§3.2 Math 22a Fall 2009 Homework 8 Solutions

12. Suppose
(4) $r_1 + r_2(4x + 3) + r_3(3x - 4) + r_4(x^2 + 2) + r_5(x - x^2) = 0$.
Then

$(r_4 - r_5)x^2 + (4r_2 + 3r_3 + r_5)x + (r_1 + 3r_2 - 4r_3 + 2r_4) = 0$.
We solve the system

$$\begin{aligned} r_4 - r_5 &= 0 \\ 4r_2 + 3r_3 \quad + r_5 &= 0 \\ r_1 + 3r_2 - 4r_3 + 2r_4 \quad &= 0. \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & -4 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 3 & -4 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right].$$

A nontrivial solution is obtained by setting $r_5 = r_3 = 1$, $r_4 = 1$, $r_2 = -1$ and $r_1 = 5$. The set of vectors is dependent.

19. Suppose $r_1 + r_2(e^x + e^{-x}) + r_3(e^x - e^{-x}) = 0$.
(6) Differentiating twice, we obtain the two additional equations

$$r_2(e^x - e^{-x}) + r_3(e^x + e^{-x}) = 0,$$
$$r_2(e^x + e^{-x}) + r_3(e^x - e^{-x}) = 0.$$

Substituting $x = 0$ yields the homogeneous linear system

$$\begin{aligned} r_1 + 2r_2 \qquad &= 0 \\ 2r_3 &= 0 \\ 2r_2 \qquad &= 0 \end{aligned}$$

having the unique solution $r_1 = r_2 = r_3 = 0$. Thus the given set of functions is independent.

20. From algebra, we see that $(x - 1)^2 = (x^2 + 1) + (-2)x$, so
(3) the set of vectors is dependent and is not a basis for P_2.

21. From algebra, we see that $(x - 1)^2 = (x + 1)^2 + (-4)x$, so
(3) the set of vectors is dependent and is not a basis for P_2.

24. Let S be the given subspace of F. Since $\cos^2 x = 1 - \sin^2 x$,
(6) we deleted $\cos^2 x$ from the given functions, and see that $S = sp(1, \sin^2 x, \cos 2x)$. If we try to conclude from $r_1(1) + r_2 \sin^2 x + r_3(\cos 2x) = 0$ that $r_1 = r_2 = r_3 = 0$, either by substituting several different values for x in this equation or its derivatives, we will find that we are unable to do so. This makes us suspicious that cos 2x is a linear combination of 1 and $\sin^2 x$. Of course, those who had a good, old-fashioned trigonometry teacher noticed that $\cos 2x = 1 - 2\sin^2 x$, and consequently deleted cos 2x. Others must reach for a trig book and look up the double angle identities to discover this. Thus $S = sp(1, \sin^2 x)$. Since $\sin^2 x$ is not a multiple of 1, the set $\{1, \sin^2 x\}$ is independent and thus is a basis for S.

29. *Generation:* Let $W = sp(\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$. Clearly
(6) $W \subseteq V$. Also

$$\mathbf{v}_1 = \mathbf{v}_1, \ \mathbf{v}_2 = -\mathbf{v}_1 + (\mathbf{v}_1 + \mathbf{v}_2), \text{ and}$$
$$\mathbf{v}_3 = -(\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3).$$

Therefore $V = sp(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \subseteq W$, so $W = V$. That is, $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ generates V.

Independence: Suppose that $r_1\mathbf{v}_1 + r_2(\mathbf{v}_1 + \mathbf{v}_2) + r_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}$. Then $(r_1 + r_2 + r_3)\mathbf{v}_1 + (r_2 + r_3)\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is independent, we obtain the system

$$\begin{aligned} r_1 + r_2 + r_3 &= 0 \\ r_2 + r_3 &= 0 \\ r_3 &= 0 \end{aligned}$$

that has only the trivial solution $r_1 = r_2 = r_3 = 0$. Thus the set $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is also independent, and thus is a basis for V.

31. Note that $(\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 - \mathbf{v}_3$, that is, $\mathbf{w}_1 - \mathbf{w}_2$
(5) $= \mathbf{w}_3$, so $sp(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = sp(\mathbf{w}_1, \mathbf{w}_2)$. To show that this is not all of V, suppose that

$$\mathbf{v}_1 = r_1\mathbf{w}_1 + r_2\mathbf{w}_2 = r_1(\mathbf{v}_1 + \mathbf{v}_2) + r_2(\mathbf{v}_2 + \mathbf{v}_3).$$

Then

$$(r_1 - 1)\mathbf{v}_1 + (r_1 + r_2)\mathbf{v}_2 + r_2\mathbf{v}_3 = \mathbf{0}$$

so

$$r_1 - 1 = 0, \quad r_1 + r_2 = 0, \quad \text{and} \quad r_2 = 0$$

by independence of the $\mathbf{v}_i$. This system is inconsistent, so $\mathbf{v}_1 \notin sp(\mathbf{w}_1, \mathbf{w}_2)$.

§3.3

4.
(4)
$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & -2 \\ 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]; \quad [1, 2, -1]$$

6.
(5)
$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 9 \\ 0 & 1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 3 & 11 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 9 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 2 & 8 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]; \quad [-1, 2, 1, 3]$$

9. By inspection,
(3) $x + x^4 = \frac{1}{2} + \frac{1}{2}(2x - 1) + (x^3 + x^4) - \frac{1}{2}(2x^3) + 0(x^2 + 2)$

$\Rightarrow$ [1/2, 1/2, 1 -1/2, 0].

10. Let $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = r_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_2\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + r_3\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + r_4\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.
(5) Then

$$\begin{aligned} r_3 \qquad &= 1 \\ r_1 - r_2 - r_3 + r_4 &= -2 \\ r_1 \qquad &= 3 \\ 3r_3 + r_4 &= 4. \end{aligned}$$

We solve this system:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & -1 & -1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Thus $r_4 = 1$, $r_3 = 1$, $r_2 = 5$, $r_1 = 3$ so the coordinate vector is [3, 5, 1, 1].

(*)20. Let $\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_n$ be the coordinate vectors of the
(0) polynomials $1, x - a, (x - a)^2, \cdots, (x - a)^n$ respectively relative to the ordered basis $(1, x, x^2, \cdots, x^n)$. Let A be the $(n + 1) \times (n + 1)$ matrix having $\mathbf{v}_k$ as kth column vector.

The term of highest degree in $(x - a)^k$ is x^k, so we see that A is an upper-triangular matrix with entries 1 on the main diagonal. Since A is an echelon form with $n + 1$ pivots, the set $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is independent, and thus the given set of polynomials is independent. Since $\dim(P_n) = n + 1$, this set of polynomials is a basis for P_n.

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