{Aut}(g)$ is therefore a homomorphism $g \rightarrow \text{Der}(g)$. We have

$$\left. \frac{d}{d\tau} \text{Ad}(\exp(\tau X)) \right|_{\tau = 0} = \left. \frac{d}{d\tau} \exp(\tau X Y \exp(-\tau X)) \right|_{\tau = 0}$$

$$= XY - YX$$

$$= [X, Y]$$

$$= \text{ad}(X) Y.$$
2. Lie theory

Example 4. Recall that $T^n = \{ (\epsilon_1, \ldots, \epsilon_n) \mid \epsilon_k = e^{2\pi i \theta_k}, \theta_k \in \mathbb{R} \}$ (Example 6 of §2.1). Let

$$f : T^n \to \mathbb{C}^n$$

be a homomorphism of matrix groups. The Lie map of $f$ is an $\mathbb{R}$-linear functional

$$\varphi : t^n = \{ (2\pi i \theta_1, \ldots, 2\pi i \theta_n) \mid \theta_k \in \mathbb{R} \} \to \mathbb{C},$$

Hence of the form

$$\varphi(X) = 2\pi i (l_1 \theta_1 + \cdots + l_n \theta_n)$$

for certain $l_k \in \mathbb{R}$. The equation

$$f(\exp X) = e^{\varphi(X)}$$

implies that $\varphi(X) \in 2\pi i \mathbb{Z}$ when $\exp X = 1$, i.e. $(l_1 \theta_1 + \cdots + l_n \theta_n) \in \mathbb{Z}$ when $\theta_k \in \mathbb{Z}$, i.e. $l_k \in \mathbb{Z}$.

Conversely, if $l_k \in \mathbb{Z}$ for $k = 1, \ldots, n$, then the above equation defines a homomorphism $f$. These homomorphisms $f$ are therefore in one-to-one correspondence with $\mathbb{Z}^n$.

Note that each such homomorphism $T \to \mathbb{C}^n$ extends uniquely to a holomorphic homomorphism $(\mathbb{C}^n)^n \to \mathbb{C}^n$, given by the same formula. Furthermore, every such holomorphic homomorphism is of this form, as one sees in the same way.

We return to the general situation of Theorem 1.

Corollary 5. Let $f : G \to H$ be a homomorphism of linear groups, $L(f) : g \to h$ its Lie map.

(a) $L(\ker f) = \ker L(f)$.

(b) $L(\text{im } f) = \text{im } L(f)$, provided $G$ has countably many connected components.

Proof.

(a) $X \in L(\ker f) \Leftrightarrow f(\exp \tau X) = 1$ for all $\tau \in \mathbb{R} \Leftrightarrow \exp \tau Lf(X) = 1$ for all $\tau \in \mathbb{R} \Leftrightarrow Lf(X) = 0 \Leftrightarrow X \in \ker L(f)$.

(b) $\langle r \rangle f(\exp \tau X) = \exp \tau Lf(X)$ implies $L(\text{im } f) \subseteq \text{im } L(f)$.

(c) It suffices to show that $\Gamma(\text{im } Lf)$ contains an open subset of $\text{im } f$. This follows from Baire’s Covering Lemma (Lemma 2, §2.5): write $G_0 = \bigcup_{j=1}^{\infty} A_j$ as in eqn (1) of §2.5. Each $A_j$ is a coordinate-ball in $G_0$. Since $G$ has countably many components, we can choose a countable family $\{a_k\}$ of elements of $G$, one from each component. Replacing the family $\{A_j\}$ by $\{a_k A_j\}$ we may assume $G = \bigcup_{j=1}^{\infty} A_j$. Then $f(G) = \bigcup_{j=1}^{\infty} f(A_j)$. Each $f(A_j)$ is closed ($A_j$ is compact), hence some $f(A_j)$ contains an open subset of $f(G)$.

Corollary 6. Let $f : G \to H$ be a homomorphism of linear groups.

(a) $f$ is locally injective if and only if $Lf$ is injective.

(b) Assume $H$ is connected and $G$ has countably many connected components. Then $f$ is surjective if and only if $Lf$ is surjective.

