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Math 28a Midterm Exam

1. Determine if the following sets G with the indicated operations $*$ form groups by checking the group axioms. If not, point out which of the axioms fail.

(a) $G =$ the set of all integers $\mathbb{Z}$ with $a * b \overset{\text{def}}{=} a + b + 2$;

$$\left.\begin{aligned} (a*b)*c &= (a+b+2)*c = a+b+c+4 \\ a*(b*c) &= a*(b+c+2) = a+b+c+4 \end{aligned}\right\} \Rightarrow \text{associativity holds}$$

$a*e = a+e+2 = a$ for all $a \Leftrightarrow e = -2$, the identity element

$a*b = a+b+2 = e = -2 \Leftrightarrow b = -4-a$, the inverse to a

$\Rightarrow$ all group axioms are satisfied, G is a group

(b) $G =$ the set $Z_{43} = \{0, \ldots, 42\}$ of all integers modulo 43, with

$$a * b \overset{\text{def}}{=} ab \pmod{43}.$$

Associativity holds, and 1 is the identity element.
However there is no $a \in G$ such that $0*a = 1$;
therefore G is <u>not</u> a group.

2. Let G be a group such that all its elements except e have order 2. Prove that G is Abelian.

Let $a, b \in G$; we need to prove that $ab = ba$

By assumption, $a^2 = b^2 = (ab)^2 = e$

$\Rightarrow abab = e$, multiply by $a = a^{-1}$ on left: $bab = a$

multiply by $b = b^{-1}$ on right: $ab = ba$.

3. Let G be a group and let H be a subset of G defined by

$$H = \{a \in G : (ax)^3 = (xa)^3 \text{ for every } x \in G\}.$$

Prove that H is a subgroup of G.

H contains e, since $(ex)^3 = x^3 = (xe)^3$ for all $x \in G$.

If $a \in H$, need to show that $a^{-1} \in H$, i.e. that

$$(a^{-1}x)^3 = (xa^{-1})^3 \quad \forall x \in G \qquad (*)$$

We know that $(ax)^3 = (xa)^3$ for all $x \in G$; therefore $(ax^{-1})^3 = (x^{-1}a)^3$ for all $x \in G$. Inverting both sides, we get:

$$\left((ax^{-1})^3\right)^{-1} = \left((ax^{-1})^{-1}\right)^3 = (xa^{-1})^3 \; ; \; \left((x^{-1}a)^3\right)^{-1} = \left((x^{-1}a)^{-1}\right)^3 = (a^{-1}x)^3,$$

and $(*)$ follows.

Finally, assume $a, b \in H$, and for every $x \in G$ write

$$(abx)^3 = (a(bx))^3 \underset{\text{since } a \in H}{=} ((bx)a)^3 = (b(xa))^3 \underset{\text{since } b \in H}{=} ((xa)b)^3 = (xab)^3$$

$\Rightarrow ab \in H$, which finishes the proof.

4. Let $G = \langle a \rangle$ be a cyclic group of order 100, and let K, L be subgroups of G defined by

$$K = \{g^{21} : g \in G\}, \quad L = \{g^{22} : g \in G\}.$$

(a) How many elements are there in K and in L? Explain.

(b) Does there exist a 21st root of a in G (that is, $b \in G$ such that $b^{21} = a$)? what about a 22nd root? Justify your answers. [Hint: use part (a).]

(a) Note that $K = \langle a^{21} \rangle$ and $L = \langle a^{22} \rangle$. By one of the theorems in Chapter 4, $K = \langle a^{\gcd(21,100)} \rangle = \langle a \rangle = G$ and $L = \langle a^{\gcd(22,100)} \rangle = \langle a^2 \rangle$; thus $|K| = 100$ and

$$|L| = \frac{100}{\gcd(22,100)} = 50$$

(b) Since $K = G$, a belongs to K, so $a = (a^k)^{21}$ for some k
$\Rightarrow a = b^{21}$ for some $b \in G$

Since $L = \langle a^2 \rangle$ and $a \notin \langle a^2 \rangle$, a 22nd root of a in G does not exist.

5. Consider the following element of S_9, written in the cycle notation:

$$\sigma = (6148)(2345)(12493)(572)$$

(a) Write it as a product of disjoint cycles.

(b) Write it as a product of 2-cycles.

(c) Prove that it is not possible to find $k \in \mathbb{N}$ such that σ^k is a 6-cycle.

(d) Compute σ^{2010}.

(a) $\sigma = (134986)(57)$

(b) For example, $\sigma = (68)(69)(64)(63)(61)(57)$

(c) σ is even, and so is σ^k for any k. However, a 6 cycle is an odd permutation $\Rightarrow$ can't be equal to σ^k

(d) The order of σ is equal to $\mathrm{lcm}(6,2) = 6$, and 2010 is divisible by 6 $\Rightarrow$ $\sigma^{2010} = e$