# Chapter 1

# Numbers, Sets, and Functions

The ancient Babylonians considered the problem of finding two numbers when given their sum and product. They expressed the solution in words, not in formulas. We begin this book by deriving the quadratic formula and using it to solve this ancient problem. We then discuss the properties of the real numbers and the basic concepts of sets and functions that enable us to state and solve mathematical problems.

## THE QUADRATIC FORMULA

Given two numbers $s$ and $p$, the Babylonians wanted to find $x$ and $y$ such that $x+y=s$ and $xy=p$. To do so, we write $y=s-x$ and substitute to obtain $x(s-x)=p$, which we rewrite as $x^2-sx+p=0$. Every solution $x$ to the problem of the Babylonians must satisfy this quadratic equation.

Solving this equation is equivalent to solving the general quadratic equation. We don't change the solutions if we multiply the equation by a nonzero constant $a$ to obtain $ax^2-asx+ap=0$, and then we can name $b=-as$ and $c=ap$ to obtain $ax^2+bx+c=0$.

The familiar **quadratic formula** expresses the solution for $x$ in terms of $a$, $b$, and $c$. First we rewrite the equation in a manner where the unknown value $x$ appears only once:

$$\begin{aligned} 0 = a(x^2+\frac{b}{a}x)+c &= a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2})-\frac{b^2}{4a}+c \\ &= a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}. \end{aligned}$$

Hence $(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$. Solving for $x$ yields the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula describes all the solutions to the general quadratic equation. When $b^2-4ac>0$, it yields two values. When $b^2-4ac=0$, these values are equal. When $b^2-4ac<0$, there is no solution in real numbers. Rewriting the solution formula in terms of $s$ and $p$ yields the expressions

$$\frac{s+\sqrt{s^2-4p}}{2},\ \frac{s-\sqrt{s^2-4p}}{2} \tag{$*$}$$

to solve the Babylonian problem. When $s^2-4p<0$, there is no solution.

The quadratic formula gave us $(*)$ as the solutions to the quadratic equation $x^2-sx+p=0$, when $s^2-4p\ge 0$. Note that the sum of the numbers in $(*)$ is $s$ and their product is $p$. This checks our solution.

For any real numbers $\alpha$ and $\beta$, we can create a quadratic polynomial that is zero at $\alpha$ and $\beta$ by letting $x-\alpha$ and $x-\beta$ be factors. Since $(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$, the product of the solutions is the constant term, and their sum is the negative of the coefficient of $x$.

What properties of numbers did we use in solving the Babylonian problem? First, we used basic rules about addition and multiplication. The result of adding several numbers does not depend on the order of writing them or on the order of performing pairwise additions. Multiplication has the same property. We also used the more subtle *distributive law*: $x(y+z)=xy+xz$.

We also used properties of subtraction and division. Every number $u$ has an *additive inverse* $-u$, and subtracting $u$ has the same effect as adding $-u$. Their sum is 0, and adding 0 causes no change. Similarly, every nonzero number $u$ has a multiplicative inverse $u^{-1}$. Their product is 1, and multiplying by 1 causes no change. An important distinction is that we cannot divide by 0. The properties of inverses allow us to cancel equal terms or *nonzero* common factors from both sides of an equality.

These rules about arithmetic are *algebraic* properties. We also used properties of *inequality* and *order*. Because the product of two nonzero numbers with the same sign is positive, square roots exist only for nonnegative numbers. Furthermore, if $u^2=v$, then also $(-u)^2=v$. Thus we write ± on the square root sign in the quadratic formula and say that there is no solution in real numbers when $b^2-4ac<0$.

The Babylonians would not have accepted our solution, because their number system did not include negative numbers! In the real number system, the formula $(-b\pm\sqrt{b^2-4ac})/(2a)$ makes sense when $b^2-4ac\ge 0$. It remains to express the square root of $b^2-4ac$ in an acceptable form. Expressing square roots in decimal form generally requires infinite nonrepeating decimal expansions. This requires a *completeness* property of the number system and is related to infinite processes and limits.

In developing mathematical ideas in this book, we take the real number system with its elementary properties as given; this allows us to focus on the logical structure of mathematical arguments. At the end of this chapter, we list the properties that characterize the real numbers, describe what we assume about them, and discuss ways to approach solving problems. Meanwhile, we discuss other background material.

## ELEMENTARY INEQUALITIES

Manipulating inequalities requires care. Multiplying both sides of an equation by the same number preserves equality, but this fails for inequalities. If $a < b$, then $ac < bc$ if and only if $c > 0$.

In this section, we derive several inequalities about real numbers. They rely on two properties: positive real numbers have positive square roots, and the square of every real number is nonnegative. We prove first that taking squares or square roots of positive numbers preserves order.

**1.1. Proposition.** If $0 < a < b$, then $a^2 < ab < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

**Proof:** Multiplying an inequality by a positive number does not change whether the inequality is true. Thus we multiply $a < b$ by $a$ to obtain $a^2 < ab$, and we multiply $a < b$ by $b$ to obtain $ab < b^2$.

We also must have $\sqrt{a} < \sqrt{b}$; otherwise, applying the first statement to $\sqrt{b} \le \sqrt{a}$ yields $b \le a$, which violates the hypothesis $a < b$. ■

We use bold type in this book for terms being defined.

**1.2. Definition.** The **absolute value** of a real number $x$, written as $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x \le 0. \end{cases}$$

We think of $|x|$ as the distance from $x$ to 0; this motivates our next proof (see Example 1.50 for another approach). Note that always $x \le |x|$ and $|xy| = |x|\,|y|$.

**1.3. Proposition.** (Triangle Inequality) If $x$ and $y$ are real numbers, then $|x + y| \le |x| + |y|$.

**Proof:** We start with the inequality $2xy \le 2\,|x|\,|y|$. By adding $x^2 + y^2$ to both sides and using $z^2 = |z|^2$, we obtain

$$x^2 + 2xy + y^2 \le x^2 + 2\,|x|\,|y| + y^2 = |x|^2 + 2\,|x|\,|y| + |y|^2 .$$

By Proposition 1.1, we may take the positive square root of both sides and preserve the inequality. Thus $|x + y| \le |x| + |y|$, as desired. ■

In order to prove a statement, we derive it from known facts. Before we find a proof, we may not know which known facts to use. To discover a proof, it may be helpful to ask what is needed to make the conclusion true. In this approach, we try to "reduce" the desired conclusion to a statement known to be true. The written proof must be a rigorous justification of the conclusion from known facts.

The next proposition illustrates this. Manipulating the desired inequality leads to a known inequality, but the proof starts with the known inequality and derives the desired one from it. The **arithmetic mean** (or "average") of $x$ and $y$ is $(x + y)/2$. The **geometric mean** of nonnegative numbers $x$ and $y$ is $\sqrt{xy}$. The term **AGM Inequality** stands for *Arithmetic Mean–Geometric Mean Inequality*; it states that the arithmetic mean of two nonnegative numbers is always at least their geometric mean.

**1.4. Proposition.** (AGM Inequality) If $x$ and $y$ are real numbers, then $2xy \le x^2 + y^2$ and $xy \le (\frac{x+y}{2})^2$. If $x$ and $y$ are also nonnegative, then $\sqrt{xy} \le (x + y)/2$. Equality holds in each only when $x = y$.

**Proof:** We begin with $0 \le (x - y)^2 = x^2 - 2xy + y^2$ and observe that equality holds only when $x = y$. Adding $2xy$ yields $2xy \le x^2 + y^2$. Adding another $2xy$ yields $4xy \le x^2 + 2xy + y^2 = (x + y)^2$, which we divide by 4 to obtain $xy \le (\frac{x+y}{2})^2$.

If $x \ge 0$ and $y \ge 0$, then also $xy \ge 0$, and we can take positive square roots in $xy \le (\frac{x+y}{2})^2$. Proposition 1.1 yields $\sqrt{xy} \le (x + y)/2$. ■

**1.5. Corollary.** If $x, y > 0$, then $\frac{2xy}{x+y} \le \sqrt{xy} \le \frac{x+y}{2}$. Equality holds in each inequality only when $x = y$.

