Math 22a Fall 2005 Homework 6 Solutions

13. Since the rowspace of A is the column space of A^T, we reduce
(4)
$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 5 & 4 & 8 \\ 7 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -7 \\ 0 & -6 & -7 \\ 0 & -12 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
The pivots in columns 1 and 2 show that [1, 3, 5, 7] and [2, 0, 4, 2] form a basis for the rowspace of A.

14. (4)
$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -6 & -6 & -12 \\ 0 & -7 & -7 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
The pivots in columns 1 and 2 show that the first two columns of A form a basis for the column space of A.

§2.1

26. (4)
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$
With pivots in columns 1,2, and 3, a basis is {[1, 2, 1], [1, 0, 0], [0, 1, 0]}.

27. (4)
$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$
With pivots in columns 1, 2, 3, and 5 a basis is {[2, 1, 1, 1], [1, 0, 1, 1], [1, 0, 0, 0], [0, 0, 1, 0]}.

38. Assume W is an n-dimensional subspace of $\mathbb{R}^n$. Then any basis
(4) for W must contain n vectors. Part 3 of Theorem 2.3 shows that these vectors must be a basis for $\mathbb{R}^n$, so $W = \mathbb{R}^n$.

§2.2

3.
$$\begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 6 & 6 & 3 \\ 0 & -7 & -7 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1) a) 3

(2) b) {[1, 0, -1, 0], [0, 1, 1, 0], [0, 0, 0, 1]}

(2) c) The set consisting of $\begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \end{bmatrix}$.

(1) d) The set consisting of the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

4.
$$\begin{bmatrix} 3 & 1 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 7 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(1) a) 4

(2) b) The set of row vectors $\{e_1, e_2, e_3, e_4\}$.

(2) c) The set consisting of $\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ or the set of column vectors $\{e_1, e_2, e_3, e_4\}$

(1) d) The null set.

10. (4)
$$\begin{bmatrix} 3 & 0 & -1 & 2 \\ 4 & 2 & 1 & 8 \\ 1 & 4 & 0 & 1 \\ 2 & 6 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -14 & 1 & 4 \\ 0 & -12 & -1 & -1 \\ 0 & -2 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 17 & 5 \\ 0 & 0 & 22 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & -7/2 \end{bmatrix}.$$
The rank is 4 so the matrix is invertible.

12. Let A be an $n \times n$ matrix. Then
(3)
$$\text{null}(A) = n - \text{rank}(A) = n - \text{rank}(A^T) = \text{null}(A^T).$$

13. Let A have column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$. Now the
(5) linear system $A\mathbf{x} = \mathbf{b}$ has a solution
if and only if $\mathbf{b} \in \text{colspace}(A)$
if and only if $\text{sp}(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{b}) = \text{sp}(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$
if and only if $\text{rank}(A \mid \mathbf{b}) = \text{rank}(A)$.

14. Let C be $m \times n$. Every vector in the column space of AC is of
(0) the form $\mathbf{v} = (AC)\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{v} = A(C\mathbf{x})$ which is a vector in the column space of A. Thus colspace(AC) $\subseteq$ colspace(A).

Two optional problems this time

15. No. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then the column space of
(0)
$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is $\text{sp}(\mathbf{e}_1)$ which is not contained in the column space $\text{sp}(\mathbf{e}_2)$ of C.

§2.3

(3) 4. No, because $T([0, 0]) = [0, 1, 0] \neq [0, 0, 0]$.

6. (3)
$$T([-3, -5]) = 3T([-1, 0]) - 5T([0, 1]) = 3[2, 3] - 5[5, 1] = [-19, 4]$$

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