

Below you will find the list of some of the topics which may appear on the exam, and some suggestions for preparation. References are made to the corresponding pages/statements in the textbook. The final exam will cover Chapters 1–4, 13–14 and the first two sections of Chapter 15.

1. Review all the homework assignments, quizzes, and the midterm exam.
2. Know the definitions of
  - (a) basic operations with sets (union, intersection, complement, containment and equality of sets, Cartesian product, power set)
  - (b) a function, injective, surjective, and bijective functions, composition of functions, inverse of a function, a graph of a function, inverse images and level sets
  - (c) bounded subsets of  $\mathbf{R}$  and bounded real-valued functions, increasing/decreasing real-valued functions
  - (d) logical operations and quantifiers ( $\forall$ ,  $\exists$ ,  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\Leftarrow$ ,  $\Leftrightarrow$ )
  - (e) finite, infinite, countable sets, the size of a finite set, two sets having the same cardinality
  - (f) upper/lower and least upper/greatest lower bounds of subsets of  $\mathbf{R}$
  - (g) limits of sequences, convergent sequences
  - (h) the canonical  $k$ -ary expansion of a real number
  - (i) Cauchy sequences, a subsequence of a sequence
  - (j) infinite series, partial sums of a series, convergent/divergent series, geometric series, harmonic series
  - (k) limits of functions, continuous functions
3. Be familiar with the following:
  - (a) the arithmetic-geometric mean inequality, the triangle inequality (4–5)
  - (b) properties of the real number system (16–17; you do not have to remember which properties are axioms and which are propositions)
  - (c) direct, contrapositive and “by contradiction” proof techniques (35–39)
  - (d) principles of induction (51–57) and strong induction (63–64), well-ordering property of  $\mathbf{N}$  and the method of descent (64–66)
  - (e) the fact that two finite sets have the same number of elements (size) if and only if there exists a bijection between them
  - (f) the fact that if two functions are injective/surjective/bijective, then so is their composition (Proposition 4.30, proved in class)
  - (g) that two finite sets have the same number of elements (size) if and only if there exists a bijection between them (Proposition 4.37, proved in class)
  - (h) examples of countable sets discussed in Chapter 4 and in class (89–90)

- (i) the Completeness Axiom (256–257), the Archimedean Property (258)
- (j) the Monotone Convergence Theorem (261)
- (k) uncountability of the set of real numbers (266)
- (l) arithmetic properties of limits, the Squeeze Theorem for limits (272–274)
- (m) the Bolzano-Weierstrass Theorem and the method of bisection, Cauchy Convergence Criterion (276–279)
- (n) a formula for the sum of geometric series (280), the divergence of harmonic series (282)
- (o) comparison/ratio/root tests for series (282–284)
- (p) the equivalence of definitions of limit and sequential limit of functions (294–296) and its consequences (296–298)
- (q) the Intermediate Value Theorem and its applications (299–300)

And here are some sample problems:

4. Using the triangle inequality, prove that if  $x$  and  $y$  are real numbers then

$$|x| - |y| \leq |x - y| \leq |x| + |y|.$$

5. Using logical connectives, quantifiers, and the usual symbols of mathematics (but no words) write down an expression without negation signs that means “the sequence  $\langle a \rangle$  does not converge.” For example,

$$(\forall \epsilon > 0) (\exists N \in \mathbf{N}) (\forall n \in \mathbf{N}) (\forall L \in \mathbf{R}) (n \geq N \Rightarrow |a_n - L| \geq \epsilon)$$

has the right form, though it is not correct.

6. Let  $A, B$  be two nonempty sets and let  $f : A \rightarrow B$  be a surjective function. Which of the following must be true? Find counterexamples to statements which are not true.
- (a) For every  $x \in A$  there is at least one  $y \in B$  such that  $f(x) = y$ .
  - (b) For every  $x \in A$  there is at most one  $y \in B$  such that  $f(x) = y$ .
  - (c) For every  $y \in B$  there is at least one  $x \in A$  such that  $f(x) = y$ .
  - (d) For every  $y \in B$  there is at most one  $x \in A$  such that  $f(x) = y$ .
7. A sequence  $\langle a \rangle$  is defined by  $a_1 = 1$ ,  $a_2 = 2$ , and  $a_{n+1} = a_n - a_{n-1}$  for  $n > 2$ . Prove by induction that for all  $n \geq 1$ ,  $a_n + a_{n+3} = 0$ .
8. Define a sequence  $\langle b \rangle$  by  $b_1 = 1$  and  $b_{n+1} = \sqrt{2b_n}$  for  $n > 1$ . Prove by induction that for all  $n \geq 1$ ,  $1 \leq b_n \leq b_{n+1} \leq 2$ . Does this sequence converge? what is its limit?

9. Let  $A$  be a nonempty finite set, and let  $F$  be the set of all functions from  $A$  to  $\mathbf{N}$ . Prove that  $F$  is countable.
10. Let  $A, B$  be two nonempty subsets of  $\mathbf{R}$  such that  $A \subset B$  and  $B$  is bounded above. Prove that  $A$  is also bounded above and  $\sup A \leq \sup B$ .
11. Prove that the following statement is not true: For any sequence  $\langle a \rangle$ , if  $a_{2n} \rightarrow L$  then  $a_n \rightarrow L$ . Is the converse true?
12. Let  $\langle a \rangle$  and  $\langle b \rangle$  be infinite sequences of real numbers such that  $a_n \rightarrow L$ ,  $b_n \rightarrow M$ , and  $a_n < b_n$  for all  $n$ . Prove that  $L \leq M$ . [Hint: Proof by contradiction.]
13. Using only the definition of a Cauchy sequence, prove that the sequence  $\langle b \rangle$  given by  $b_n = (n^2 + 1)/n^2$  is Cauchy.
14. Does there exist
  - (a) a Cauchy sequence that is not monotone?
  - (b) a monotone sequence that is not Cauchy?
  - (c) a Cauchy sequence with a divergent subsequence?
  - (d) an unbounded sequence containing a sequence which is Cauchy?Give examples if yes, provide a justification if no.
15. Using only the definition of a limit prove that if  $\langle a \rangle$  converges to  $L$  and  $b_n = a_{n+1}$  for all  $n$  then  $\langle b \rangle$  converges to  $L$ .
16. Let  $\langle a \rangle$  be a sequence with  $a_n \geq 0$  for all  $n$  and such that  $\sum_{n=1}^{\infty} a_n$  converges. Prove that  $\sum_{n=1}^{\infty} a_n^2$  also converges.
17. Using only the definition of continuity, prove that  $f(x) = \sqrt{1+x}$  is continuous at  $x = 3$ .
18. Let  $f$  and  $g$  be continuous on  $(-1, 1)$  and suppose that  $f(x) < g(x)$  for all  $0 < x < 1$ . Prove that  $f(0) \leq g(0)$ . Show that  $f(0) < g(0)$  might be false.