2.6. Homomorphisms and coverings of linear groups

Explanation. $f : G \to H$ is locally injective if every $a \in G$ has a neighborhood on which $f$ is one-to-one. This is the case if and only if $f$ is one-to-one on some neighborhood of $a = 1$, which happens if and only if $L(\ker f) = \{0\}$. For example, the homomorphism $\theta \to e^{2\pi i \theta}$, from $\mathbb{C}, +$ to $\mathbb{C}^\times, \times$ (or from $\mathbb{R}$ to $T$) is locally injective, but certainly not globally injective.

$f : G \to H$ is locally surjective if any $a \in G$ has a neighborhood which gets mapped onto a neighborhood of $f(a)$. The statement (a) holds with ‘locally injective’ replaced by ‘locally surjective’, but the global statement (b) is usually more interesting. In any case, a locally surjective homomorphism into a connected group is surjective, because a connected group is generated by any open subset. This explains the asymmetry in the statements (a) and (b).

Proof.

(a) Since $\exp$ is one-to-one on a neighborhood of 0, the equation

$$f(\exp X) = \exp Lf(X)$$

shows that $f$ is one-to-one in a neighborhood of 1 in $G \Leftrightarrow Lf$ is one-to-one in a neighborhood of 0 in $g$, which happens $\Leftrightarrow Lf$ is injective.

(b) $\text{im } f = H \Leftrightarrow L(\text{im } f) = h \mid [H \text{ is connected}] \Leftrightarrow \text{im } (Lf) = h \mid [\text{preceding corollary}].$

Example 7 (Adjoint representation). Let $G$ be a connected linear group. For fixed $a \in G$ we have the inner automorphism $c(a) : G \to G, b \to aba^{-1}$. Its Lie map is $\text{Ad}(a) : g \to g$. The kernel of $c(a)$ is $Z_G(a) = \{ b \in G \mid ab = ba \}$, the centralizer of $a$ in $G$. By part (a) of Corollary 5, the Lie algebra of $Z_G(a)$ is $z_G(a) = \{ X \in g \mid \text{Ad}(a)X = X \}$.

Now consider the homomorphism $\text{Ad} : G \to \text{Aut}(g)$. Its Lie map is $\text{ad} : g \to \text{Der}(g)$. By part (b) of Corollary 5, the Lie algebra of $\text{Ad}(G)$ is $\text{ad}(g)$, the Lie algebra of inner derivations of $g$. The kernel of $\text{Ad} : G \to \text{Aut}(g)$ is $Z(G) = \{ c \in G \mid ca = ac \text{ for all } a \in G \}$, the center of $G$. By part (a) of the same corollary, the Lie algebra of $Z(G)$ is the kernel of $\text{ad} : g \to \text{Der}(g)$, which is $z(g) = \{ X \in g \mid [X,Y] = 0 \text{ for all } Y \in g \}$, the center of $g$.

A homomorphism $f : G \to H$ is locally bijective if every $a \in G$ has a neighborhood that gets mapped bijectively onto a neighborhood of $f(a)$.

Lemma 8. The following conditions on a homomorphism of linear groups, $f : G \to H$, are equivalent.

(a) $f$ is locally bijective.

(b) $Lf : L(G) \to L(H)$ is bijective.

For such a homomorphism, $\ker f$ is a discrete subgroup of $G$.

Proof. (a) $\Rightarrow$ (b). If $f$ is locally bijective, then $Lf$ must be bijective, because $Lf(X) = 0$ implies $f(\exp X) = 1$.

(b) $\Rightarrow$ (a). If $Lf$ is bijective, then $f$ is locally bijective at 1 (Inverse Function Theorem), hence locally bijective at any point (because $f(a_0a) = f(a_0)f(a))$.

For such an $f$ there is a neighborhood of 1 in $G$ where $f(a) = 1$ implies $a = 1$, which says that $\ker f$ is discrete. QED
A locally bijective homomorphism \( p : \hat{G} \to G \) between connected linear groups called a covering. A covering is necessarily surjective with discrete kernel, and these properties characterize coverings. \( G \) together with \( p \) is the called a covering group of \( G \); the covering homomorphism \( p \) need not be mentioned explicitly.

The kernel of a covering \( \hat{G} \to G \) is necessarily contained in the center of \( \hat{G} \), because generally a discrete normal subgroup \( Z \) of a connected group \( H \) is contained in the center of \( H \): by assumption, any \( a \in H \) may be connected to 1 by a continuous path \( a(\tau) \); for \( z \in Z \), the continuous function \( \tau \to a(\tau)za(\tau)^{-1} \) on \([0,1]\) takes on values in the discrete set \( Z \), hence must be constant. Thus \( a(\tau)za(\tau)^{-1} \equiv z \) and in particular \( za^{-1} = z \). As a consequence, the groups \( \hat{G} \) covered by a given group \( \hat{G} \) are isomorphic with \( \hat{G}/Z \) (as abstract groups), where \( Z \) is a discrete central subgroup of \( \hat{G} \).

