

Math 311a, Spring 2006, Homework # 1

Ergodic Theory

In the following problems, (X, \mathcal{B}, μ) is a measure space, $\mu(X) = 1$, and φ is a measurable self-map of X .

1. Prove that φ preserves μ (that is, $\mu(\varphi^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$) if and only if the following equivalent conditions hold:

- (a) $\mu(\varphi^{-1}(A)) \leq \mu(A)$ for all $A \in \mathcal{B}$;
- (b) $\mu(\varphi^{-1}(A)) \geq \mu(A)$ for all $A \in \mathcal{B}$;
- (c) $\varphi(A) \in \mathcal{B}$ and $\mu(\varphi(A)) \geq \mu(A)$ for all $A \in \mathcal{B}$.

In the remaining problems, assume that φ preserves μ .

2. Let $f \in L^1(X, \mathcal{B}, \mu)$ be such that $f(\varphi(x)) \leq f(x)$ for μ -a.e. $x \in X$. Prove that $f(\varphi(x)) = f(x)$ for μ -a.e. x (that is, f is essentially φ -invariant).

3. Prove that if $(X, \mathcal{B}, \mu, \varphi)$ is ergodic and f is a nonnegative measurable function on X such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k(x)) < \infty$$

for μ -a.e. x , then $f \in L^1$.

4. Let U_φ be the unitary operator on $L^2(X, \mathcal{B}, \mu)$ defined by $U_\varphi f = f \circ \varphi$. Suppose that $(X, \mathcal{B}, \mu, \varphi)$ is ergodic and μ is non-atomic (that is, $\mu(\{x\}) = 0$ for all $x \in X$). Then every point on the unit circle in \mathbb{C} is an *approximate eigenvalue* of U_φ , that is, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ there exists a sequence $\{f_n\} \subset L^2$ with $\|f_n\|_2 = 1$ for all n and $\|U_\varphi f_n - \lambda f_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Consequently, the spectrum of U_φ is the entire unit circle.

5. Prove that φ is mixing if and only if

$$\lim_{n \rightarrow \infty} \mu(\varphi^{-n}(A) \cap A) \rightarrow \mu(A)^2$$

for any $A \in \mathcal{B}$ (that is, it is enough to consider only “diagonal” matrix coefficients). [Hint: consider the space generated by the constant functions together with all the shifts $1_A \circ \varphi^n$ of 1_A , and for every $B \in \mathcal{B}$ project 1_B onto this space and onto its orthogonal complement.]

6. Show that there is no such concept as “uniform mixing”: if

$$\mu(\varphi^{-n}(A) \cap B) \rightarrow \mu(A)\mu(B)$$

as $n \rightarrow \infty$ *uniformly* for all $A, B \in \mathcal{B}$, then every set in \mathcal{B} has measure 0 or 1.