Math 22a Fall 2005 Homework 7 Answers and Solutions

§2.3

9. Reducing $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & 1 & 10 \end{array}\right]$, we find that
(3) $[0, 10] = 4[-1, 2] + 2[2, 1]$. Thus $T([0, 10]) = 4T([-1, 2]) + 2T([2, 1]) = 4[1, 0, 0] + 2[0, 1, 2] = [4, 2, 4]$.

12. We need to find scalars x_1, x_2, x_3 such that
(5)

$$\begin{bmatrix} -3 \\ 11 \\ -4 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\mathbf{x}.$$

Using A^{-1} found in Example 7 of Section 1.5, we have

$$\mathbf{x} = A^{-1}\begin{bmatrix} -3 \\ 11 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 11 \\ -4 \end{bmatrix} = \begin{bmatrix} 64 \\ -23 \\ -27 \end{bmatrix}. \text{ Thus}$$

$$T\left(\begin{bmatrix} -3 \\ 11 \\ -4 \end{bmatrix}\right) = 64T\left(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right) - 23T\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right) - 27T\left(\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}\right)$$
$$= 64(8) - 23(-5) - 27(17) = 168.$$

14. $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$
(2)

19. The matrix A associated with T is $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$ and the matrix
(5)
A' associated with T' is $A' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, so the matrix associated with $T' \circ T$ is $A'A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$.

$(T' \circ T)([x_1, x_2]) = [2x_1, 3x_1 + x_2]$.

34. See the proof of Theorem 3.7 in the text.
(5)

§3.1

2. Consider Axiom A1. We have
(4)

$$([1,2] \tilde{+} [1,4]) \tilde{+} [1,6] = [6,2] \tilde{+} [1,6] = [8,7]$$

while

$$[1,2] \tilde{+} ([1,4] \tilde{+} [1,6]) = [1,2] \tilde{+} [10,2] = [4,11]$$

so this axiom does not hold. We do not have a vector space.

3. We check the eight axioms.
(8)

A1 $([a, b] \tilde{+} [c, d]) \tilde{+} [e, f]$
$= [a + c + 1, b + d] \tilde{+} [e, f]$
$= [a + c + e + 2, b + d + f]$,
$[a, b] \tilde{+} ([c, d] \tilde{+} [e, f])$
$= [a, b] \tilde{+} [c + e + 1, d + f]$
$= [a + c + e + 2, b + d + f]$.

A2 $[a, b] \tilde{+} [c, d] = [a + c + 1, b + d]$
$= [c + a + 1, d + b] = [c, d] \tilde{+} [a, b]$.

A3 $[-1, 0] \tilde{+} [a, b] = [-1 + a + 1, 0 + b] = [a, b]$.

A4 $[a, b] \tilde{+} [-a - 2, -b]$
$= [a - a - 2 + 1, b - b] = [-1, 0]$.

S1 $r([a, b] \tilde{+} [c, d]) = r[a + c + 1, b + d]$
$= [ra + rc + r + r - 1, rb + rd]$
$= [ra + r - 1, rb] \tilde{+} [rc + r - 1, rd]$
$= r[a, b] \tilde{+} r[c, d]$.

S2 $(r + s)[a, b] = [ra + sa + r + s - 1, rb + sb]$
$= [ra + r - 1, rb] \tilde{+} [sa + s - 1, sb]$
$= r[a, b] \tilde{+} s[a, b]$.

S3 $r(s[a, b]) = r[sa + s - 1, sb]$
$= [rsa + rs - r + r - 1, rsb]$
$= [rsa + rs - 1, rsb] = (rs)[a, b]$.

S4 $1[a, b] = [a + 1 - 1, b] = [a, b]$.

All eight axioms hold, so we do have a vector space.

12. It is easily checked that this set of matrices is closed
(3) under matrix addition and scalar multiplication so we do have a vector space.

14. The set of rational numbers is not closed under scalar
(4) multiplication by real numbers, since 1 is a rational number and if r is any real number that is not rational, then $r1 = r$ is not rational.

19. Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are vectors such that $\mathbf{v} + \mathbf{y}_1 = \mathbf{0}$ and
(0) $\mathbf{v} + \mathbf{y}_2 = \mathbf{0}$. Then

$$\begin{aligned} \mathbf{v} + \mathbf{y}_1 &= \mathbf{v} + \mathbf{y}_2, \\ \mathbf{y}_1 + (\mathbf{v} + \mathbf{y}_1) &= \mathbf{y}_1 + (\mathbf{v} + \mathbf{y}_2), \\ \mathbf{y}_1 + \mathbf{0} &= \mathbf{y}_2 + (\mathbf{v} + \mathbf{y}_1), \quad \text{(Axioms A1 and A2)} \\ \mathbf{y}_1 + \mathbf{0} &= \mathbf{y}_2 + \mathbf{0}, \\ \mathbf{y}_1 &= \mathbf{y}_2. \quad \text{(Axiom A3)} \end{aligned}$$

23. Let $r\mathbf{v} = \mathbf{0}$ and assume that $r \neq 0$. Then
(0)

$$\begin{aligned} \tfrac{1}{r}(r\mathbf{v}) &= \tfrac{1}{r}\mathbf{0} = \mathbf{0}, \quad \text{(part iv of Theorem 3.1)} \\ (\tfrac{1}{r}r)\mathbf{v} &= \mathbf{0}, \quad \text{(Axiom S3)} \\ 1\mathbf{v} &= \mathbf{0}, \\ \mathbf{v} &= \mathbf{0}. \quad \text{(Axiom S4)} \end{aligned}$$

24. Suppose that $\mathbf{v} = s(\mathbf{v} + \mathbf{w})$. Note that $s \neq 1$ since $s = 1$ would
(4) yield $\mathbf{w} = \mathbf{0}$, contrary to hypothesis. Thus we have $(1 - s)\mathbf{v} = s\mathbf{w}$ which we can write as as $\mathbf{v} = \frac{s}{1 - s}\mathbf{w}$, which is again contrary to hypothesis. Thus $\mathbf{v} \neq s(\mathbf{v} + \mathbf{w})$ for any scalar s.

§3.2

2. No (the sum of polynomials of degree 4
(3) can be a nonzero polynomial of degree < 4)

4. Let $W = \{f \in F \mid f(1) = 0\}$ which is a nonempty set. If
(4) $f, g \in W$, then

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

so $f + g \in W$. Also,

$$(rf)(1) = rf(1) = (r)0 = 0$$

so $rf \in W$. Thus W is a subspace of F.

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