**Proof:** Proposition 1.4 yields $\sqrt{xy} \le \frac{x+y}{2}$. We obtain the other inequality from this by multiplying both sides by the positive number $\frac{2\sqrt{xy}}{x+y}$. ■

**1.6. Application.** The expression $\frac{2xy}{x+y}$ is the **harmonic mean** of $x$ and $y$. It arises in the study of average rates. When we travel a distance $d$ at rate $r$ in time $t$, we have $d = rt$, in appropriate units.

If we travel a distance $d$ at a rate $r_1$ in time $t_1$ and make the return trip at rate $r_2$ in time $t_2$, then $r_1 t_1 = d = r_2 t_2$. What is the average rate $r$ for the full trip? The computation is $2d = r(t_1 + t_2)$, and hence

$$r = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{r_1} + \frac{d}{r_2}} = \frac{2r_1 r_2}{r_1 + r_2}.$$

Thus the average rate for the full trip is the harmonic mean of the rates in the two directions. By Corollary 1.5, the rate for the full trip is less than the average of the two rates one-way when those rates differ.

For example, if the rate one way on a plane trip is 380 mph, and the return rate over the same distance is 420 mph, then the average rate is

$$\frac{2(380)(420)}{380+420} = \frac{800(19)(21)}{800} = (20-1)(20+1) = 399 \text{ mph}$$

This is less than 400 because more time is spent at the slower rate. ■

In this book, we reserve the label **Example** for direct illustrations of mathematical concepts. We use **Solution** and **Application** to designate examples that incorporate additional reasoning. Results that can be used to solve problems here and elsewhere have the labels **Definition**, **Proposition**, **Lemma**, **Theorem**, and **Corollary**.

## SETS

We begin our formal development with basic notions of set theory. Our most primitive notion is that of a **set**. This notion is so fundamental that we do not attempt to give a precise definition. We think of a set as a collection of distinct objects with a precise description that provides a way of deciding (in principle) whether a given object is in it.

**1.7. Definition.** The objects in a set are its **elements** or **members**. When $x$ is an element of $A$, we write $x \in A$ and say "$x$ **belongs to** $A$". When $x$ is not in $A$, we write $x \notin A$. If every element of $A$ belongs to $B$, then $A$ is a **subset** of $B$, and $B$ **contains** $A$; we write $A \subseteq B$ or $B \supseteq A$.

When we list the elements of a set explicitly, we put braces around the list; "$A = \{-1, 1\}$" specifies the set $A$ consisting of the elements $-1$ and 1. Writing the elements in a different order does not change a set. We write $x, y \in S$ to mean that both $x$ and $y$ are elements of $S$.

**1.8. Example.** By convention, we use the special characters $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ to name the sets of **natural numbers**, **integers**, **rational numbers**, and **real numbers**, respectively. Each set in this list is contained in the next, so we write $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

We take these sets as familiar. We use the convention that 0 is not a natural number; $\mathbb{N} = \{1, 2, 3, \ldots\}$. The set of integers is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The set $\mathbb{Q}$ of rational numbers is the set of real numbers that can be written as $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. ■

**1.9. Definition.** Sets $A$ and $B$ are **equal**, written $A = B$, if they have the same elements. The **empty set**, written $\varnothing$, is the unique set with no elements. A **proper subset** of a set $A$ is a subset of $A$ that is not $A$ itself. The **power set** of a set $A$ is the set of all subsets of $A$.

Note that the empty set is a subset of every set.

**1.10. Example.** Let $S$ be the set {Kansas, Kentucky}. Let $T$ be the set of states in the United States whose names begin with "K". The sets $S$ and $T$ are equal. The set $S$ has four subsets: $\varnothing$, {Kansas}, {Kentucky}, and {Kansas, Kentucky}. These four are the elements of the power set of $S$. ■

**1.11. Remark.** *Specifying a set.* In Example 1.10, we specified a set both by listing its elements and by describing it as a subset of a larger set. In order to specify a set $S$ consisting of the elements in a set $A$ that satisfy a given condition, we write "$\{x \in A : \text{condition}(x)\}$". We read this as "the set of $x$ in $A$ such that $x$ satisfies 'condition' ". For example, the expression $S = \{x \in \mathbb{R} : ax^2 + bx + c = 0\}$ specifies $S$ as the set of real numbers satisfying the equation $ax^2 + bx + c = 0$, where $a, b, c$ are known constants. We may omit specifying the universe $A$ when the context makes it clear. ■

**1.12. Remark.** What must be done to determine the solutions to a mathematical problem? In order to prove that the set of solutions is $T$, we must prove that every solution belongs to $T$, and we must prove that every member of $T$ is a solution.

Letting $S$ denote the set of solutions, our goal is to prove that $S = T$, where $T$ is a list or has a simple description. The statement "$S = T$" conveys two pieces of information: "$S \subseteq T$ and $T \subseteq S$". The first containment states that every solution belongs to $T$, and the second states that every member of $T$ is a solution. ■

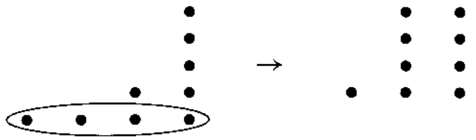
**1.13. Example.** *Equality of sets.*

1) *The inequality* $x^2 < x$. Let $S = \{x \in \mathbb{R}: x^2 < x\}$, and let $T = \{x \in \mathbb{R}: 0 < x < 1\}$. We claim that $S = T$. To prove this, we show that $T \subseteq S$ and that $S \subseteq T$. First consider $x \in T$. Since $x > 0$, we can multiply the known inequality $x < 1$ by $x$ to obtain $x^2 < x$, so $x \in S$. Conversely, consider $x \in S$. Since $x^2 < x$, we have $0 > x^2 - x = x(x-1)$. This requires that $x$ and $x - 1$ are nonzero and have opposite signs, which yields $x \in T$.

2) *The quadratic equation* $ax^2 + bx + c = 0$. Let $S$ be the set of solutions, and let $T = \{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\}$. When we proved $S = T$, we could have proved both $S \subseteq T$ and $T \subseteq S$. The latter involves plugging each purported solution into the original equation and checking that it works. Our reasoning was more efficient; we operated on the equation in ways that preserved the set of solutions. This produced a string of equalities of sets, starting with $S$ and ending with $T$. Note that plugging the members of $T$ into the equation enables us to check that we have not made an error in manipulating the equation. ■

Our next example again illustrates the process of describing a solution set by proving two containments. We are proving that an object is a solution *if and only if* it belongs to the desired set.

**1.14. Application.** *The Penny Problem.* Given piles of pennies, we remove one coin from each pile to make one new pile. Each original pile shrinks by one, so each pile of size one disappears: 1,1,2,5 becomes 1,4,4, for example. We consider different orderings of the same list of sizes to be equivalent, so we restrict our attention to lists of positive integers in nondecreasing order. Let $S$ be the set of lists that do not change.



Let $a$ be a list with $n$ piles, and let $b$ be the resulting new list. If $a \in S$, meaning that $a$ and $b$ are the same, then $b$ also has $n$ piles. Since we introduce one new pile, exactly one pile must disappear. Thus $a$ has exactly one pile of size 1. Thus $b$ also has exactly one pile of size 1. This forces $a$ to have exactly one pile of size 2.

We continue this reasoning for $i$ from 1 to $n-1$. From $a$ having one pile of size $i$, we conclude that $b$ has one pile of size $i$, and therefore that $a$ has one pile of size $i+1$. This gives us one pile of each size 1 through $n$.

Let $T$ be the set of lists consisting of one pile of each size from 1 through some natural number $n$. We have shown that every unchanged configuration has this form, so $S \subseteq T$. To complete the solution, we also check that all elements of $T$ remain unchanged.