We now return to the situation of Theorem 1. The equation

\[
f(\exp X) = \exp \varphi(X)
\]

may be interpreted as saying that in exponential coordinates a group homomorphism \( f : \hat{G} \to H \) becomes a Lie algebra homomorphism \( \varphi : \mathfrak{g} \to \mathfrak{h} \). Suppose, conversely, we start with a homomorphism \( \varphi : \mathfrak{g} \to \mathfrak{h} \) between linear Lie algebras. Then the equation \( f(\exp X) = \exp \varphi(X) \) defines a map from a neighborhood of 1 in the connected linear group \( G = \Gamma(\mathfrak{g}) \) to \( H = \Gamma(\mathfrak{h}) \). Furthermore, it is clear from the Campbell–Baker–Hausdorff Formula that

\[
f(ab) = f(a)f(b).
\]

for \( a, b \) in some neighborhood of 1 in \( G \). Such a map \( f \) is called a local homomorphism from \( \mathfrak{g} \) to \( \mathfrak{h} \). There remains the question if such a local homomorphism \( G \to H \) extends to a global homomorphism \( G \to H \), defined on all of \( G \). In general the answer is no. For example, \( a \to \log a \) defines a local homomorphism of \( \mathbb{C}^x, \times \to \mathbb{C}, + \) (or from \( \mathbb{T} \) to \( \mathbb{R} \)) that does not extend to a global homomorphism.

It is here that coverings come in. We introduce the following terminology. Let \( G \) and \( H \) be connected linear groups, \( \varphi : \mathfrak{g} \to \mathfrak{h} \) a homomorphism between their Lie algebras. Let \( p : \hat{G} \to G \) be a covering of \( G \). We say \( \varphi : \mathfrak{g} \to \mathfrak{h} \) lifts to \( \hat{G} : \hat{G} \to H \) (or simply lifts to \( H \)) if there is a homomorphism \( f : \hat{G} \to H \) so that \( L(f) = \varphi \circ L(p) : \hat{G} \to \mathfrak{h} \). This means the Lie map of \( f \) is the given \( \varphi \) when one identifies \( \hat{G} \) with \( G \) by the isomorphism \( L(p) : \hat{G} \to \mathfrak{g} \). If such an \( f \) exists it is unique since a homomorphism between connected linear groups is determined by its Lie map. If the local homomorphism \( G \to H \) corresponding to \( \varphi : \mathfrak{g} \to \mathfrak{h} \) does not extend to a global homomorphism \( f : G \to H \), then we may take \( \hat{G} = G \) and \( \varphi : \mathfrak{g} \to \mathfrak{h} \) lifts to \( f : \hat{G} \to H \). This is not always the case, as we observed, but \( \varphi \) always lifts to some covering \( \hat{G} \) of \( G \). This is an important fact, which we list as a theorem, even though it is very easy to prove:

**Theorem 9.** Let \( G \) and \( H \) be linear groups, \( G \) connected, \( \varphi : \mathfrak{g} \to \mathfrak{h} \) a homomorphism between their Lie algebras. Then there is a linear covering group \( \hat{G} \) of \( G \) so that \( \varphi \) lifts to \( \hat{G} \to H \).

**Proof.** Given \( G, H, \) and \( \varphi : \mathfrak{g} \to \mathfrak{h} \), let \( \hat{G} = \{ (X,Y) \in \mathfrak{g} \times \mathfrak{h} \mid Y = \varphi(X) \} \), the graph of \( \varphi \). \( \hat{G} \) is a linear Lie algebra. Let \( \hat{G} \) be the corresponding connected linear group. It is a subgroup of \( G \times H \); let \( p : \hat{G} \to G \) be the restriction to \( \hat{G} \) of the projection \( G \times H \to G \), and \( f : \hat{G} \to H \) the restriction of the projection \( G \times H \to H \). The Lie map \( L(p) \) of \( p \) is the map \( \hat{G} \to \mathfrak{g}, (X, \varphi(X)) \to X \), which is an isomorphism. Thus \( p : \hat{G} \to G \) is a covering. The Lie map of \( f \) is the map \( \hat{G} \to \mathfrak{h}, (X, \varphi(X)) \to \varphi(X) \), from which one sees that \( f : \hat{G} \to H \) lifts \( \varphi : \mathfrak{g} \to \mathfrak{h} \).

