Math 23b Midterm Solutions and Extra Problems April 2, 2008

1. [12 points] Let x_1, x_2, x_3, x_4, and x_5 be real numbers such that $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 100$.

(a) Prove that $x_5 \leq 20$.

(b) Prove that $x_1 + x_2 \geq 40$.

(a) Suppose $x_5 > 20$; then $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5 > 20$, so $x_i > 20$ for all $i = 1, \ldots, 5$. Adding, we get $x_1 + x_2 + x_3 + x_4 + x_5 > 5 \cdot 20 = 100$, a contradiction.

It is also easy to come up with a direct proof: $100 = x_1 + x_2 + x_3 + x_4 + x_5 \geq x_1 + x_2 + x_3 + 2x_5 \geq x_1 + x_2 + 3x_5 \geq x_1 + 4x_5 \geq 5x_5$ (each of the steps uses the fact thast $x_5 \leq x_i$ for all $i = 1, \ldots, 4$), therefore $x_5 \leq 20$.

(b) From (a) we know that $x_5 \leq 20$; therefore $x_1 + x_2 + x_3 + x_4 = 100 - x_5 \geq 80$. Since $x_1 \geq x_3$ and $x_2 \geq x_4$, we have $x_1 + x_2 + x_1 + x_2 \geq 80$; dividing by 2 we arrive an the conclusion $x_1 + x_2 \geq 40$. [Exercise: write down a proof by contradiction.]

Problem 1+. Let $y_1, \ldots, y_{10}$ be 10 real numbers such that $y_1 \cdot \ldots \cdot y_{10} = 1024$. Suppose that $y_1 \leq y_i$ and $y_{10} \geq y_i$ for all $i = 1, \ldots, 10$ (that is, y_1 is a minimum and y_{10} is a maximum of those numbers). Prove or find a counterexample to the following assertions:

(a) $y_1 \leq 2$.

(b) $y_{10} \geq 2$.

If you choose to write a proof, do it in two different ways, i.e. directly and by contradiction, following the pattern of the solution to Problem 1 written above.

2. [12 points] Define a function $h : \mathbf{N} \to \mathbf{N}$ by $h(1) = 1$ and for $n > 1$, $h(n) = h(n-1) + h(\lfloor \frac{n}{2} \rfloor)$. (Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Prove by induction that for all $n \geq 1$, $h(n) < 2^n$. (You must decide which type of induction to use.)

Basis: $h(1) = 1 < 2^1 = 2$. Now take $k > 1$ and assume that $h(i) < 2^i$ for all $1 \leq i < k$ (that is, use strong induction). Consider $h(k) = h(k-1) + h(\lfloor \frac{k}{2} \rfloor)$. By the induction assumption, $h(k-1) < 2^{k-1}$ and $h(\lfloor \frac{k}{2} \rfloor) < 2^{\lfloor \frac{k}{2} \rfloor}$. However $\lfloor \frac{k}{2} \rfloor \leq \frac{k}{2} \leq k - 1$ for all $k \geq 2$, therefore $2^{\lfloor \frac{k}{2} \rfloor} \leq 2^{k-1}$, and we can write $h(k) = h(k-1) + h(\lfloor \frac{k}{2} \rfloor) < 2^{k-1} + 2^{k-1} = 2^k$.

Problem 2+. Define a function $q : \mathbf{N} \to \mathbf{N}$ by $q(1) = 1$, $q(2) = 1$, and for $n > 2$,

$$q(n) = q(n-1) + 2q(n-2)\,.$$

Prove by (strong) induction that for all $n \geq 1$, $2^{n-2} \leq q(n) \leq 2^{n-1}$. [Note: you need a proof for each of the inequalities. Alternatively, you can try to guess and then prove by induction an exact formula for $q(n)$.]

3. [12 points] Let $f : A \to B$ and $g : B \to C$ be functions. Show that if $g \circ f$ is injective and f is surjective, then g is injective.