Consider the element of $T$ with piles of sizes $1, 2, \dots, n$. For each $i$ from 2 to $n$, the pile of size $i$ becomes a pile of size $i-1$. The pile of size 1 disappears, and the $n$ piles each contribute one coin to form a new pile of size $n$. The result is the original list. Now we have proved that $S \subseteq T$ and $T \subseteq S$, so $S = T$. We have described all the unchanged lists. ■

The next three definitions introduce notation and terminology for special sets that we will use throughout this book.

**1.15. Definition.** *Sets of integers.* When $a, b \in \mathbb{Z}$ with $a \le b$, we use $\{a, \dots, b\}$ to denote $\{i \in \mathbb{Z} : a \le i \le b\}$. When $n \in \mathbb{N}$, we write $[n]$ for $\{1, \dots, n\}$. The set of **even numbers** is $\{2k : k \in \mathbb{Z}\}$. The set of **odd numbers** is $\{2k+1 : k \in \mathbb{Z}\}$.

Note that 0 is an even number. Every integer is even or odd, and no integer is both. The **parity** of an integer states whether it is even or odd. We say "even" and "odd" for numbers *only* when discussing integers. Similarly, when we say that a number is positive without specify the number system containing it, we mean that it is a positive real number. Thus, "consider $x > 0$" means "let $x$ be a positive real number".

**1.16. Definition.** *Intervals.* When $a, b \in \mathbb{R}$ with $a \le b$, the **closed interval** $[a, b]$ is the set $\{x \in \mathbb{R}: a \le x \le b\}$. The **open interval** $(a, b)$ is the set $\{x \in \mathbb{R}: a < x < b\}$.

Consider $S \subseteq \mathbb{R}$. If an element $x$ belonging to $S$ is at least as large as every element of $S$, then $x$ is a **maximum** of $S$. A set can only have one maximum. The concept of **minimum** is defined analogously. The open interval $(a, b)$ has no maximum and no minimum.

There are several natural ways to obtain new sets from old sets.

**1.17. Definition.** *k-tuples and Cartesian product.* A **list** with entries in $A$ consists of elements of $A$ in a specified order, with repetition allowed. A **$k$-tuple** is a list with $k$ entries. We write $A^k$ for the set of $k$-tuples with entries in $A$.

An **ordered pair** is a list with two entries. The **Cartesian product** of sets $S$ and $T$, written $S \times T$, is the set $\{(x, y): x \in S, y \in T\}$.

Note that $A^2 = A \times A$ and $A^k = \{(x_1, \dots, x_k): x_i \in A\}$. We read "$x_i$" as "$x$ sub $i$". Since we use the notation $(a, b)$ for ordered pairs, we often write "the interval $(a, b)$" to avoid confusion when specifying an open interval.

When $S = T = \mathbb{R}$, the Cartesian product $S \times T$ or $\mathbb{R}^2$ can be viewed as the set of all points in the plane, designated by horizontal and vertical coordinates, called the **Cartesian coordinates** of the point. The concept of Cartesian product is named for René Descartes (1596–1650). The Cartesian product of two intervals in $\mathbb{R}$ is a rectangle in the plane.

**1.18. Definition.** *Set operations.* Let $A$ and $B$ be sets. Their **union**, written $A \cup B$, consists of all elements in $A$ or in $B$. Their **intersection**, written $A \cap B$, consists of all elements in both $A$ and $B$. Their **difference**, written $A - B$, consists of the elements of $A$ that are not in $B$. Two sets are **disjoint** if their intersection is the empty set $\emptyset$. If a set $A$ is contained in some universe $U$ under discussion, then the **complement** $A^c$ of $A$ is the set of elements of $U$ *not* in $A$.

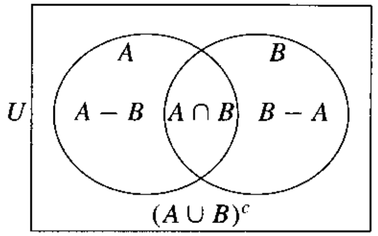
**1.19. Example.** Let $E$ and $O$ denote the sets of even numbers and odd numbers. We have $E \cap O = \emptyset$ and $E \cup O = \mathbb{Z}$. Within $\mathbb{Z}$, we have $E^c = O$. ■

Pictures give life to mathematical concepts and illuminate essential ideas. We encourage the reader to draw pictures to clarify concepts. We do this for the operations in Definition 1.18. Diagrams illustrating sets and their relationships are named for John Venn (1834–1923), though he was not the first person to use them.

**1.20. Remark.** *Venn diagrams.* In a **Venn diagram**, an outer box represents the universe under consideration, and regions within the box correspond to sets. Non-overlapping regions correspond to disjoint sets. The four regions in the Venn diagram for two sets $A$ and $B$ represent $A \cap B$, $(A \cup B)^c$, $A - B$, and $B - A$.

Since $A - B$ consists of the elements in $A$ and not in $B$, we have $A - B = A \cap B^c$. Similarly, the diagram suggests that $B^c$ is the union of $A - B$ and $(A \cup B)^c$, which are disjoint. Also $A - B$ and $B - A$ are disjoint.

Slightly more subtle is $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. A rigorous proof shows that an element belongs to one set if and only if it belongs to the other. Exercise 41 lists other elementary relationships. ■



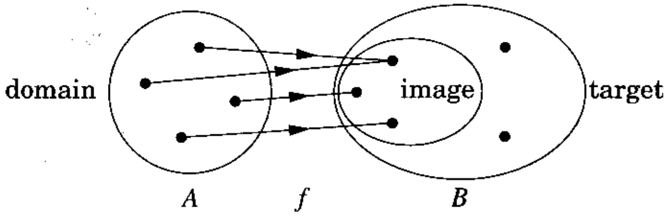
## FUNCTIONS

"Function" is the name we use for a mathematical machine with inputs and outputs. The inputs are the elements from one set; the outputs are elements of a (possibly) different set. Familiar ways to specify a function include an algebraic formula, a list of the outputs associated with the inputs, a description in words of how an input determines its output, and various graphical representations.

**1.21. Definition.** A **function** $f$ *from* a set $A$ *to* a set $B$ assigns to each $a \in A$ a single element $f(a)$ in $B$, called the **image** of $a$ under $f$. For a function $f$ from $A$ to $B$ (written $f\colon A \to B$), the set $A$ is the **domain** and the set $B$ is the **target**. The **image** of a function $f$ with domain $A$ is $\{f(a)\colon a \in A\}$.

**1.22. Remark.** *Schematic representation.* A function $f\colon A \to B$ is **defined on** $A$ and **maps** $A$ into $B$. To visualize a function $f\colon A \to B$, we draw a region representing $A$ and a region representing $B$, and from each $x \in A$ we draw an arrow to $f(x)$ in $B$.

The image of a function is contained in its target. Thus we draw the region for the image inside the region for the target. ■



There are many ways to describe a function; for each $a \in A$, we must specify $f(a)$. We can list the pairs $(a, f(a))$, provide a formula for computing $f(a)$ from $a$, or describe the rule for obtaining $f(a)$ from $a$ in words. Note that $f(a)$ denotes an element of the target of $f$ and does not denote the function $f$. Thus $x^2$ is a number (when we know $x$); it should be distinguished from the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

**1.23. Example.** *Descriptions of functions.*

*Functions given by formulas.* The "squaring" function $S\colon \mathbb{R} \to \mathbb{R}$ is defined by $S(x) = x \cdot x = x^2$. The "addition" and "multiplication" functions are defined from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ by $A(x, y) = x + y$ and $M(x, y) = xy$.

*A function given by listing its values.* Define $g\colon [7] \to \mathbb{N}$ by listing $g(1) = 6$, $g(2) = 6$, $g(3) = 7$, $g(4) = 9$, $g(5) = 8$, $g(6) = 6$, $g(7) = 8$.

*A function given by words.* Define $h\colon [7] \to \mathbb{N}$ by letting $h(n)$ be the number of letters in the English word for the $n$th day of the week, starting with Sunday. The function $h$ is the same as $g$ defined above. ■

**1.24. Remark.** *The meaning of "well-defined".* A function $f\colon A \to B$ may be specified by different rules on different subsets of $A$. The statement "$f$ is well-defined" means that the rules assign to each element of $A$ exactly one element, belonging to $B$. When different rules apply to an element of $A$, we must check that they give the same element of $B$ (see Exercise 45).

