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Write clearly, using complete sentences. If necessary, use additional sheets of paper with your name on them. Remember that every induction proof requires a basis step and an induction hypothesis. In induction problems, you must decide which form of induction to use.

1. [15 points] Given any positive integer greater than 1, we obtain a new number by the following procedure:

If the number is odd we add 1, and if the number is even we divide it by 2.

If the new number is greater than 1, then we repeat this procedure, over and over, stopping only if we reach 1. For example, suppose we start with 5. Since 5 is odd, we add 1 to get 6. Since 6 is even, we divide it by 2, to get 3. Since 3 is odd, we add 1 to get 4. We divide 4 by 2 to get 2, then we divide 2 by 2 to get 1 and then we stop. Prove by induction that if we start with an integer greater than 1 then we will eventually reach 1. (Remember that the point of this problem is to write a correct induction proof!)

Let  $P(n)$  be the statement that if we start with  $n$  then we will eventually reach 1. Then  $P(2)$  is true, since if we start with 2, we divide it by 2 to get 1. Now suppose that  $k \geq 3$  and that  $P(i)$  is true for all  $i$  with  $2 \leq i < k$ . We must show that  $P(k)$  is true. If we start with  $k$  and  $k$  is even, then we first divide by 2 to get  $k/2$ . Since  $2 \leq k/2 < k$ ,  $P(k/2)$  is true by the induction hypothesis, so we will eventually reach 1. If we start with  $k$  and  $k$  is odd, then we first add 1 to get  $k+1$ , which is even, and then we divide by 2 to get  $(k+1)/2$ . Since  $2 \leq (k+1)/2 < k$ ,  $P((k+1)/2)$  is true by the induction hypothesis, so again we will eventually reach 1. Therefore, in either case,  $P(k)$  is true. So by the principle of strong induction,  $P(n)$  is true for all  $n \geq 2$ .

2. [10 points] A sequence  $\langle u \rangle$  is defined by the recurrence  $u_0 = 2$ ,  $u_1 = 3$ ,  $u_2 = 5$ , and  $u_n = 7u_{n-2} - 6u_{n-3}$  for  $n \geq 3$ . Prove by induction that for every nonnegative integer  $n$ ,  $u_n = 2^n + 1$ .

First we check that  $u_0 = 2 = 2^0 + 1$ ,  $u_1 = 3 = 2^1 + 1$ , and  $u_2 = 5 = 2^2 + 1$ . Now suppose that  $k \geq 3$  and that for all  $i$  with  $0 \leq i < k$ , we have  $u_i = 2^i + 1$ . In particular, since  $k \geq 3$ , this formula holds for  $i = k-2$  and  $i = k-3$ . Then we have

$$\begin{aligned} u_k &= 7u_{k-2} - 6u_{k-3} \\ &= 7(2^{k-2} + 1) - 6(2^{k-3} + 1) \\ &= 14 \cdot 2^{k-3} + 7 - 6 \cdot 2^{k-3} - 6 \\ &= 8 \cdot 2^{k-3} + 1 \\ &= 2^k + 1, \end{aligned}$$

so the formula to be proved holds for  $i = k$ . So by strong induction,  $u_n = 2^n + 1$  for all  $n \geq 0$ .