Math 22a Fall 2005 Homework 5 Solutions

2. Not a linear transformation since $T(x) = 5^2 = 25$ while
(2) $T(2x) = (10)^2 = 100 \neq 50 = 2(T(x))$.

10. a) Using Eq.(8), we see that $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$.
(2)

b) Using column vector notation, we have
(2)

$$T(\mathbf{v})_{B'} = A\mathbf{v}_B = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ -24 \\ -20 \end{bmatrix}.$$

c)
(2)

$$\left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 14 & -8 & -1 \\ 0 & 1 & 0 & -17 & 10 & 1 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right].$$

Thus T is invertible and the matrix respresentation of T^{-1} relative to B', B is $A^{-1} = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}$.

d) $T^{-1}(\mathbf{v}')_B = A^{-1}\mathbf{v}'_{B'} = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -25 \\ 30 \\ 33 \end{bmatrix}$.
(2)

e) Using Eq.(8) with part (c), we see that
(2) $T^{-1}(\mathbf{b}'_1) = 14\mathbf{b}_1 - 17\mathbf{b}_2 - 19\mathbf{b}_3$,
$T^{-1}(\mathbf{b}'_2) = -8\mathbf{b}_1 + 10\mathbf{b}_2 + 11\mathbf{b}_3$,
$T^{-1}(\mathbf{b}'_3) = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$.

21. a) Since
(2)

$$\begin{aligned} T(x^3) &= 0x^3 + 3x^2 + 0x + 0 \\ T(x^2) &= 0x^3 + 0x^2 + 2x + 0 \\ T(x) &= 0x^3 + 0x^2 + 0x + 1 \\ T(1) &= 0x^3 + 0x^2 + 0x + 0 \end{aligned}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

b) Multiplying A times the coordinate vector of
(2) $4x^3 - 5x^2 + 10x - 13$ relative to B, we obtain

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \\ 10 \\ -13 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ -10 \\ 10 \end{bmatrix}. \text{ Thus}$$

$T(4x^3 - 5x^2 + 10x - 13) = 0x^3 + 12x^2 - 10x + 10$.

c) Let $p(x) = -5x^3 + 8x^2 - 3x + 4$. Since $D^2 = T \circ T$, we can
(2) compute $D^2(p(x))$ by left multiplying its coordinate vector by A twice, that is, by left multiplying its coordinate vector by A^2. We obtain

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2\begin{bmatrix} -5 \\ 8 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -30 \\ 16 \end{bmatrix}.$$

Thus $D^2(p(x)) = 0x^3 + 0x^2 - 30x + 16$.

24. a) $T(x^3) = D((2x + 1)^3) = 6(2x + 1)^2 = 24x^2 + 24x + 6$,
(2) $T(x^2) = D((2x + 1)^2) = 4(2x + 1) = 0x^2 + 8x + 4$,
$T(x) = D(2x + 1) = 2 = 0x^2 + 0x + 2$,
$T(1) = D(1) = 0 = 0x^2 + 0x + 0$.

Thus $A = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 24 & 8 & 0 & 0 \\ 6 & 4 & 2 & 0 \end{bmatrix}$.

b) Multiplying A by the coordinate vector of
(2) $4x^3 - 5x^2 + 4x - 7$ relative to B, we have

$$\begin{bmatrix} 24 & 0 & 0 & 0 \\ 24 & 8 & 0 & 0 \\ 6 & 4 & 2 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 96 \\ 56 \\ 12 \end{bmatrix}. \text{ Thus}$$

$T(4x^3 - 5x^2 + 4x - 7) = 96x^2 + 56x + 12$.

26. We compute
(4) $T(x) = 0 - 4(1) + x = x - 4 = 5x - 4(1 + x)$,
$T(1 + x) = 0 - 4(1) + 1 + x = x - 3 = 4x - 3(1 + x)$,
$T(x + x^2) = 2 - 4(1 + 2x) + x + x^2 = x^2 - 7x - 2$
$= -6x - 2(x + 1) + 1(x^2 + x)$.
(The coefficients -2 and 1 are necessary choices to achieve -2 and x^2 respectively, and the -6 then is needed to supply the -7x . Alternatively, the coefficients can be found by solving a linear system formed by equating like powers of x.)
$T(x^3) = 6x - 4(3x^2) + x^3 = x^3 - 12x^2 + 6x$
$= 18x + 0(1 + x) - 12(x + x^2) + 1(x^3)$.

