Math 23b Review Problems Solutions May 2008

4. The triangle inequality states that for any real numbers α and β, $|\alpha+\beta| \le |\alpha| + |\beta|$. For the first inequality we have $|x| = |(x-y)+y| \le |x-y| + |y|$, so $|x-y| \ge |x| - |y|$. For the second inequality we have $|x-y| = |x+(-y)| \le |x| + |-y| = |x| + |y|$.

5. $$(\forall\, L \in \mathbf{R})\,(\exists\, \epsilon > 0)\,(\forall\, N \in \mathbf{N})(\exists\, n \in \mathbf{N})\big(n \ge N \wedge |a_n - L| > \epsilon\big)$$

6. (a) and (b) are part of the definition of f being a function, (c) is exactly the surjectivity of f, and (d) is its injectivity, so any injective and non-surjective function will constitute a counterexample.

7. To check the basis ($n = 1$) we need to add a_1 and a_4, so need to compute $a_3 = 2 - 1 = 1$ and $a_4 = 1 - 2 = -1 = -a_1$. In fact, since the recursion involves not one but two previous terms we will need strong induction, and therefore will need to also check $n = 2$: $a_5 = -1 - 1 = -2 = -a_2$. Now the induction step: if $a_{n-1} + a_{n+2} = 0$ and $a_n + a_{n+3} = 0$, then $a_{n+1} + a_{n+4} = (a_n - a_{n-1}) + (a_{n+3} - a_{n+2}) = a_n + a_{n+3} - (a_{n-1} + a_{n+2}) = 0$.

8. The basis: $1 \le 1 \le \sqrt{2} \le 2$. Now suppose that $1 \le b_n \le b_{n+1} \le 2$. Then $b_{n+1} = \sqrt{2b_n} \ge \sqrt{2} > 1$, and also $b_{n+2} = \sqrt{2b_{n+1}} \le \sqrt{2 \cdot 2} = 2$. Finally, $b_{n+1} \le b_{n+2}$ is equivalent to $b_{n+1} \le \sqrt{2b_{n+1}} \Leftrightarrow \sqrt{b_{n+1}} \le \sqrt{2}$, which also follows from the induction assumption.

   Then the fact that $L = \lim_{n\to\infty} b_n$ exists and is equal to the supremum of the set $\{b_n\}$ follows from the Monotone Convergence Theorem.

   However to identify the limit one needs some additional argument. Informally, since for large n both b_n and b_{n+1} are very close to L, and $b_{n+1} = \sqrt{2b_n}$, this L must satisfy $L = \sqrt{2L}$, which implies $L = 2$. Here is a rigorous argument: for any $\epsilon > 0$ there exist n such that $b_n > L(1-\epsilon)$ and $b_{n+1} < L(1+\epsilon)$. Since $b_{n+1} = \sqrt{2b_n}$, we get $\sqrt{2L(1-\epsilon)} < L(1+\epsilon)$, which translates into $L > 2\frac{1-\epsilon}{(1+\epsilon)^2}$. Since ϵ is arbitrary, it follows that $L \ge 2$, and $L \le 2$ is clear from the first part of the problem.

9. The set of functions from a set A of size k to $\mathbf{N}$ is in one-to-one correspondence with the set of ordered k-tuples of natural numbers, that is, with the set $\mathbf{N}^k$ (the Cartesian product of k copies of $\mathbf{N}$). Indeed, if $A = \{a_1, \dots, a_k\}$, any function $A \to \mathbf{N}$ corresponds to the k-tuple $\big(f(a_1), \dots, f(a_k)\big)$, and a k-tuple $(n_1, \dots, n_k)$ defines the function f by $f(a_i) = n_i$ for each i. The fact that $\mathbf{N}^k$ is countable can be easily proved by induction and using Theorem 4.4.

10. Let $b = \sup B$, so that $x \le b$ for every $x \in B$. Since any element of A is also an element of B, it follows that $x \le b$ for every $x \in A$, i.e. A is also bounded from above by b. Now suppose that $a = \sup A > b$. Then there must exist an element a' of A which is greater than b (otherwise a would not be the least upper bound of A); but a' is also an element of B, which contradicts to the fact that b is an upper bound for B.