QED

One will ask whether every connected group admits a universal covering group, meaning a single covering group \( \hat{G} \) of \( G \) so that every Lie algebra homomorphism \( \varphi : \mathfrak{g} \to \mathfrak{h} \) lifts to this group. If such a universal covering group exists, it is unique up to an isomorphism compatible with the covering map: if \( \hat{G} \) and \( \hat{G}' \) are two universal covering groups, then the Lie algebra homomorphisms \( \mathfrak{g} \to \mathfrak{g}' \) and \( \mathfrak{g} \to \mathfrak{h} \) lift to homomorphisms \( \hat{G} \to \hat{G}' \) and \( \hat{G} \to \hat{G}' \), which are inverses of each other because the Lie map of their composite is the identity \( \mathfrak{g} \to \mathfrak{g} \). One may therefore speak of the universal covering group. Universal covering groups always exist when one admits general Lie groups (to be defined in §4.3) for the covering groups, but not when the covering groups are themselves required to be linear groups. For example, it can be shown that \( \text{SL}(2, \mathbb{R}) \) does not admit a linear group as universal covering group (Example 7, §4.3).

Within the category of general Lie groups, universal covering groups are characterized by the topological property of being simply connected, meaning any continuous, closed path can be continuously deformed into a point. The precise definition is not needed here, but we take for granted the following fact from the theory of covering spaces (see Massey (1967), for example): A simply connected topological space admits no non-trivial coverings, i.e. the only coverings of such a space are homeomorphisms. (The space is here assumed locally arcwise connected, i.e. every neighborhood of a point contains an arcwise connected neighborhood, as is evidently the case for linear groups). In particular, the only coverings of simply connected linear groups are isomorphisms. (Coverings of linear groups, as defined above are coverings in the sense of algebraic topology.) Theorem 9 therefore implies:

**Theorem 10.** Let \( G \) and \( H \) be linear groups, \( G \) simply connected, \( \varphi : \mathfrak{g} \to \mathfrak{h} \) a homomorphism between their Lie algebras. Then \( \varphi \) lifts to a homomorphism \( f : \hat{G} \to H \).

**Example 11 (SU(2) is simply connected).** \( SU(2) \) consists of the matrices

\[
\begin{pmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{pmatrix}
\]

with \( \alpha + \beta = 1 \) and is therefore homeomorphic with the sphere \( S^3 \) in \( \mathbb{C}^2 = \mathbb{R}^4 \), hence simply connected.

**Example 12 (The covering \( SU(2) \to SO(3) \)).** Recall the homomorphism \( p : SU(2) \to SO(3) \) of Example 2, §2.1, which is just the adjoint representation when \( \mathfrak{su}(2) \) is identified with \( \mathbb{R}^3 \) by a suitable basis. We have seen that its image
Proposition 12A. The homomorphism \( p : \text{SU}(2) \to \text{SO}(3) \) is a covering with kernel \( \{ \pm 1 \} \).

**Proof.** \( \text{SU}(2) \) is connected, as we know. Thus the image of \( \text{SU}(2) \) by \( p \) is connected as well, hence contained in the connected component \( \text{SO}(3) \) of \( O(3) \). Let \( \pi : \text{su}(2) \to \text{so}(3) \) be the Lie map of \( p \). Since \( p = \text{Ad} \), \( \pi = \text{ad} \) and therefore the kernel of \( \pi \) consists of the \( X \in \text{su}(2) \) satisfying \( XY - YX = 0 \) for all skew Hermitian \( Y \in M_2(\mathbb{C}) \) of trace 0, hence for all skew Hermitian \( Y \in M_2(\mathbb{C}) \), hence for all \( Y \in M_2(\mathbb{C}) \) (as one sees by writing \( Y = iy' + iy'' \) with \( y' \) and \( y'' \) skew Hermitian). Thus \( X \) is a scalar and therefore equals 0, because \( tr X = 0 \). Since the kernel of \( \pi \) is \( \{0\} \), \( \pi \) is an isomorphism by dimension-count. It follows that \( p \) is a covering. Its kernel consists of all \( a \in \text{SU}(2) \) satisfying \( aYa^{-1} = Y \) for all skew Hermitian \( Y \), hence are scalar, hence equal to \( \pm 1 \). QED

Example 13. \( \text{SO}(3) \) is not simply connected, but admits a simply connected double covering. There are two ways of seeing this: (a) \( \text{SU}(2) \to \text{SO}(3) \) is the required double covering.