For example, the absolute value of $x$ (Definition 1.2) is defined using two rules, both applying when $x = 0$. Since $0 = -0$, the rules agree at 0, and thus absolute value is well-defined. ■

**1.25. Definition.** A function $f$ is **real-valued** if its image is a subset of $\mathbb{R}$; in this case $f(x)$ is a number. For real-valued functions $f$ and $g$ with domain $A$, the **sum** $f + g$ and **product** $fg$ are real-valued functions on $A$ defined by $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$.

**1.26. Definition.** A (real) **polynomial** in one variable is a function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = c_0 + c_1x^1 + \ldots + c_kx^k$, where $k$ is a nonnegative integer and $c_0, \ldots, c_k$ are real numbers called the **coefficients** of $f$. The **degree** of $f$ is the largest $d$ such that $c_d \neq 0$; the polynomial with all coefficients 0 has no degree. Polynomials of degrees 0,1,2,3, are **constant**, **linear**, **quadratic**, **cubic**, respectively.

We can study polynomials in more variables. A **monomial** in variables $x_1, \ldots, x_n$ is an expression $cx_1^{a_1} \cdots x_n^{a_n}$, where $c$ is a real number and each $a_j$ is a nonnegative integer. A polynomial in $n$ variables is a finite sum of monomials in $n$ variables. For example, the function $f$ defined by

$$f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

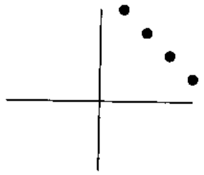
is a polynomial in three variables. It is a polynomial in the single variable $x$ when $y$ and $z$ are held constant.

We can also describe functions using geometric ideas.

**1.27. Definition.** The **graph** of a function $f: A \to B$ is the subset of $A \times B$ consisting of the ordered pairs $\{(x, f(x)): x \in A\}$.

**1.28. Remark.** *Pictures of graphs.* Let $f$ be a real-valued function defined on a set $A \subseteq \mathbb{R}$. We draw two copies of $\mathbb{R}$ as horizontal and vertical axes, associating the horizontal axis with the domain. The graph of $f$ is then a set of points in the plane. A set $S$ of points in the plane is the graph of a function if and only if it contains at most one element $(x, y)$ for each real number $x$; in other words, each vertical line intersects $S$ at most once. ∎

**1.29. Example.** *Alternative representations.* We describe a particular function $f: [4] \to [4]$ using each method we have discussed. Define $f$ by $f(n) = 5 - n$ to give a formula. Define $f$ by $f(1) = 4$, $f(2) = 3$, $f(3) = 2$, $f(4) = 1$, listing values. Define $f$ by saying that $f$ interchanges 1 and 4 and interchanges 2 and 3. The graph of $f$ is $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$. ∎



For a function defined by a formula, the image may not be obvious.

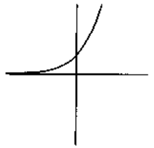
**1.30. Example.** *For the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x/(1 + x^2)$, the image is the interval $[-1/2, 1/2]$.* To prove that this is the image, we show first that $|f(x)| \le 1/2$ for $x \in \mathbb{R}$. This claim is equivalent to $|x| \le (1 + x^2)/2$, which follows from $(1 - |x|)^2 \ge 0$.

We have proved that the interval contains the image; we must also prove that the image contains the interval. For $y \in [-1/2, 1/2]$, we prove that there exists $x \in \mathbb{R}$ such that $f(x) = y$. Note that $f(0) = 0$. For $y \neq 0$ and $y \in [-1/2, 1/2]$, we set $y = x/(1 + x^2)$ and solve for $x$ in terms of $y$. Applying the quadratic formula to $yx^2 - x + y = 0$ yields $x = (1 \pm \sqrt{1 - 4y^2})/2y$. Since $|y| \le 1/2$, we now have $x \in \mathbb{R}$ such that $f(x) = y$. ∎

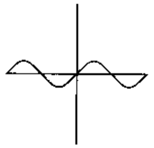
**1.31. Definition.** A set $S \subseteq \mathbb{R}$ is **bounded** if there exists $M \in \mathbb{R}$ such that $|x| \le M$ for all $x \in S$. A set is **unbounded** if no such $M$ exists. A **bounded function** is a real-valued function whose image is bounded; that is, a real-valued function $f$ for which there is some $M$ in $\mathbb{R}$ such that $|f(x| \le M$ for all $x$ in the domain.

**1.32. Definition.** Let $f: \mathbb{R} \to \mathbb{R}$, and let $A$ be a set of real numbers. We say that $f$ is **increasing** (on $A$) if $f(x) < f(x')$ whenever $x < x'$ and $x, x' \in A$. It is **nondecreasing** (on $A$) if $f(x) \le f(x')$ whenever $x < x'$ and $x, x' \in A$. Changing $<$ to $>$ and $\le$ to $\ge$ yields definitions for **decreasing** and **nonincreasing**. A function is **monotone** on $A$ if it is nondecreasing on $A$ or if it is nonincreasing on $A$.

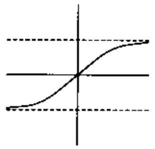
The properties "increasing" and "nondecreasing" are also called **strictly increasing** and **weakly increasing**, respectively. Similarly, a function is **strictly monotone** on $A$ if it is increasing on $A$ or if it is decreasing on $A$. We use the word "monotone" to avoid repetition; many results apply in both cases. A function that is increasing on one interval and decreasing on another is *not* monotone. The function of Example 1.30 is bounded but not monotone.



unbounded, increasing



bounded, not monotone



bounded, increasing

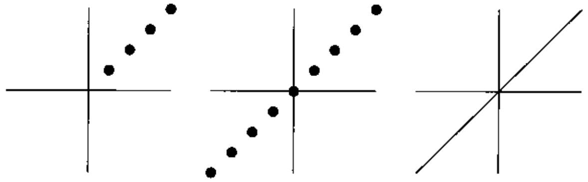
**1.33. Remark.** *Geometric interpretations.* A function from $\mathbb{R}$ to $\mathbb{R}$ is increasing if and only if for every horizontal line intersecting its graph, the graph is above that line to the right of the intersection and below it to the left. The function is bounded if and only if every point in the graph lies in the band between some pair of horizontal lines. ∎

The use of "if" in Definitions 1.31–1.32 has the same meaning as the use of "if and only if" in Remark 1.33. In *defining* $X$, we often say that $X$ occurs "if" some property holds, yet we mean that the new concept and the condition are equivalent. This is a convention; in some sense the concept does not exist until it is defined, so the implication can only hold in one direction. In this book, the definition usage of "if" is recognizable by the use of bold type for the concept being defined.

**1.34. Definition.** The **identity** function on a set $S$ is the function $f: S \to S$ defined by $f(x) = x$ for all $x \in S$. A **fixed point** of a function $f: S \to S$ is an element $x \in S$ such that $f(x) = x$.

Every element of $S$ is a fixed point for the identity function on $S$. In the Penny Problem (Application 1.14), we studied a function from the set of nondecreasing lists of natural numbers to itself; our aim was to find all fixed points. A function $f$ from $\mathbb{R}$ to $\mathbb{R}$ has a fixed point if and only if the line $\{(x, x)\}$ through the origin intersects the graph of $f$.

The identity functions on $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are graphed below. The graphs show that these are different functions; a function cannot be specified by a formula alone. Two functions are **equal** if they have the same domain, have the same target, and agree in value at each element of the domain.



## INVERSE IMAGE AND LEVEL SETS

We can interpret solution sets for equations using the language of functions. For any function $f$ and value $y$ in its target, we consider the set of solutions to $f(x) = y$.

**1.35. Definition.** Given $f\colon A \to B$ and $y \in B$, the **inverse image** of $y$ under $f$, written $I_f(y)$, is the set $\{x \in A\colon f(x) = y\}$.

If $f(p)$ is the temperature at the point $p$, then $I_f(32)$ is the set of all points where the temperature is 32. The inverse image is called an *isotherm*; sketches of isotherms appear on most weather maps.

