4. The triangle inequality states that for any real numbers $\alpha$ and $\beta$, $|\alpha + \beta| \leq |\alpha| + |\beta|$. For the first inequality we have $|x| = |(x - y) + y| \leq |x - y| + |y|$, so $|x - y| \geq |x| - |y|$. For the second inequality we have $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$.

5. $(\forall L \in \mathbb{R}) (\exists \epsilon > 0) (\exists N \in \mathbb{N})(\exists n \in \mathbb{N}) (n \geq N \land |a_n - L| > \epsilon)$

6. (a) and (b) are part of the definition of $f$ being a function, (c) is exactly the surjectivity of $f$, and (d) is its injectivity, so any injective and non-surjective function will constitute a counterexample.

7. To check the basis ($n = 1$) we need to add $a_1$ and $a_4$, so need to compute $a_3 = 2 - 1 = 1$ and $a_4 = 1 - 2 = -1 = -a_1$. In fact, since the recursion involves not one but two previous terms we will need strong induction, and therefore will need to also check $n = 2$: $a_5 = -1 - 1 = -2 = -a_2$. Now the induction step: if $a_{n-1} + a_{n+2} = 0$ and $a_n + a_{n+3} = 0$, then $a_{n+1} + a_{n+4} = (a_n - a_{n-1}) + (a_{n+3} - a_{n+2}) = a_n + a_{n+3} - (a_{n-1} + a_{n+2}) = 0$.

8. The basis: $1 \leq 1 \leq \sqrt{2} \leq 2$. Now suppose that $1 \leq b_n \leq b_{n+1} \leq 2$. Then $b_{n+1} = \sqrt{2b_n} \geq \sqrt{2} > 1$, and also $b_{n+2} = \sqrt{2b_{n+1}} \leq \sqrt{2 \cdot 2} = 2$. Finally, $b_{n+1} \leq b_{n+2}$ is equivalent to $b_{n+1} \leq \sqrt{2b_{n+1}} \iff \sqrt{b_{n+1}} \leq \sqrt{2}$, which also follows from the induction assumption.

Then the fact that $L = \lim_{n \to \infty} b_n$ exists and is equal to the supremum of the set $\{b_n\}$ follows from the Monotone Convergence Theorem.

However to identify the limit one needs some additional argument. Informally, since for large $n$ both $b_n$ and $b_{n+1}$ are very close to $L$, and $b_{n+1} = \sqrt{2b_n}$, this $L$ must satisfy $L = \sqrt{2L}$, which implies $L = 2$. Here is a rigorous argument: for any $\epsilon > 0$ there exist $n$ such that $b_n > L(1 - \epsilon)$ and $b_{n+1} < L(1 + \epsilon)$. Since $b_{n+1} = \sqrt{2b_n}$, we get $\sqrt{2L(1-\epsilon)} < L(1+\epsilon)$, which translates into $L > \frac{1-\epsilon}{2(1+\epsilon)}L$. Since $\epsilon$ is arbitrary, it follows that $L \geq 2$, and $L \leq 2$ is clear from the first part of the problem.

9. The set of functions from a set $A$ of size $k$ to $\mathbb{N}$ is in one-to-one correspondence with the set of ordered $k$-tuples of natural numbers, that is, with the set $\mathbb{N}^k$ (the Cartesian product of $k$ copies of $\mathbb{N}$). Indeed, if $A = \{a_1, \ldots, a_k\}$, any function $A \to \mathbb{N}$ corresponds to the $k$-tuple $(f(a_1), \ldots, f(a_k))$, and a $k$-tuple $(n_1, \ldots, n_k)$ defines the function $f$ by $f(a_i) = n_i$ for each $i$. The fact that $\mathbb{N}^k$ is countable can be easily proved by induction and using Theorem 4.4.

10. Let $b = \sup B$, so that $x \leq b$ for every $x \in B$. Since any element of $A$ is also an element of $B$, it follows that $x \leq b$ for every $x \in A$, i.e. $A$ is also bounded from above by $b$. Now suppose that $a = \sup A > b$. Then there must exist an element $a'$ of $A$ which is greater than $b$ (otherwise $a$ would not be the least upper bound of $A$); but $a'$ is also an element of $B$, which contradicts to the fact that $b$ is an upper bound for $B$.