(b) \( \text{SO}(3) \) is homeomorphic with the closed unit ball in \( \mathbb{R}^3 \), antipodal points on its surface being identified (Example 1, §2.1). This is a model of real projective 3-space, well-known to have a simply connected double covering.

It follows from general facts of homotopy theory that there must be a continuous loop \( C_0 \) on \( \text{SO}(3) \) that cannot be continuously deformed into a point, while \( 2C_0 \) (\( C_0 \) traversed twice) can so be deformed; and any loop may be deformed either into \( C_0 \) or into the constant loop. The one-parameter group of rotations about a fixed axis, traversed once, provides a model for such a path \( C_0 \). That \( 2C_0 \) may be deformed into the constant loop may be seen thus, according to Weyl (1939):

Take two solid straight circular cones of aperture \( \alpha \) with common vertex and touching each other along a generator, the one fixed in space, the other rolling on the first. The roller describes a closed motion which is \( 2C_0 \) for \( \alpha = 90^\circ \) and approaches rest as \( \alpha \to 180^\circ \). By continuous variation of the parameter \( \alpha \) one thus deforms \( 2C_0 \) into the point 1.

(The initial motion \( 2C_0 \) in this picture is not exactly a double rotation about a fixed axis but homotopic thereto; see problem 11, §2.1.)

Example 14. \( \text{SL}(2, \mathbb{C}) \): the covering \( \text{SL}(2, \mathbb{C}) \to \text{SO}_0(3,1) \). \( \text{SL}(2, \mathbb{C}) = \{ a \in M_2(\mathbb{C}) | \det a = 1 \} \) operates on the real vector space of Hermitian \( 2 \times 2 \) matrices \( X \in M_2(\mathbb{C}) | X^* = X \) by \( X \to aXa^* \). Writing

\[
X = \begin{bmatrix} \xi_1 + \xi_3 & \xi_1 - i \xi_2 \\ \xi_1 + i \xi_2 & \xi_1 - \xi_3 \end{bmatrix},
\]

and \( \bar{X} = (\xi_1, \xi_2, \xi_3, \xi_4) \), one obtains an operation of \( a \in \text{SL}(2, \mathbb{C}) \) on \( \mathbb{R}^4 \) denoted \( p(a) \):

\[
p(a) \bar{X} = \bar{aXa^*}.
\]

Problems for §2.6

1. Let \( (\mathbb{C}^*)^n = \{ t = (\tau_1, \ldots, \tau_n) | 0 \neq \tau_k \in \mathbb{C} \} \). Show that every holomorphic homomorphism \( f : (\mathbb{C}^*)^n \to \mathbb{C}^* \) is of the form

\[
f(t) = (\text{sgn} \tau_1)^{\delta_1} \cdots (\text{sgn} \tau_n)^{\delta_n} |\tau_1|^{l_1} \cdots |\tau_n|^{l_n}
\]

for certain \( l_k \in \mathbb{Z} \) and certain \( \delta_k \in \mathbb{Z}_2 \).

2. Show that \( \text{Aut}(\mathbb{C}^*)^n = \text{GL}(n, \mathbb{Z}) = \{ a \in M_n(\mathbb{Z}) | a^{-1} \in M_n(\mathbb{Z}) \} = \{ a \in M_n(\mathbb{Z}) | \det a = \pm 1 \} \).

[Suggestion: consider Lie maps.]
3. Let $G$ be a connected linear group. Show that any finite number of homomorphisms $\varphi_k : g \to h_k$ of the Lie algebra $g$ of $G$ to Lie algebras $h_k$ of linear groups $H_k$ lift simultaneously to homomorphisms $f_k : G \to H_k$ of a single linear covering group $G$.

4. Give direct proofs of the surjectivity of the maps $SU(2) \to SO(3)$ and $SL(2,\mathbb{C}) \to SO_o(3,1)$, without appealing to the connectedness of $SO(3)$ and $SO_o(3,1)$. [Suggestion: use Lemma 1.1. of §2.1 and Lemma 4C, §2.4.]