The inverse image of $y$ under a function $f\colon A \to B$ is a subset of the domain $A$. Generally speaking, inverse image is not a function from $B$ to $A$, because it may associate many elements of $A$ with an element of $B$.

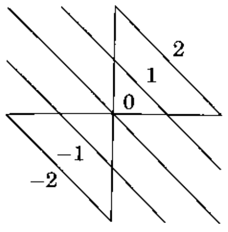
Real-valued functions often arise as measurements. Consider for example the function $h$ that assigns to a point in the United States the height of this point above sea level. A topographical map shows points with the same height above sea level connected by a *level curve* (the curve may have many pieces). Each level curve for $h$ is $I_h(c)$ for some $c$; the number $c$ gives the height above sea level.

**1.36. Definition.** For $h\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the **level set** of $h$ with value $c$ is $I_h(c)$.

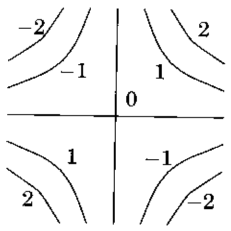
**1.37. Example.** Let $A(x, y) = x + y$. For each $c$, $I_A(c)$ is a line in $\mathbb{R}^2$. The level sets are parallel lines whose union is all of $\mathbb{R}^2$.

Let $M(x, y) = xy$. The level set $I_M(0)$ consists of the two coordinate axes. For $c \neq 0$, the level sets are hyperbolas; each has two branches.

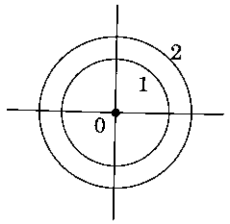
Let $D(x, y) = x^2 + y^2$. The level set $I_D(c)$ is empty when $c < 0$, consists of one point when $c = 0$, and is a circle of radius $\sqrt{c}$ when $c > 0$. The figure shows these level sets when $c \in \{-2, -1, 0, 1, 2\}$. ■



$I_A(c)$



$I_M(c)$

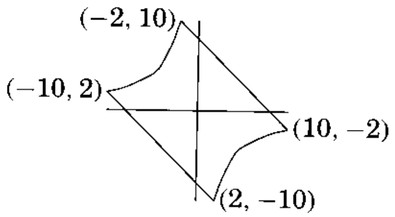


$I_D(c)$

**1.38. Application.** Given two real numbers whose sum is between −8 and 8 and whose product is between −20 and 20, what is the largest that one of these numbers can be?

We use level sets to solve this problem. We are given $|x + y| \le 8$ and $|xy| \le 20$. In the graph of the solution set the boundary is determined by the level sets $x + y = 8$, $x + y = -8$, $xy = 20$, and $xy = -20$. The level sets for intermediate values lie between them.

By plotting the level sets, we see that the largest value $x$ can have (when both inequalities hold) occurs when $xy = -20$ and $x + y = 8$. Solving these equations as in the discussion of the Babylonian problem of Chapter 1 yields $x = 10$ and $y = -2$. Thus the maximum value is 10. ■



## THE REAL NUMBER SYSTEM

The real numbers satisfy a short list of properties called **axioms** from which all the other properties can be derived. In this section we state these properties and some of their consequences. Our purpose here is *not* to study these in detail, but rather to state our starting point and clarify what the student may assume when solving exercises.

A structure satisfying Definitions 1.39–1.41 below is a **complete ordered field**. In Appendix A, we prove that all such structures are essentially equivalent. Furthermore, we build such a structure and verify that it satisfies the axioms. The construction begins with $\mathbb{N}$ (satisfying appropriate axioms) and successively builds $\mathbb{Z}$, $\mathbb{Q}$, and finally what we call $\mathbb{R}$, each time defining the new objects in terms of the previous objects.

These constructions are somewhat formal and dull. In the text, we instead begin with the real numbers and their properties and emphasize techniques of reasoning. *We assume that the real number system $\mathbb{R}$ exists and satisfies the properties in Definitions 1.39–1.41.* These imply all other properties of real numbers, such as those in Propositions 1.43–1.46. For now we treat $\mathbb{N}$ informally; in Chapter 3 we give a formal definition of $\mathbb{N}$ as a subset of $\mathbb{R}$. Whether we begin with $\mathbb{N}$ and define $\mathbb{R}$ as in Appendix A, or begin with $\mathbb{R}$ as in the text, the same results hold. In each case, the real number system satisfies Definitions 1.39–1.41.

**1.39. Definition.** *Field Axioms.* A set $S$ with operations $+$ and $\cdot$ and distinguished elements 0 and 1 with $0 \neq 1$ is a **field** if the following properties hold for all $x, y, z \in S$.

| | | |
|---|---|---|
| A0: $x + y \in S$ | M0: $x \cdot y \in S$ | Closure |
| A1: $(x+y)+z = x+(y+z)$ | M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ | Associativity |
| A2: $x + y = y + x$ | M2: $x \cdot y = y \cdot x$ | Commutativity |
| A3: $x + 0 = x$ | M3: $x \cdot 1 = x$ | Identity |
| A4: given $x$, there is a $w \in S$ such that $x + w = 0$ | M4: for $x \neq 0$, there is a $w \in S$ such that $x \cdot w = 1$ | Inverse |
| | DL: $x \cdot (y + z) = x \cdot y + x \cdot z$ | Distributive Law |

The operations $+$ and $\cdot$ are called **addition** and **multiplication**. The elements 0 and 1 are the **additive identity element** and the **multiplicative identity element**.

It follows from these axioms that the additive inverse and multiplicative inverse (of a nonzero $x$) are unique. The additive inverse of $x$ is the **negative** of $x$, written as $-x$. To define **subtraction** of $y$ from $x$, we let $x - y = x + (-y)$. The multiplicative inverse of $x$ is the **reciprocal** of $x$, written as $x^{-1}$. The element 0 has no reciprocal. To define **division** of $x$ by $y$ when $y \neq 0$, we let $x/y = x \cdot (y^{-1})$. We write $x \cdot y$ as $xy$ and $x \cdot x$ as $x^2$. We use parentheses where helpful to clarify the order of operations.

**1.40. Definition.** *Order Axioms.* A **positive set** in a field $F$ is a set $P \subseteq F$ such that for $x, y \in F$,

| | |
|---|---|
| P1: $x, y \in P$ implies $x + y \in P$ | Closure under Addition |
| P2: $x, y \in P$ implies $xy \in P$ | Closure under Multiplication |
| P3: $x \in F$ implies exactly one of $x = 0,\ x \in P,\ -x \in P$ | Trichotomy |

An **ordered field** is a field with a positive set $P$. In an ordered field, we define $x < y$ to mean $y - x \in P$. The relations $\leq$, $<$, and $\geq$ have analogous definitions in terms of $P$.

Note that $P = \{x \in F : x > 0\}$. Another phrasing of trichotomy is that each ordered pair $(x, y)$ satisfies exactly one of $x < y$, $x = y$, $x > y$.

If $S \subseteq F$, then $\beta \in F$ is an **upper bound** for $S$ if $x \leq \beta$ for all $x \in S$.

**1.41. Definition.** *Completeness Axiom.* An ordered field $F$ is **complete** if every nonempty subset of $F$ that has an upper bound in $F$ has a least upper bound in $F$.

Until Part IV, we do not need the Completeness Axiom for $\mathbb{R}$, except to be aware that it ensures the existence of square roots of positive real numbers. The axioms in Definitions 1.39–1.40 imply that arithmetic has its familiar properties. We list some of these below. We agree to assume all these properties of numbers. Note that $\mathbb{Q}$ also is an ordered field, and thus the properties listed below also hold for arithmetic in $\mathbb{Q}$. The set $\mathbb{Z}$ of integers satisfies all the field and order axioms except the existence of multiplicative inverses.

**1.42. Proposition.** *Arithmetic in* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$. Each of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is closed under addition and multiplication, $\mathbb{Z}$ and $\mathbb{Q}$ are closed under subtraction, and the set of nonzero numbers in $\mathbb{Q}$ is closed under division.

