Math 23b	Quiz 1 Solutions	February 13, 2008

Your name: Dmitry Kleinbock

Write clearly, using complete sentences. If necessary, use additional sheets of paper with your name on them.

1. [10 points] Let f and g be functions from $\mathbf{R}$ to $\mathbf{R}$ with the property that for all x in $\mathbf{R}$, $f(x) \geq 0$ and $g(x) \geq 0$. Show that if $f+g$ is bounded then both f and g are bounded.

   Assume that $f+g$ is bounded. Then there exists a real number M such that for every $x \in \mathbf{R}$, $|f(x)+g(x)| \leq M$. So for every real number x, since $f(x) \geq 0$ and $g(x) \geq 0$, we have
   $$|f(x)| = f(x) \leq f(x) + g(x) = |f(x)+g(x)| \leq M.$$
   Therefore f is bounded; the same argument applies to g in place of f.

2. [10 points] Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \dfrac{1}{1+x^2}$. Find the image of f and prove that your answer is correct.

   The image of f is the interval $(0,1] = \{\, x \in \mathbf{R} : 0 < x \leq 1 \,\}$.
   To prove this, let A be the image of f. First we prove that $A \subseteq (0,1]$. Let y be any element of A. Then there exists $x \in \mathbf{R}$ such that $y = f(x) = 1/(1+x^2)$. Since $1+x^2 \geq 1 > 0$, we have $y = 1/(1+x^2) > 0$. Also, since $1+x^2 \geq 1$ we have $y = 1/(1+x^2) \leq 1$. Therefore $y \in (0,1]$.
   Next we prove that $(0,1] \subseteq A$. Let y be any element of $(0,1]$. Then $\frac{1}{y} \geq 1$, so $\frac{1}{y} - 1$ is nonnegative and therefore has a square root. Let $x = \sqrt{\frac{1}{y} - 1}$. Then $x^2 = \frac{1}{y} - 1$ so $y = 1/(1+x^2) = f(x)$. Therefore $y \in A$.
   We have shown that $A \subseteq (0,1]$ and $(0,1] \subseteq A$, so $A = (0,1]$.

3. [5 points] Suppose that you are going to prove the following statement by contradiction. How would the proof start? (You do not have to finish the proof!)

   If $|a| < 2$ *then for all* $x \in \mathbf{R}$, $x^2 + ax + 1 \geq 0$.

   (In this statement, a is a real number.)

   Assume that $|a| < 2$ and that there exists $x \in \mathbf{R}$ such that $x^2 + ax + 1 < 0$.