5. Let $F$ be the real vector space of all polynomial functions $f(x) = f(\xi_1, \ldots, \xi_n)$ of degree $\leq d$ on $\mathbb{R}^n$ (some $d$ some $n$). Define a homomorphism $T : GL(n,\mathbb{R}) \to GL(F)$ of $GL(n,\mathbb{R})$ into the group $GL(F)$ of invertible linear transformations of $F$ by the formula

$$T(a)f(x) = f(a^{-1}x).$$

Notation: for $a \in GL(n,\mathbb{R})$ and $x \in \mathbb{R}^n$,

$$ax = \sum_{ij} a_{ij} \xi_j e_i \quad \text{if} \quad x = \sum_j \xi_j e_j.$$

(a) Show that the Lie map $\tau : gl(n,\mathbb{R}) \to gl(F)$ of $T$ is given by

$$\tau(X)f(x) = \sum_{ij} -X_{ij} \xi_j \frac{\partial f}{\partial \xi_i}$$

(b) Show that a polynomial function $f(x) = f(\xi_1, \xi_2, \xi_3)$ on $\mathbb{R}^3$ is invariant under the rotation group $SO(3)$, i.e.

$$f(ax) = f(x) \quad \text{for all} \quad a \in SO(3) \quad \text{and all} \quad x \in \mathbb{R}^3$$

if and only if

$$\xi_i \frac{\partial f}{\partial \xi_j} - \xi_j \frac{\partial f}{\partial \xi_i} = 0$$

for all $i \neq j, (i,j = 1,2,3)$. Can you generalize this?

6. Construct a double covering $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$.

7. (a) Construct a double covering $SL(2,\mathbb{R}) \to SO_o(2,1)$.

(b) Let $SU(1,1)$ be the group of all complex $2 \times 2$ matrices of the form

$$\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad \alpha \bar{\alpha} - \beta \bar{\beta} = 1.$$  

Construct a double covering $SU(1,1) \to SO_o(2,1)$.

(c) Find an isomorphism $SL(2,\mathbb{R}) \approx SU(1,1)$ compatible with (a) and (b).

[Suggestion: for (c), consider conjugation by suitable $2 \times 2$ matrix.]

8. (a) The double covering $SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SO(4,\mathbb{C})$. Construct this double covering.

(b) The double covering $SL(2,\mathbb{C}) \to SO_o(1,3)$. Construct this double covering.

[Suggestion: For (a) consider $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ acting on $M_2(\mathbb{C}) \approx \mathbb{C}^2$ by $X \mapsto \alpha X \beta^{-1}$. For (b) consider $SL(2,\mathbb{C})$ acting $\{ X \in M_2(\mathbb{C}) \mid X^* = X \} \approx \mathbb{R}^3$ by $X \mapsto \alpha X \alpha^*$. These actions preserve the quadratic form $\det(X)$. Show that the Lie maps of the group homomorphisms defined by these actions are 1-1. Note that $SL(2,\mathbb{C}) = KB$ is connected, see Example 4 of §2.1.]

9. The double covering $SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$. This problem assumes familiarity with the wedge product of alternating forms. [See Hoffmann-Kunze (1961), section 5.7, for example.]

$SL(4,\mathbb{C})$ acts in a natural way on the six-dimensional space $F$ of all skew-symmetric bilinear forms on $\mathbb{C}^4$. Denote the linear transformation of $F$ corresponding to $a \in SL(4,\mathbb{C})$ by $f(a)$. $F$ carries a non-degenerate symmetric bilinear form $(\varphi, \psi)$ defined by

$$\varphi \wedge \psi = (\varphi, \psi)e,$$

where $e$ is a fixed non-zero four-form. Show that $f(a) \in SO(F) \approx SO(6,\mathbb{C})$ and that the map $f : SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$ is a double covering with kernel $\{ \pm 1 \}$. [Suggestion: start by showing that the Lie map of $f$ is 1-1; then count dimensions. Use connectedness.]