The next four propositions state properties of an ordered field $F$. All statements apply for each choice of $x, y, z, u, v \in F$.

**1.43. Proposition.** *Elementary consequences of the field axioms.*

a) $x + z = y + z$ implies $x = y$
b) $x \cdot 0 = 0$
c) $(-x)y = -(xy)$
d) $-x = (-1)x$
e) $(-x)(-y) = xy$
f) $xz = yz$ and $z \neq 0$ imply $x = y$
g) $xy = 0$ implies $x = 0$ or $y = 0$

**1.44. Proposition.** *Properties of an ordered field.*

| | |
|---|---|
| O1: $x \leq x$ | Reflexive Property |
| O2: $x \leq y$ and $y \leq x$ imply $x = y$ | Antisymmetric Property |
| O3: $x \leq y$ and $y \leq z$ imply $x \leq z$ | Transitive Property |
| O4: at least one of $x \leq y$ and $y \leq x$ holds | Total Ordering Property |

**1.45. Proposition.** *More properties of an ordered field.*

| | |
|---|---|
| F1: $x \leq y$ implies $x + z \leq y + z$ | Additive Order Law |
| F2: $x \leq y$ and $0 \leq z$ imply $xz \leq yz$ | Multiplicative Order Law |
| F3: $x \leq y$ and $u \leq v$ imply $x + u \leq y + v$ | Addition of Inequalities |
| F4: $0 \leq x \leq y$ and $0 \leq u \leq v$ imply $xu \leq yv$ | Multiplication of Inequalities |

**1.46. Proposition.** *Still more properties of an ordered field.*

a) $x \leq y$ implies $-y \leq -x$
b) $x \leq y$ and $z \leq 0$ imply $yz \leq xz$
c) $0 \leq x$ and $0 \leq y$ imply $0 \leq xy$
d) $0 \leq x^2$
e) $0 < 1$
f) $0 < x$ implies $0 < x^{-1}$
g) $0 < x < y$ implies $0 < y^{-1} < x^{-1}$

Properties (a) and (b) of Proposition 1.46 tell us that multiplying an inequality by a negative number requires reversing the inequality.

There are other equivalent formulations of the axioms. Hence it is not important to remember which are axioms and which are consequences in our list; we take the entire list as our starting point.

## HOW TO APPROACH PROBLEMS

In this chapter, we discussed mathematical objects; in Chapter 2, we will discuss mathematical statements. As a warmup, the exercises here begin with translations between mathematics and English. Most problems in this chapter demand precise understanding of language but require little calculation. Computations become just one part of a mathematical tool box. We will develop more tools in later chapters.

We mention some simple strategies to help students get started on unfamiliar problems. Although they seem self-evident, these strategies will be helpful throughout the book; keep them in mind.

1) Understand the problem and approach it logically.

2) Substitutions allow us to simplify expressions or to introduce useful new expressions.

3) When there are only a few possibilities, analysis by cases may help eliminate all possibilities except the desired conclusion.

4) Check whether answers are reasonable.

**Understanding problems.**

Exercises 1–26 provide practice in translating words into mathematical concepts. One must also understand the definitions of mathematical concepts used (see Exercise 18 and beyond).

To gain an understanding of a problem, one sometimes analyzes a special case. For example, one could analyze the Penny Problem for small values of $n$ to discover a pattern and then extend the argument that works for special values to prove the desired result in general.

Distinguish what is given or known from what is to be shown. Understand what is needed to obtain the desired conclusion from the known information. Break a complicated problem into simpler steps.

**Substitution.**

As change for a dollar we might receive four quarters, or we might exchange four quarters for a dollar. Mathematical equations also can be read two ways. Substitution is the process of replacing a mathematical expression by a more convenient expression with the same value. Substitution has many facets; we substitute when we apply a general formula in a special case, when we wish to simplify, or when we eliminate variables.

**1.47. Example.** Since $x^2-y^2=(x+y)(x-y)$ holds for all $x$ and $y$, we may replace one side of this equality by the other. For example, to multiply 598 by 602 mentally, think

$$(600-2)(600+2)=600^2-2^2=360000-4=359996.$$

Here it was convenient to replace $(x+y)(x-y)$ with $x^2-y^2$. On the other hand, to find the roots to an equation we might replace $x^2-y^2$ with its factored form $(x+y)(x-y)$. ∎

Substitution can sometimes be used to eliminate an irrelevant variable. In Application 1.6, we wrote $r=2d/(t_1+t_2)$, but we wanted to express $r$ in terms of $r_1$ and $r_2$. We substituted expressions for $t_1$ and $t_2$ in terms of the desired variables, and the dependence on $d$ canceled out.

**1.48. Example.** In Exercise 31, the hint for part (a) suggests using the inequality $2tu \le t^2+u^2$ from Proposition 1.4. In fact, we use six instances of this inequality, substituting various quantities for $t$ and $u$, to obtain the inequality $4xyzw \le x^4+y^4+z^4+w^4$.

To obtain $3abc \le a^3+b^3+c^3$ from this, we again use substitution. We want to reduce the expression from four variables to three variables with symmetric roles; letting $w=(xyz)^{1/3}$ accomplishes this in a useful way. After this, substituting $a, b, c$ for appropriate expressions in $x, y, z$ yields the desired identity.

The substitutions in the last step are natural, but finding the substitution $w=(xyz)^{1/3}$ is more difficult. Experience, intelligent guessing, and trial and error all help decide what substitutions might be useful. ∎

**Analysis by cases.**

The form of an answer may depend on the values of the variables; the cases in Exercise 37 arise in this way. Alternatively, deductions we want to make might be valid only for restricted choices of the variables.

**1.49. Example.** We seek all integer solutions to $a^2b>2a$. In other words, we seek an explicit description of the set $\{(a,b)\in\mathbb{Z}^2\colon a^2b>2a\}$. We rewrite the inequality as $a(ab-2)>0$. This inequality holds if and only if both factors have the same sign. Thus we are led to the two cases below.

1) $a>0$ and $ab>2$

2) $a<0$ and $ab<2$

The first case contains all integer pairs in the first quadrant except $(1,1)$, $(1,2)$, and $(2,1)$. The second case contains all integer pairs in the second quadrant and also $(-1,-1)$ from the third quadrant. The answer is the union of the sets of solutions in the two cases. ∎

**1.50. Example.** Analysis by cases may arise when studying the absolute value function. For real numbers $x, y$, we can prove the Triangle Inequality $|x+y|\le|x|+|y|$ (Proposition 1.3) in this way.

When $x, y$ are both nonnegative, both sides equal $x + y$. When $x, y$ are both nonpositive, both sides equal $-x - y$. When $x, y$ have opposite signs, we may assume that $x > 0 > y$. The inequality then holds because

$$|x + y| = \max\{x + y, -x - y\} < x - y = |x| + |y|. \qquad \blacksquare$$

Finding a way to avoid analysis by cases can lead to deeper understanding of a problem or method. Many questions related to absolute value and distance are best understood by studying squared distance.

The use of sets facilitates analysis by cases. The word "or" corresponds to union of sets, and the word "and" corresponds to intersection.

**Checking answers.**

Checking answers can expose errors in reasoning. When finding a general answer, one should check it in special cases. When a formula describes areas or lengths (as in Exercise 19), the resulting values must be nonnegative. We recommend checking answers for reasonableness; how to do this depends on the problem.

## EXERCISES

Words like "determine", "show", "obtain", or "construct" include a request for justification; these are very similar to "prove". Answers to problems in this book should be given full explanations. Explanations include *sentences*; reasoning cannot be explained without words.

Easier problems are indicated by "(−)", harder problems by "(+)". Those designated "(!)" are particularly interesting or instructive.

**1.1.** (−) We have many tables and many chairs. Let $t$ be the number of tables, and let $c$ be the number of chairs. Write down an inequality that means "We have at least four times as many chairs as tables."

**1.2.** (−) Fill in the blanks. The equation $x^2 + bx + c = 0$ has exactly one solution when ______________, and it has no solutions when ______________.