11. The sequence $\langle a \rangle$ given by $a_n = (-1)^n$ is a counterexample, since $a_{2n} \to 1$ but $\langle a \rangle$ does not converge. The converse is true since any subsequence of a convergent sequence converges.

12. Assume that $a_n \to L$, $b_n \to M$, and $a_n < b_n$ for all n, but $L > M$. Now let $\epsilon = (L-M)/2$. Since $L > M$, we have $\epsilon > 0$. Therefore there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|a_n - L| < \epsilon$ and there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $|b_n - M| < e$. Now let $N = \max(N_1, N_2)$. Then for $n \geq N$ we have $|a_n - L| < \epsilon$ and $|b_n - M| < \epsilon$, so
$$b_n < M + \epsilon = M + (L-M)/2 = (L+M)/2 = L - (L-M)/2 = L - \epsilon < a_n,$$
and this contradicts $a_n < b_n$.

13. Given $\epsilon > 0$, let N be a natural number greater than $2/\epsilon$. This implies that $1/N < \epsilon/2$. So if $m \geq N$ then $|1/m^2| = 1/m^2 \leq 1/m \leq 1/N < \epsilon/2$. Now suppose that $m, n \geq N$. Then
$$|b_m - b_n| = \left|\left(1 + \frac{1}{m^2}\right) - \left(1 + \frac{1}{n^2}\right)\right| = \left|\frac{1}{m^2} - \frac{1}{n^2}\right| \leq \left|\frac{1}{m^2}\right| + \left|\frac{1}{n^2}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Thus $\langle b \rangle$ is a Cauchy sequence.

14. (a) Yes – any convergent sequence is Cauchy, so for example $a_n = (-1)^n/n$ works.
    (b) Yes: any monotone unbounded sequence, such as $a_n = n$, gives an example.
    (c) No: any Cauchy sequence converges, and any subsequence of a convergent sequence converges.
    (d) Yes: an unbounded sequence can contain a convergent subsequence; for example $a_n = n$ if n is odd, and 0 if n is even.

15. Suppose that $\langle a \rangle$ converges to L and that $b_n = a_{n+1}$ for all n. Let ϵ be a positive real number. Since $a_n \to L$, there exists a natural number N such that $n \geq N$ implies $|a_n - L| < \epsilon$. Then $n \geq N$ implies $n + 1 \geq N$ so $|b_n - L| = |a_{n+1} - L| < \epsilon$. Thus $\langle b \rangle$ converges to L.

16. Since $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$ (Lemma 14.27) , in particular $a_n < 1$ for large enough n, i.e. there exists $N \in \mathbf{N}$ such that $a_n < 1$ for $n > N$. Then $a_n^2 < a_n$ for $n > N$, therefore, by the Comparison Test (Proposition 14.29), $\sum_{n=N+1}^{\infty} a_n^2$ converges, and thus so does $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n^2$.

17. We need to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x - 3| < \delta$ implies that $|f(x) - f(3)| = |\sqrt{1+x} - 2| < \epsilon$. Let us rewrite this as
$$\begin{aligned} 2 - \epsilon < \sqrt{1+x} < 2 + \epsilon &\Leftrightarrow (2-\epsilon)^2 < 1 + x < (2+\epsilon)^2 \\ &\Leftrightarrow 3 - 2\epsilon + \epsilon^2 = (2-\epsilon)^2 - 1 < x < (2+\epsilon)^2 - 1 = 3 + 2\epsilon + \epsilon^2\,. \end{aligned}$$
If $\epsilon < 2$ (which we can assume) then the left (respectively, right) hand side of the last inequality is strictly less (respectively, greater) than 3. Now the proof proceeds as follows: given $0 < \epsilon < 2$, choose δ smaller than $2\epsilon - \epsilon^2$; then as long as $|x - 3| < \delta$, we would have $3 - 2\epsilon + \epsilon^2 < x < 3 + 2\epsilon + \epsilon^2$, which is equivalent to $|f(x) - f(3)| < \epsilon$.

18. Consider $h = g - f$, this is a function continuous at $0 < x < 1$. Suppose that the conclusion fails to hold, that is, $h(0) < 0$; then, by Lemma 15.18, there exists $\delta > 0$ such that $|x| < \delta$ implies that $h(x) = g(x) - f(x) < 0$, which contradicts to the assumption of the problem. For the second part, consider $f(x) = x^2$ and $g(x) = x$.