10. The double covering $Sp(2,\mathbb{C}) \to SO(5,\mathbb{C})$. Denote by $\sigma$ the non-degenerate skew-form on $\mathbb{C}^4$ defining $Sp(2,\mathbb{C})$. Show that the map $SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$ defined in problem 9 induces a double covering $Sp(2,\mathbb{C}) \to SO(5,\mathbb{C})$, $SO(5,\mathbb{C})$ realized as the subgroup of $SO(6,\mathbb{C})$ of transformations that preserve the five-dimensional hyperplane $\{ \varphi, \sigma \} = 0$ in $F = \mathbb{C}^6$ and have $\det = 1$ therein. [Prove also that the group described is $SO(5,\mathbb{C})$.]

2.7 Closed subgroups

As we know, the group topology of a linear group does not necessarily coincide with its relative topology in matrix space: one should keep in mind $GL(n,\mathbb{Q})$ or the irrational line on the torus. More generally, the group topology on a subgroup $H$ of a linear group $G$ need not coincide with its relative topology in $G$, i.e. the open sets of $H$ need not be exactly the intersections of open sets of $G$ with $H$. This phenomenon cannot occur if $H$ is closed in $G$, as may seem plausible from the examples mentioned. The essential point is the following theorem.

**Theorem 1 (Closed subgroup theorem).** Let $G$ be a linear group, $H$ a closed subgroup of $G$. Let $s$ be a vector space complement for the Lie algebra $h$ of $H$ in the Lie algebra $g$ of $G : g = s \oplus h$. There is an open neighborhood $U$ of $0$ in $s$ so that the map

$$U \times H \to G, \quad (X,b) \to \exp(X)b$$

is an analytic bijection onto an open neighborhood of $h$ in $G$. 
Proof. We first show that the differential of \( s \times H \rightarrow G \), \((X, b) \rightarrow \exp(X)b\), is invertible at every point \((X, b)\) with \(X\) in a neighborhood \(U\) of 0 in \(s\). It suffices to show this for points \((X, 1)\) with \(X\) in a neighborhood of 0 in \(s\) (by composing with right translation by \(b^{-1}\)), and in fact only for the point \((0,1)\) (by continuity in \(X\)). But at \((0,1)\) the differential is simply the map \(s \times h \rightarrow g, (X,Y) \rightarrow X + Y\), which is invertible by the choice of \(s\). It follows from the Inverse Function Theorem that the map \(U \times H \rightarrow G\) is locally bi-analytic onto a neighborhood of \(H\) in \(G\).

To prove the theorem it remains to show that this map \(U \times H \rightarrow G, (X, b) \rightarrow \exp(X)b\) is one-to-one when \(X\) is restricted to a suitable (possibly smaller) neighborhood \(\{X \in s \mid \|X\| < \epsilon\}\) of 0 in \(s\). Suppose this were not the case, then for any \(\epsilon > 0\) we would have an equation

\[
\exp(X_1)b_1 = \exp(X_2)b_2
\]

with \(X_1 \neq X_2 \in s, \|X_1\|, \|X_2\| < \epsilon, b_1, b_2 \in H\). Then

\[
\exp(-X_1)\exp(X_2) = b_1b_2^{-1} \in H.
\]

Since \(s \times h \rightarrow G, (X, Y) \rightarrow \exp(X)\exp(Y),\) has an invertible differential at \((0,0)\), we can write uniquely (for small \(\epsilon\)):

\[
\exp(-X_1)\exp(X_2) = \exp(X)\exp(Y)
\]

with \((X, Y)\) in a neighborhood of \((0,0)\) in \(s \times h\). Thus \(\exp(X) \in H\) as well, and taking \(\epsilon\) small enough we may pick such an \(X\) in any neighborhood of 0 in \(s\). Also, \(X \neq 0\), since otherwise \(\exp(X_2) = \exp(X_1)\exp(Y)\) would give \(X_2 = X_1\), because of the uniqueness of the \(\exp(X)\exp(Y)\)-form. So we can choose a sequence \(\{X_k\}\) in \(s\) with \(X_k \neq 0, \|X_k\| \rightarrow 0\), and \(\exp(X_k) \in H\). Passing to a subsequence, we may further assume that \(X_k/\|X_k\|\) converges to some \(X\), which is necessarily nonzero and in \(s\). We shall show that also \(X \in H\), contradicting \(s \times h = 0\).