Consequently $A = \begin{bmatrix} 5 & 4 & -6 & 18 \\ -4 & -3 & -2 & 0 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

28. We compute
(3) $T(e^{2x}) = \dfrac{e^{2x}}{2}$, $T(e^{4x}) = \dfrac{e^{4x}}{4}$, $T(e^{8x}) = \dfrac{e^{8x}}{8}$.

Consequently $A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$.

48. Let A be the standard matrix representation of T. The
(*) equation $T(T(\mathbf{x})) = T(\mathbf{x}) + T(\mathbf{x}) + 3\mathbf{x}$ then corresponds to the matrix equation $A^2 = A + A + 3I$. Thus $A^2 - 2A = 3I$ so $A(\frac{1}{3}(A - 2I)) = I$. This shows that A is invertible and $A^{-1} = \frac{1}{3}(A - 2I)$. Therefore T is an isomorphism, and consequently is a one-to-one mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

14. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 1 & 4 \\ 2 & 1 & -3 \end{vmatrix} = (-3 - 4)\mathbf{i} - (15 - 8)\mathbf{j} + (-5 - 2)\mathbf{k}$
(2) $= -7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$

20. $\begin{vmatrix} -1 & 4 \\ 2 & 3 \end{vmatrix} = -3 - 8 = -11$. **The area is $|-11| = 11$.**
(2)

26. Taking the point (1, 1) as a new origin, vectors from that point falling on the sides of the triangle are a = [5, 7] - [1, 1] = [4, 6] and b = [3, -4] - [1, 1] = [2, -5].
(6) $\begin{vmatrix} 4 & 6 \\ 2 & -5 \end{vmatrix} = -20 - 12 = -32$. **The area of the triangle is thus $|-32|/2 = 16$, since the area of the triangle is half the area of the parallelogram determined by a and b.**

32. Solving simultaneously, we find that
(4) the lines $x - 2y = 3$ and $2x + 3y = -1$ meet at (1, -1),
the lines $x - 2y = 3$ and $2x + 3y = -8$ meet at (-1, -2),
the lines $x - 2y = 10$ and $2x + 3y = -1$ meet at (4, -3),
the lines $x - 2y = 10$ and $2x + 3y = -8$ meet at (2, -4).

If we select (1, -1) as new origin, we obtain the correct answer by finding the area of the parallelogram having any two of the remaining three points as vertices adjacent to (1, -1). We let **a** = [4, -3] - [1, -1] = [3, -2] and **b** = [-1, -2] - [1, -1] = [-2, -1]. $\begin{vmatrix} 3 & -2 \\ -2 & -1 \end{vmatrix} = -3 - 4 = -7$. Area = $|-7| = 7$.

38. $\begin{vmatrix} 2 & 1 & -4 \\ 3 & -1 & 2 \\ 1 & 3 & -8 \end{vmatrix} = 2(8 - 6) - 1(-24 - 2) + (-4)(9 + 1)$
(3) $= 4 + 26 - 40 = -10$. Volume $= |-10| = 10$.

48. ← (3)

Three points are collinear if and only if the "triangle" having them as vertices has area 0, or equivalently, if and only if the "parallelogram" determined by the vectors emanating from one point to the two others has area 0. We let **a** = [2, 3] - [1, -4] = [1,7] and **b** = [6, 2] - [1, -4] = [5, 6], and we find that $\begin{vmatrix} 1 & 7 \\ 5 & 6 \end{vmatrix} = 6 - 35 = -29 \neq 0$. Thus the points are not collinear.

52. We let
(4) **a** = [3, 3, 4] - [1, 2, 1] = [2, 1, 3],
b = [2, 2, 2] - [1, 2, 1] = [1, 0, 1], and
c = [4, 3, 5] - [1, 2, 1] = [3, 1, 4].
$\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 2(0 - 1) - 1(4 - 3) + 3(1 - 0)$
$= -2 - 1 + 3 = 0$ so the points are coplanar.

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