11. The sequence $\langle a \rangle$ given by $a_n = (-1)^n$ is a counterexample, since $a_{2n} \to 1$ but $\langle a \rangle$ does not converge. The converse is true since any subsequence of a convergent sequence converges.
12. Assume that $a_n \to L$, $b_n \to M$, and $a_n < b_n$ for all $n$, but $L > M$. Now let $\epsilon = (L-M)/2$. Since $L > M$, we have $\epsilon > 0$. Therefore there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - L| < \epsilon$ and there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|b_n - M| < \epsilon$. Now let $N = \max(N_1, N_2)$. Then for $n \geq N$ we have $|a_n - L| < \epsilon$ and $|b_n - M| < \epsilon$, so

$$b_n < M + \epsilon = M + (L - M)/2 = (L + M)/2 = L - (L - M)/2 = L - \epsilon < a_n,$$

and this contradicts $a_n < b_n$.

13. Given $\epsilon > 0$, let $N$ be a natural number greater than $2/\epsilon$. This implies that $1/N < \epsilon/2$. So if $m \geq N$ then $|1/m^2| = 1/m^2 \leq 1/m \leq 1/N < \epsilon/2$. Now suppose that $m, n \geq N$. Then

$$|b_m - b_n| = \left| \left(1 + \frac{1}{m^2}\right) - \left(1 + \frac{1}{n^2}\right) \right| = \left| \frac{1}{m^2} - \frac{1}{n^2} \right| \leq \frac{1}{m^2} + \frac{1}{n^2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $(b)$ is a Cauchy sequence.

14. (a) Yes – any convergent sequence is Cauchy, so for example $a_n = (-1)^n/n$ works.

(b) Yes: any monotone unbounded sequence, such as $a_n = n$, gives an example.

(c) No: any Cauchy sequence converges, and any subsequence of a convergent sequence converges.

(d) Yes: an unbounded sequence can contain a convergent subsequence; for example $a_n = n$ if $n$ is odd, and 0 if $n$ is even.

15. Suppose that $(a)$ converges to $L$ and that $b_n = a_{n+1}$ for all $n$. Let $\epsilon$ be a positive real number. Since $a_n \to L$, there exists a natural number $N$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. Then $n \geq N$ implies $n + 1 \geq N$ so $|b_n - L| = |a_{n+1} - L| < \epsilon$. Thus $(b)$ converges to $L$.

16. Since $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$ (Lemma 14.27), in particular $a_n < 1$ for large enough $n$, i.e. there exists $N \in \mathbb{N}$ such that $a_n < 1$ for $n > N$. Then $a_n^2 < a_n$ for $n > N$, therefore, by the Comparison Test (Proposition 14.29), $\sum_{n=N+1}^{\infty} a_n^2$ converges, and thus so does $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n^2$.

17. We need to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x - 3| < \delta$ implies that $|f(x) - f(3)| = |\sqrt{1+x} - 2| < \epsilon$. Let us rewrite this as

$$2 - \epsilon < \sqrt{1+x} < 2 + \epsilon \iff (2 - \epsilon)^2 < 1 + x < (2 + \epsilon)^2 \iff 3 - 2\epsilon + \epsilon^2 < 1 < x < (2 + \epsilon)^2 - 1 = 3 + 2\epsilon + \epsilon^2.$$

If $\epsilon < 2$ (which we can assume) then the left (respectively, right) hand side of the last inequality is strictly less (respectively, greater) than 3. Now the proof proceeds as follows: given $0 < \epsilon < 2$, choose $\delta$ smaller than $2\epsilon - \epsilon^2$; then as long as $|x - 3| < \delta$, we would have $3 - 2\epsilon + \epsilon^2 < x < 3 + 2\epsilon + \epsilon^2$, which is equivalent to $|f(x) - f(3)| < \epsilon$.

18. Consider $h = g - f$, this is a function continuous at $0 < x < 1$. Suppose that the conclusion fails to hold, that is, $h(0) < 0$; then, by Lemma 15.18, there exists $\delta > 0$ such that $|x| < \delta$ implies that $h(x) = g(x) - f(x) < 0$, which contradicts to the assumption of the problem. For the second part, consider $f(x) = x^2$ and $g(x) = x$. 