To prove that g is injective, we must prove that for any b_1 and b_2 in B, if $g(b_1) = g(b_2)$ then $b_1 = b_2$. To do this, suppose that b_1 and b_2 are in B and that $g(b_1) = g(b_2)$. Since f is surjective, there exist a_1 and a_2 in A such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then $g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2))$. Since $g \circ f$ is injective, this implies that $a_1 = a_2$. Thus $b_1 = f(a_1) = f(a_2) = b_2$.

Problem 3+. Again, let $f : A \to B$ and $g : B \to C$ be functions. Show that:

(a) if $g \circ f$ is injective, then f is injective (without any additional assumptions on g);

(b) if $g \circ f$ is surjective, then g is surjective (without any additional assumptions on f);

(c) if $g \circ f$ is surjective and f is injective, then f is surjective.

Also, (by constructing counterexamples) show that the following statements are in general not true:

(d) if $g \circ f$ is injective, then g is injective (that is, the additional assumption "f is surjective" in Problem 3 cannot be omitted);

(e) if $g \circ f$ is surjective, then f is surjective (that is, the additional assumption "f is injective" in (c) cannot be omitted).

4. [14 points] Consider the set

$$F = \{\, S \mid S \text{ is a finite subset of } \mathbf{N} \,\}$$

of all finite subsets of $\mathbf{N}$. Prove that F is countable. [Hint: One way to do this is to express F as a countable union of finite (or perhaps countable) sets. There are also other ways.]

Let $A_0 = \{\emptyset\}$, and for $n = 1, \ldots,$ let A_n be the set of subsets of $\mathbf{N}$ in which the largest element is n. Then $F = A_0 \cup A_1 \cup A_2 \cup \cdots$ and each A_n is finite. Thus F is expressed as a countable union of finite sets, so F is countable. Put another way, we can list all the finite subsets of $\mathbf{N}$ by starting with the empty set, then listing all the subsets whose largest element is 1, then the subsets whose largest element is 2, and so on.

Another way to do this problem is to arrange finite subsets of $\mathbf{N}$ according to their size. That is, write $F = B_0 \cup B_1 \cup B_2 \cup \cdots$, where $B_0 = \{\emptyset\}$ consists of all the subsets of size 0, B_1 consists of all the subsets of size 1 (so it can be identified with $\mathbf{N}$), B_2 consists of all the subsets of size 2 (so it can be identified with a subset of $\mathbf{N} \times \mathbf{N}$, that is, with the set of pairs (a, b) of natural numbers $a < b$), and so on. This way each B_n can be identified with a subset of the Cartesian product of n copies of $\mathbf{N}$, and hence each of them is finite or countable (in fact, all except B_0 are countable). So F is expressed as a countable union of countable sets, and therefore is countable.

Problem 4+. Consider the following function f from F as above to $\mathbf{N}$: $f(\emptyset) = 1$ and

$$f(\{n_1, n_2, \ldots, n_k\}) = 1 + 2^{n_1 - 1} + 2^{n_2 - 1} + \cdots + 2^{n_k - 1}\,.$$

Prove that it is a bijection (this gives another solution of Problem 4). [Hint: this amounts to saying that every positive integer can be expressed uniquely as a sum of distinct powers of 2, see Theorem 4.6 – a basis for the binary number system. You may use this theorem in your proof. Note that the problem has two separate parts: proving the injectivity and the surjectivity of f.]

Problem 5+. Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... be the sequence of all prime numbers. Consider the following function g from F as above to $\mathbf{N}$: $f(\emptyset) = 1$ and, given a nonempty finite subset $S = \{n_1, n_2, \ldots, n_k\}$ of $\mathbf{N}$ with $n_1 < n_2 < \cdots < n_k$, define $g(S) = p_1^{n_1} p_2^{n_2} \cdots p_k^{s_k}$. That is, all one-elements sets correspond to powers of 2, all two-element sets to products of powers of 2 and 3 and so on. Prove that it is an injection (this gives another solution of Problem 4). [Hint: you will need to use the Fundametal Theorem of Arithmetic, that is, the unique factorization of positive integers into primes – Theorem 6.9.] Additional problem: show that g is not a surjection, and describe its image explicitly.