**1.3.** (−) Given that $x + y = 100$, what is the maximum value of $xy$?

**1.4.** (−) Explain why the square has the largest area among all rectangles with a given perimeter.

**1.5.** (−) Consider the Celsius (C) and Fahrenheit (F) temperature scales.

| C | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| F | 32 | 41 | 50 | 59 | 68 | 77 | 86 |

Express the sentence "The temperature was 10° C and increased by 20° C" using the Fahrenheit scale.

**1.6.** (−) At a given moment, let $f$ and $c$ be the values of the temperature on the Fahrenheit and Celsius scales, respectively. These values are related by $f = (9/5)c + 32$. At what temperatures do the following events occur?

a) The Fahrenheit and Celsius values of the temperature are equal.
b) The Fahrenheit value is the negative of the Celsius value.
c) The Fahrenheit value is twice the Celsius value.

**1.7.** (−) The statement below is not always true for $x, y \in \mathbb{R}$. Give an example where it is false, and add a hypothesis on $y$ that makes it a true statement.

"If $x$ and $y$ are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."

**1.8.** (!) In the morning section of a calculus course, 2 of the 9 women and 2 of the 10 men receive the grade of A. In the afternoon section, 6 of the 9 women and 9 of the 14 men receive A. Verify that, in each section, a higher proportion of women than of men receive A, but that, in the combined course, a lower proportion of women than of men receive A. Explain! (See Exercises 9.19–9.20 for related exercises and Example 9.20 for a real-world example.)

**1.9.** (−) If a stock declines 20% in one year and rises 23% in the next, is there a net profit? What if it goes up 20% in the first year and down 18% in the next?

**1.10.** (−) On July 4, 1995, the *New York Times* reported that the nation's universities were awarding 25% more Ph.D. degrees than the economy could absorb. The headline concluded that there was a 1 in 4 chance of underemployment. Here "underemployment" means having no job or having a job not requiring the Ph.D. degree. What should the correct statement of the odds have been?

**1.11.** (−) A store offers a 15% promotional discount for its grand opening. The clerk believes that the law requires the discount to be applied first and then the tax computed on the resulting amount. A customer argues that the discount should be applied to the total after the 5% sales tax is added, expecting to save more money that way. Does it matter? Explain.

**1.12.** (−) A store offers an "installment plan" option, with no interest to be paid. There are 13 monthly payments, with the first being a "down payment" that is half the size of the others, so payment is completed one year after purchase. If a customer buys a $1000 stereo, what are the payments under this plan?

**1.13.** (−) Let $A$ be the set of integers expressible as $2k - 1$ for some $k \in \mathbb{Z}$. Let $B$ be the set of integers expressible as $2k + 1$ for some $k \in \mathbb{Z}$. Prove that $A = B$.

**1.14.** (−) Let $a, b, c, d$ be real numbers with $a < b < c < d$. Express the set $[a, b] \cup [c, d]$ as the difference of two sets.

**1.15.** (−) For what conditions on sets $A$ and $B$ does $A - B = B - A$ hold?

**1.16.** (−) Starting with a single pile of 5 pennies, determine what happens when the operation of Application 1.14 is applied repeatedly. Determine what happens when the initial configuration is a single pile of 6 pennies.

**1.17.** (−) What are the domain and the image of the absolute value function?

**1.18.** (−) Determine which real numbers exceed their reciprocals by exactly 1.

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**1.19.** What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers $p$ and $a$, under what conditions does there exist a rectangular carpet with perimeter $p$ and area $a$?

**1.20.** Suppose that $r$ and $s$ are distinct real solutions of the equation $ax^2+bx+c=0$. In terms of $a, b, c$, obtain formulas for $r+s$ and $rs$.

**1.21.** Let $a, b, c$ be real numbers with $a \neq 0$. Find the flaw in the following "proof" that $-b/2a$ is a solution to $ax^2+bx+c=0$.

> Let $x$ and $y$ be solutions to the equation. Subtracting $ay^2+by+c=0$ from $ax^2+bx+c=0$ yields $a(x^2-y^2)+b(x-y)=0$, which we rewrite as $a(x+y)(x-y)+b(x-y)=0$. Hence $a(x+y)+b=0$, and thus $x+y=-b/a$. Since $x$ and $y$ can be any solutions, we can apply this computation letting $y$ have the same value as $x$. With $y=x$, we obtain $2x=-b/a$, or $x=-b/(2a)$.

**1.22.** We have two identical glasses. Glass 1 contains $x$ ounces of wine; glass 2 contains $x$ ounces of water ($x \geq 1$). We remove 1 ounce of wine from glass 1 and add it to glass 2. The wine and water in glass 2 mix uniformly. We now remove 1 ounce of liquid from glass 2 and add it to glass 1. Prove that the amount of water in glass 1 is now the same as the amount of wine in glass 2.

**1.23.** A digital 12-hour clock is defective: the reading for hours is always correct, but the reading for minutes always equals the reading for hours. Determine the minimum number of minutes between possible correct readings of the clock.

**1.24.** Three people register for a hotel room; the desk clerk charges them \$30. The manager returns and says this was an overcharge, instructing the clerk to return \$5. The clerk takes five \$1 bills, but pockets \$2 as a tip and returns only \$1 to each guest. Of the original \$30 payment, each guest actually paid \$9, and \$2 went to the attendant. What happened to the "missing" dollar?

**1.25.** A census taker interviews a woman in a house. "Who lives here?" he asks. "My husband and I and my three daughters," she replies. "What are the ages of your daughters?" "The product of their ages is 36 and the sum of their ages is the house number." The census taker looks at the house number, thinks, and says, "You haven't given me enough information to determine the ages." "Oh, you're right," she replies, "Let me also say that my eldest daughter is asleep upstairs." "Ah! Thank you very much!" What are the ages of the daughters? (The problem requires "reasonable" mathematical interpretations of its words.)

**1.26.** (+) Two mail carriers meet on their routes and have a conversation. A: "I know you have three sons. How old are they?" B: "If you take their ages, expressed in years, and multiply those numbers, the result will equal your age." A: "But that's not enough to tell me the answer!" B: "The sum of these three numbers equals the number of windows in that building." A: "Hmm [pause]. But it's still not enough!" B: "My middle son is red-haired." A: "Ah, now it's clear!" How old are the sons? (Hint: The ambiguity at the earlier stages is needed to determine the solution for the full conversation.) (G. P. Klimov)

**1.27.** Determine the set of real solutions to $|x/(x+1)| \leq 1$.

**1.28.** (!) *Application of the AGM Inequality.*
a) Use Proposition 1.4 to prove that $x(c-x)$ is maximized when $x=c/2$.
b) For $a>0$, use part (a) to find the value of $y$ maximizing $y(c-ay)$.

**1.29.** Let $x, y, z$ be nonnegative real numbers such that $y+z \geq 2$. Prove that $(x+y+z)^2 \geq 4x+4yz$. Determine when equality holds.

**1.30.** (!) Let $x, y, u, v$ be real numbers.
a) Prove that $(xu+yv)^2 \leq (x^2+y^2)(u^2+v^2)$.
b) Determine precisely when equality holds in part (a).

**1.31.** (+) *Extensions of the AGM Inequality.*
a) Prove that $4xyzw \leq x^4+y^4+z^4+w^4$ for real numbers $x, y, z, w$. (Hint: Use the inequality $2tu \leq t^2+u^2$ repeatedly.)
b) Prove that $3abc \leq a^3+b^3+c^3$ for nonnegative $a, b, c$. (Hint: In the inequality of part (a), set $w$ equal to the cube root of $xyz$.)

**1.32.** (!) Assuming only arithmetic (not the quadratic formula or calculus), prove that $\{x \in \mathbb{R}\colon x^2-2x-3<0\} = \{x \in \mathbb{R}\colon -1<x<3\}$.

**1.33.** Let $S=\{(x,y) \in \mathbb{N}^2\colon (2-x)(2+y) > 2(y-x)\}$. Prove that $S=T$, where $T=\{(1,1),(1,2),(1,3),(2,1),(3,1)\}$.

