Math 22a Fall 2005 Homework 5 Solutions

§1.6

2. Let $L = \{[x, x+1] \mid x \in L\}$ which is a nonempty subset of
(4) $\mathbb{R}^2$. Clearly $[0, 0]$ is not in L. Thus L is not a subspace. The only lines in $\mathbb{R}^2$ that are subspaces are those that contain the origin.

8. Let $P = \{[2x, x+y, y] \mid x, y \in \mathbb{R}\}$ which is a nonempty
(4) subset of $\mathbb{R}^3$. Let $\mathbf{v} = [2a, a+b, a]$ and $\mathbf{w} = [2c, c+d, d]$ be in P. Then

$$\mathbf{v} + \mathbf{w} = [2a + 2c, a + b + c + d, b + d] = [2(a + c), (a + c) + (b + d), b + d]$$

which has the form $[2x, x + y, y]$ and is in P. Also

$$r[2a, a + b, b] = [2ra, ra + rb, rb]$$

which is in P. Thus P is a subspace of $\mathbb{R}^3$.

§1.6
18.
(4) $$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & 4/5 \\ 0 & 0 & 1 & -4/5 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} -3r/5 \\ -4r/5 \\ 4r/5 \\ r \end{bmatrix} = \frac{r}{5}\begin{bmatrix} -3 \\ -4 \\ 4 \\ 5 \end{bmatrix},$$ so $\{[-3, -4, 4, 5]\}$ is a basis for the solution space.

24.
(3) $$\begin{bmatrix} -1 & 1 & 1 \\ 3 & 5 & 13 \\ 4 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 8 & 16 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$ The matrix is not invertible so by Theorem 1.16, the given set is not a basis for $\mathbb{R}^3$.

26.
(4) $$\begin{bmatrix} 2 & 2 & 3 & 5 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 2 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix is invertible so the set is a basis for $\mathbb{R}^4$ by Theorem 1.16.

32,
(5) $$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -6 & -6 & -12 \\ 0 & -7 & -7 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ The solution space is given by $$\begin{bmatrix} -2s - r \\ -s - 2r \\ s \\ r \end{bmatrix}$$ for all scalars r and s.

A basis is $$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

34.
(4) $$\mathbf{x} = \begin{bmatrix} 7 + 2t - r - 5s \\ t \\ r \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2t - r - 5s \\ t \\ r \\ s \end{bmatrix}$$

37.
(4) $$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 5 \\ 1 & -1 & 2 & 1 & 0 \\ 4 & -1 & 7 & 2 & 5 \\ -1 & -2 & -1 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & -2 & 5 \\ 0 & 3 & -1 & -2 & 5 \\ 0 & -3 & 1 & 2 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 5/3 & 1/3 & 5/3 \\ 0 & 1 & -1/3 & -2/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]; \quad \mathbf{x} = \begin{bmatrix} 5/3 \\ 5/3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -(5r + s)/3 \\ (r + 2s)/3 \\ r \\ s \end{bmatrix}$$

47.
(6) Clearly $W_1 \cap W_2$ is nonempty; it contains 0. Let $\mathbf{v}, \mathbf{w} \in (W_1 \cap W_2)$. Then $\mathbf{v}, \mathbf{w} \in W_1$ and $\mathbf{v}, \mathbf{w} \in W_2$ so $\mathbf{v} + \mathbf{w} \in W_1$ and $\mathbf{v} + \mathbf{w} \in W_2$ since W_1 and W_2 are subspaces. Thus $\mathbf{v} + \mathbf{w} \in (W_1 \cap W_2)$. Similarly, $r\mathbf{v} \in W_1$ and $r\mathbf{v} \in W_2$ since W_1 and W_2 are subspaces, so $r\mathbf{v} \in (W_1 \cap W_2)$. Thus $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

Please give partial credits

§2.1
10.
(3) $$\begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 7 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ The pivots in columns 1 and 2 show that $\{[-2, 3, 1], [3, -1, 2]\}$ is a basis.

24.
(3) $$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 5 & 13 \\ -1 & 6 & 4 \\ 3 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 9 & 9 \\ 0 & 5 & 5 \\ 0 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ Each column has a pivot so the set of vectors is independent.

32. Suppose that $r_1[1, 0, 1] + r_2[2, s, 3] + r_3[2, 3, 1] =$
(6) $[0, 0, 0]$ so that $[r_1 + 2r_2 + 2r_3, sr_2 + 3r_3, r_1 + 3r_2 + r_3] = [0, 0, 0]$. We solve the system

$$\begin{aligned} r_1 + 2r_2 + 2r_3 &= 0, \\ sr_2 + 3r_3 &= 0, \\ r_1 + 3r_2 + r_3 &= 0. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & s & 3 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & s & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3+s & 0 \end{array}\right].$$ We see that we have a dependence relation if and only if $s = -3$. Thus the given vectors are independent for all $s \neq -3$.

37. Suppose that $r_1(C\mathbf{v}_1) + r_2(C\mathbf{v}_2) + \cdots + r_k(C\mathbf{v}_k) = 0$. Then
(0) $C(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = 0$. Multiplying on the left by C^{-1}, we have $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = 0$, so $r_1 = r_2 = \cdots = r_k = 0$ by the independence of the $\mathbf{v}_i$. Hence the vectors $C\mathbf{v}_1, C\mathbf{v}_2, \ldots, C\mathbf{v}_k$ are independent.

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