

## Math 311a, Spring 2006, Homework # 2

### Groups, measures and lattices

In the first two problems,  $G$  is a locally compact  $\sigma$ -compact separable topological group and  $\mu$  is a left-invariant Haar measure on  $G$ .

1. Prove that  $\mu(G) < \infty$  iff  $G$  is compact, and  $\mu(\{e\}) > 0$  iff  $G$  is discrete.
2. Let  $\Delta$  be the *modular function* of  $G$ , defined by  $\mu(Ag^{-1}) = \Delta(g)\mu(A)$  for all  $g \in G$  and measurable  $A \subset G$ .
  - (a) Prove that  $\Delta$  is a continuous homomorphism  $G \rightarrow \mathbb{R}_+$ .
  - (b) Show that the measure  $\nu$  given by  $d\nu(g) = \Delta(g) d\mu(g)$  is right-invariant.
  - (c) The measure  $A \mapsto \mu(A^{-1})$  is also right-invariant, hence is equal to  $c\nu$  for some  $c > 0$ , where  $\nu$  is as in (b). Find  $c$ .

In the next two problems,  $G = \mathrm{SL}_n(\mathbb{R})$ .

3. Prove that  $G$  is simple; in fact, any its normal subgroup has at most two elements.
4. Let  $N_-, A, N_+$  be the subgroups of  $G$  consisting of strictly lower-triangular, diagonal, strictly upper-triangular matrices respectively. Prove that their product  $N_-AN_+$  is a proper open subset of  $G$ . Identify  $G \setminus N_-AN_+$  when  $n = 2$ .
5. Prove that the following are equivalent for a discrete subgroup  $\Lambda$  of  $\mathbb{R}^n$ :
  - (a)  $\mathbb{R}^n/\Lambda$  is compact;
  - (b)  $\mathbb{R}^n/\Lambda$  has finite volume (that is,  $\Lambda$  is a lattice in  $\mathbb{R}^n$ );
  - (c)  $\mathbb{R}\Lambda = \mathbb{R}^n$ ;
  - (d) the rank of  $\Lambda$  as a  $\mathbb{Z}$ -module is equal to  $n$ ;
  - (e)  $\Lambda = g\mathbb{Z}^n$  for some  $g \in \mathrm{GL}_n(\mathbb{R})$ .
6. Let  $\Lambda$  be a lattice in  $\mathbb{R}^n$ . Prove that the following are equivalent for  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Lambda$ :
  - (a)  $\mathbf{v}_1, \dots, \mathbf{v}_n$  generate  $\Lambda$  as a  $\mathbb{Z}$ -module;
  - (b) the matrix with columns  $\mathbf{v}_1, \dots, \mathbf{v}_n$  has determinant equal to the volume of  $\mathbb{R}^n/\Lambda$ ;
  - (c)  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly independent, and the closed  $n$ -simplex with vertices  $0, \mathbf{v}_1, \dots, \mathbf{v}_n$  contains no other points of  $\Lambda$  except for its vertices.