§4.1 Math 22a Fall 2009 Homework 10 Solutions

55. Let θ be the angle between a and b. Then
(0) $\|a \times b\|^2 + |a \cdot b|^2 = (\|a\|\|b\| \sin\theta)^2 + (\|a\|\|b\| \cos\theta)^2$
$= \|a\|^2\|b\|^2(\sin^2\theta + \cos^2\theta)$
$= \|a\|^2\|b\|^2$.

56. $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) =$
(0) $(a \cdot c)b - (a \cdot b)c + (b \cdot a)c - (b \cdot c)a + (c \cdot b)a - (c \cdot a)b = 0$.

§4.2 8. (4)
$$\begin{vmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 6 & 1 & 4 \\ 8 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 & 7 \\ 6 & 1 & 4 \\ 4 & 1 & 2 \end{vmatrix}$$ (expanding by minors down the third column)
$= -[4(8 + 8) - 0 + 2(-12 - 8)]$ (expanding down column 3)
$-0 + 1(4 - 28) - 1(8 - 42)$ (expanding down column 2)
$= -[4(16) + 2(-20)] + (-24) + 34 = -24 - 24 + 34 = -14$

18. (4) If any one of the three rows of A is multiplied by 2, the resulting matrix has a determinant two times as great. Thus multiplying the rows one at a time by 2, we see that $\det(A + A) = \det(2A) = (2)(2)(2)\det(A) = 8\det(A) = 16$.

22. (4)
$$\begin{vmatrix} a+a & & \\ a+b & & \\ a+c & & \end{vmatrix} = \begin{vmatrix} 2a & & \\ a+b & & \\ a+c & & \end{vmatrix}$$
$$= \begin{vmatrix} 2a & & \\ b & & \\ c & & \end{vmatrix}$$ (Add -1/2 times row 1 to rows 2 and 3.)
$$= 2\begin{vmatrix} a & & \\ b & & \\ c & & \end{vmatrix} = 2\det(A) = (2)(3) = 6$$

29. (4)
$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 3 \\ 0 & 4 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 4-\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$$
$= (1 - \lambda)(\lambda^2 - 4\lambda - 12) = (1 - \lambda)(\lambda - 6)(\lambda + 2)$
$= 0$ when $\lambda = 1, 6, -2$.
These are the values that make the matrix singular.

§4.3 4. (4)
$$\begin{vmatrix} 3 & -5 & -1 & 7 \\ 0 & 3 & 1 & -6 \\ 2 & -5 & -1 & 8 \\ -8 & 8 & 2 & -9 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & -2 & -1 & 2 \\ -8 & 2 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ -8 & 2 & 3 \end{vmatrix} = -2\begin{vmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 0 & -6 & 11 \end{vmatrix} =$$
$$2\begin{vmatrix} 1 & -2 \\ -6 & 11 \end{vmatrix} = 2(-1) = -2$$

7. (4)
$$\begin{vmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & -3 \\ 0 & -2 & 1 & 0 & 0 \\ 5 & -3 & 2 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 9 & -3 \\ 0 & -2 & 1 & 0 & 0 \\ 5 & -3 & 2 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 2 & 9 \\ 0 & -2 & 1 & 0 \\ 5 & -3 & 2 & 0 \\ -3 & 4 & 0 & 0 \end{vmatrix} =$$
$$-9\begin{vmatrix} 0 & -2 & 1 \\ 5 & -3 & 2 \\ -3 & 4 & 0 \end{vmatrix} = -9\begin{vmatrix} 0 & 0 & 1 \\ 5 & 1 & 2 \\ -3 & 4 & 0 \end{vmatrix} = -9\begin{vmatrix} 5 & 1 \\ -3 & 4 \end{vmatrix} = -9(23) = -207$$

8. (4)
$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -4 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 3[(2)(7)(1)(10)] = 420$$

18. (4) We find that
$a'_{11} = (-1)^2(6) = 6$ $a'_{21} = (-1)^3(-3) = 3$ $a'_{31} = (-1)^4(-3) = -3$
$a'_{12} = (-1)^3(6) = -6$ $a'_{22} = (-1)^4(27) = 27$ $a'_{32} = (-1)^5(-18) = 18$
$a'_{13} = (-1)^4(9) = 9$ $a'_{23} = (-1)^5(3) = -3$ $a'_{33} = (-1)^6(3) = 3$
and $\det(A) = a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} = 3(6) + 0 + 3(9) = 45$.
Thus $A^{-1} = \frac{1}{45}\begin{bmatrix} 6 & 3 & -3 \\ -6 & 27 & 15 \\ 9 & -3 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 2 & 1 & -1 \\ -2 & 9 & 6 \\ 3 & -1 & 1 \end{bmatrix}$.

23. (4) We have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \pm\operatorname{adj}(A)$. Since the a_{ij} are integers, each cofactor a'_{ij} is also an integer, being a sum of products of integers. Thus A^{-1} has integer entries.

28. (5) $\det(A) = \begin{vmatrix} 5 & -2 & 1 \\ 0 & 1 & 1 \\ 1 & 6 & -1 \end{vmatrix} = 0 + 1(-6) - 1(32) = -38$. Then
$$x_1 = \frac{\begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 4 & 6 & -1 \end{vmatrix}}{-38} = \frac{1(-7) - 0(-4) + 4(-3)}{-38} = \frac{-19}{-38} = \frac{1}{2},$$
$$x_2 = \frac{\begin{vmatrix} 5 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 4 & -1 \end{vmatrix}}{-38} = \frac{-0(-5) + 0(-6) - 1(19)}{-38} = \frac{-19}{-38} = \frac{1}{2}, \text{ and}$$
$$x_3 = \frac{\begin{vmatrix} 5 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 6 & 4 \end{vmatrix}}{-38} = \frac{-0(-14) + 1(19) - 0(32)}{-38} = \frac{19}{-38} = -\frac{1}{2}.$$

32. (4) We have
$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -3 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 5 & -6 & -1 \\ 3 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -3 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & -6 & -1 \\ 3 & 1 & 0 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 1 & -3 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 5 & -6 & -1 \\ 3 & 1 & 0 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -3 & 1 \\ 0 & -1 & 6 & 0 \\ 0 & 1 & -6 & -1 \\ 0 & -2 & 9 & -2 \end{vmatrix}} = \frac{-2\begin{vmatrix} 1 & -3 & 0 \\ 5 & -6 & -1 \\ 1 & 0 & -2 \end{vmatrix}}{\begin{vmatrix} -1 & 6 & 0 \\ 1 & -6 & -1 \\ -2 & 9 & -2 \end{vmatrix}}$$
$$= \frac{-2\begin{vmatrix} 1 & -3 & 2 \\ 5 & -6 & 9 \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -2 & -3 & -2 \end{vmatrix}} = \frac{-2\begin{vmatrix} -3 & 2 \\ -6 & 9 \end{vmatrix}}{-\begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix}} = \frac{-2(-27 + 12)}{-(0 - 3)} = \frac{30}{3} = 10.$$

38. (5) Since $A(\operatorname{adj}(A)) = \det(A)I$, then taking the determinant of both sides and remembering that I is $n \times n$, we have
$$\det(A)\det(\operatorname{adj}(A)) = (\det(A))^n.$$
If $\det(A) \neq 0$, then the relation $\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$ follows at once. If $\det(A) = 0$, then A is singular, and Exercise 37 shows that $\operatorname{adj}(A)$ is also singular so $\det(\operatorname{adj}(A)) = 0$ also. Thus $\det(\operatorname{adj}(A)) = (\det(A))^{n-1} = 0$.

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