To see this, take \(\tau \in \mathbb{R}\), and pick integers \(n_k\) so that \(n_k\|X_k\| \rightarrow \tau\). (Take \(n_k\|X_k\| < \tau < (n_k + 1)\|X_k\|\)) Then

\[
(\exp X_k)^{n_k} = \exp(n_kX_k) = \exp \left( n_k\|X_k\| \frac{X_k}{\|X_k\|} \right) \rightarrow \exp(\tau X).
\]

Since \(H\) is closed, \(\exp \tau X \in H\), and this for all \(\tau \in \mathbb{R}\). Thus \(X \in H\), which is the desired contradiction.

QED

**Corollary 2.** Let \(G\) be a linear group, \(H\) a closed subgroup of \(G\). Any point \(a_0 \in H\) has a neighborhood \(U\) in \(G\) that lies in the domain of a coordinate system \((\xi_1, \ldots, \xi_{m+1}, \ldots, \xi_n)\) on \(G\) so that the elements of \(H\) in \(U\) are precisely those for which \(\xi_{m+1} = 0, \ldots, \xi_n = 0\). Here \(n = \dim G\) and \(m = \dim H\).

**Terminology.** We say \(H\) is given by the equations \(\xi_{m+1} = 0, \ldots, \xi_n = 0\) locally around \(a_0\).

**Proof.** In a neighborhood of \(a_0 \in H\) in \(G\), write \(a = \exp(X)\exp(Y)a_0\) with \(X \in s\) and \(Y \in h\) close to 0. Then \(X\) and \(Y\) (or their components with respect to bases) give a coordinate system in a neighborhood of \(a_0\) on \(G\) in which \(H\) is given by the equation \(X = 0\), as required.

QED

**Corollary 3.** Let \(G\) be linear group, \(H\) a closed subgroup of \(G\). The open subsets of \(H\) are the intersections of open subsets of \(G\) with \(H\), i.e. the group-topology of \(H\) is its relative topology in \(G\).

**Proof.** Evident from Corollary 2 and the definition of 'open'.

QED

The theorem and its corollaries apply in particular to closed subgroups of \(GL(E)\). As a consequence, the group-topology on a closed subgroup of \(GL(E)\) is its relative topology in the matrix space \(M\).

Recall that a subset \(C\) of \(\mathbb{R}^N\) is compact if every open cover of \(C\) has a finite subcover. One can use the same definition to define compact linear groups (with 'open' understood in the sense of the group-topology), but such a group is necessarily a compact subset of the matrix space: the image of a compact space (in this sense) under a continuous map is again compact, as is clear from the definition. This fact may be applied to the inclusion of a compact group in the matrix space. The basic compactness criterion is the Heine-Borel Theorem: A subset of \(\mathbb{R}^N\) is compact if and only if it is closed and bounded. This criterion applies in particular to linear groups as subset of the matrix space. For example, the \(SO(n)\) and \(SU(n)\) are evidently closed and bounded subset of the matrix space.

**Problems for §2.7**

1. Show that a locally closed subgroup of a linear group is closed. [A subset \(A\) of \(G\) is locally closed if every point of \(A\) has a neighborhood \(U\) in \(G\) so that \(A \cap U\) is closed in \(U\). Suggestion: assume \(a_k \in H, a_k \rightarrow a\) in \(G\). If \(H\) is locally closed in \(G\) so is \(aH\).]

2. Prove the converse of Corollary 3: Let \(G\) be linear group, \(H\) a subgroup of \(G\). Suppose the open subsets of \(H\) are the intersections of open subsets of \(G\) with \(H\). Show that \(H\) is closed in \(G\). [Suggestion: preceding problem and Inverse Function Theorem.]

3. (a) Let \(G\) be a linear group, suppose \(H\) is a subgroup of \(G\) that can be written in the form \(AB = \{ab \mid a \in A, b \in B\}\) where \(A\) is a compact subgroup and \(B\) a closed subset of \(G\). Show that \(H\) is closed in \(G\). [Suggestion: Assume \(a_k b_k \rightarrow c\). Choose a convergent subsequence of \(\{a_k\}\).

(b) Show that the conclusion of (a) need not hold if \(A\) is only assumed to be closed in \(G\). [Suggestion: consider upper and lower triangular matrices in \(SL(2, \mathbb{R})\).]

4. Let \(G\) be a linear group \(g\) its Lie algebra. Let \(h\) be a Lie sub-algebra of \(g\) with the property that the only elements \(X \in g\) satisfying \([X, h] \subset h\) are those of \(h\). Show that \(\Gamma(h)\) is closed in \(G\).