**1.34.** Let $S=\{(x,y) \in \mathbb{R}^2 : (1-x)(1-y) \geq 1-x-y\}$. Give a simple description of $S$ involving the signs of $x$ and $y$.

**1.35.** (!) Determine the set of ordered pairs $(x, y)$ of nonzero real numbers such that $x/y+y/x \geq 2$.

**1.36.** Let $S=[3]\times[3]$ (the Cartesian product of $\{1,2,3\}$ with itself). Let $T$ be the set of ordered pairs $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ such that $0 \leq 3x+y-4 \leq 8$. Prove that $S \subseteq T$. Does equality hold?

**1.37.** Determine the set of solutions to the general quadratic inequality $ax^2+bx+c \leq 0$. Express the answer using linear inequalities or intervals. (Use the quadratic formula; the complete solution involves many cases.)

**1.38.** Let $S=\{x \in \mathbb{R}\colon x(x-1)(x-2)(x-3)<0\}$. Let $T$ be the interval $(0,1)$, and let $U$ be the interval $(2,3)$. Obtain a simple set equality relating $S, T, U$.

**1.39.** (!) Given $n \in \mathbb{N}$, let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1<a_2<\cdots<a_n$. Express $\{x \in \mathbb{R}\colon (x-a_1)(x-a_2)\cdots(x-a_n)<0\}$ using the notation for intervals. (For convenience, use $(-\infty, a)$ to denote $\{x \in \mathbb{R}\colon x<a\}$.)

**1.40.** Let $A$ and $B$ be sets. Explain why the two sets $(A-B)\cup(B-A)$ and $(A\cup B)-(A\cap B)$ must be equal. Check this when $A$ is the set of states in the United States whose names begin with a vowel and $B$ is the set of states in the United States whose names have at most six letters.

**1.41.** (−) Let $A, B, C$ be sets. Explain the relationships below. Use the definitions of set operations and containment, with Venn diagrams to guide the argument.
a) $A \subseteq A\cup B$, and $A\cap B \subseteq A$.
b) $A-B \subseteq A$.
c) $A\cap B = B\cap A$, and $A\cup B = B\cup A$.
d) $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$.
e) $A\cap(B\cap C) = (A\cap B)\cap C$.
f) $A\cup(B\cup C) = (A\cup B)\cup C$.

**1.42.** Let $A=\{\text{January, February}, \ldots, \text{December}\}$. Given $x \in A$, let $f(x)$ be the number of days in $x$. Does $f$ define a function from $A$ to $\mathbb{N}$?

**1.43.** (−) Let $S=\{(x,y) \in \mathbb{R}^2\colon 2x+5y \leq 10\}$. Graph $S$. Explain how the answer changes when the constraint is $2x+5y<10$.

**1.44.** (!) Let $S=\{(x,y) \in \mathbb{R}^2\colon x^2+y^2 \leq 100\}$. Let $T=\{(x,y) \in \mathbb{R}^2\colon x+y \leq 14\}$.
a) Graph $S\cap T$.
b) Count the points in $S\cap T$ whose coordinates are both integers.

**1.45.** (−) Determine whether the rules below define functions from $\mathbb{R}$ to $\mathbb{R}$.

a) $f(x) = |x-1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$.
b) $f(x) = |x-1|$ if $x < 2$ and $f(x) = |x| - 1$ if $x > -1$.
c) $f(x) = ((x+3)^2 - 9)/x$ if $x \neq 0$ and $f(x) = 6$ if $x = 0$.
d) $f(x) = ((x+3)^2 - 9)/x$ if $x > 0$ and $f(x) = x + 6$ if $x < 7$.
e) $f(x) = \sqrt{x^2}$ if $x \geq 2$, $f(x) = x$ if $0 \leq x \leq 4$, and $f(x) = -x$ for $x < 0$.

**1.46.** Determine the images of the functions $f\colon \mathbb{R} \to \mathbb{R}$ defined as follows:

a) $f(x) = x^2/(1+x^2)$. b) $f(x) = x/(1+|x|)$.

**1.47.** Let $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by $f(a,b) = (a+1)(a+2b)/2$.

a) Show that the image of $f$ is contained in $\mathbb{N}$.
b) (+) Determine exactly which natural numbers are in the image of $f$. (Hint: Formulate a hypothesis by trying values.)

**1.48.** Give several descriptions of the function $f\colon [0,1] \to [0,1]$ defined by $f(x) = 1 - x$. Compare with Example 1.29.

**1.49.** (!) Let $f$ and $g$ be functions from $\mathbb{R}$ to $\mathbb{R}$. For the sum and product of $f$ and $g$ (see Definition 1.25), determine which statements below are true. If true, provide a proof; if false, provide a counterexample.

a) If $f$ and $g$ are bounded, then $f + g$ is bounded.
b) If $f$ and $g$ are bounded, then $fg$ is bounded.
c) If $f + g$ is bounded, then $f$ and $g$ are bounded.
d) If $fg$ is bounded, then $f$ and $g$ are bounded.
e) If both $f + g$ and $fg$ are bounded, then $f$ and $g$ are bounded.

**1.50.** (!) For $S$ in the domain of a function $f$, let $f(S) = \{f(x) : x \in S\}$. Let $C$ and $D$ be subsets of the domain of $f$.

a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.
b) Give an example where equality does not hold in part (a).

**1.51.** When $f\colon A \to B$ and $S \subseteq B$, we define $I_f(S) = \{x \in A : f(x) \in S\}$. Let $X$ and $Y$ be subsets of $B$.

a) Determine whether $I_f(X \cup Y)$ must equal $I_f(X) \cup I_f(Y)$.
b) Determine whether $I_f(X \cap Y)$ must equal $I_f(X) \cap I_f(Y)$.

(Hint: Explore this using the schematic representation described in Remark 1.22.)

**1.52.** Let $M$ and $N$ be nonnegative real numbers. Suppose that $|x + y| \leq M$ and $|xy| \leq N$. Determine the maximum possible value of $x$ as a function of $M$ and $N$.

**1.53.** Solve Application 1.38 by using inequalities rather than graphs.

**1.54.** (!) Let $S = \{(x,y) \in \mathbb{R}^2 : y \leq x \text{ and } x + 3y \geq 8 \text{ and } x \leq 8\}$.

a) Graph the set $S$.
b) Find the minimum value of $x + y$ such that $(x,y) \in S$. (Hint: On the graph from part (a), sketch the level sets of the function $f$ defined by $f(x,y) = x + y$.)

**1.55.** (+) Let $\mathbf{F}$ be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for $\mathbf{F}$.

**1.56.** (+) Is there a field with exactly four elements? Is there a field with exactly six elements?

# Chapter 2

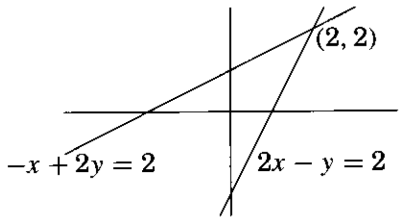
# Language and Proofs

Understanding mathematical reasoning requires familiarity with the precise meaning of words like "every", "some", "not", "and", "or", etc.; these arise often in analyzing mathematical problems. Relevant aspects of language include word order, quantifiers, logical statements, and logical symbols. With these, we can discuss elementary techniques of proof.

## TWO THEOREMS ABOUT EQUATIONS

We begin with two problems that illustrate both the need for careful use of language and the variety of techniques in proofs.

**2.1. Definition.** A **linear equation** in two variables $x$ and $y$ is an equation $ax + by = r$, where the coefficients $a, b$ and the constant $r$ are real numbers. A **line** in $\mathbb{R}^2$ is the set of pairs $(x,y)$ satisfying a linear equation whose coefficients $a$ and $b$ are not both 0.



Geometric intuition suggests three possibilities for a pair of linear equations in two variables. If each equation describes a line, then the lines may intersect in one point, may be parallel, or may be identical. The equations then have one, none, or infinitely many common solutions, respectively. We can analyze this without relying on geometric intuition, because we have defined "line" using only arithmetic of real numbers.