Math 311a, Spring 2006, Homework # 3

Diophantine approximation

For $Y \in M_{m,n}(\mathbb{R})$ (the set of $m \times n$ matrices with real entries) define the *Diophantine exponent* $\omega(Y)$ of Y by

$$\omega(Y) \stackrel{\text{def}}{=} \sup \left\{ v \,\middle|\, \begin{array}{c} \mathrm{dist}(Y\mathbf{q}, \mathbb{Z}^m) < \|\mathbf{q}\|^{-v} \\ \text{for } \infty \text{ many } \mathbf{q} \in \mathbb{Z}^n \end{array} \right\},$$

where 'dist' is induced by some norm on $\mathbb{R}^m$. (A reminder: $\omega(Y) \geq n/m$ for all Y, and $= n/m$ for Lebesgue-a.e. Y.)

1. Prove that $\omega(Y)$ in the above definition does not depend on the choices of norms on $\mathbb{R}^m$ and $\mathbb{R}^n$.

2. Prove that

$$\begin{aligned} \omega(Y) &= \limsup_{\mathbf{q}\to\infty,\, \mathbf{q}\in\mathbb{Z}^n} \frac{-\log\big(\mathrm{dist}(Y\mathbf{q}, \mathbb{Z}^m)\big)}{\log \|\mathbf{q}\|} \\ &= \limsup_{T\to\infty} \frac{-\log\big(\min\{\mathrm{dist}(Y\mathbf{q}, \mathbb{Z}^m) \mid \mathbf{q} \in \mathbb{Z}^n,\, \|\mathbf{q}\| \leq T\}\big)}{\log T} \end{aligned}$$

3. Prove that $\omega(Y) = \omega(AY+B) = \omega(YC+B)$ for amy $A \in \mathrm{GL}_m(\mathbb{Q})$, $C \in \mathrm{GL}_n(\mathbb{Q})$, $B \in M_{m,n}(\mathbb{Q})$. [Hint: first work with matrices with integer coefficients.]

4. Let $a, b \in \mathbb{R}$ be given. Prove that for any $x \in \mathbb{R}$ one has

$$\omega\big(\begin{pmatrix} x & ax+b \end{pmatrix}\big) \geq \omega\left(\begin{pmatrix} a \\ b \end{pmatrix}\right).$$

Note that here $\begin{pmatrix} x & ax+b \end{pmatrix} \in M_{1,2}(\mathbb{R})$ is a row matrix and $\begin{pmatrix} a \\ b \end{pmatrix} \in M_{2,1}(\mathbb{R})$ is a column matrix.

Now for $x \in \mathbb{R}$ and $n \in \mathbb{N}$ define $\omega_n(x) \stackrel{\text{def}}{=} \omega\big(\begin{pmatrix} x & x^2 & \dots & x^n \end{pmatrix}\big)$, where $\begin{pmatrix} x & x^2 & \dots & x^n \end{pmatrix} \in M_{1,n}(\mathbb{R})$ is understood to be a row matrix.

5. Prove that

$$\begin{aligned} \omega_n(x) &= \limsup_{P\to\infty,\, P\in\mathbb{Z}_n[X]} \frac{-\log|P(x)|}{\log H(P)} \\ &= \limsup_{T\to\infty} \frac{-\log\big(\min\big\{|P(x)| \;\big|\; P \in \mathbb{Z}_n[X],\, H(P) \leq T\big\}\big)}{\log T}, \end{aligned}$$

where $\mathbb{Z}_n[X]$ stands for the set of polynomials of degree $\leq n$ with integer coefficients, and $H(P)$ is the *height* of P, that is, the maximum of absolute values of its coefficients.

(OVER)

6. Prove that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $\omega_n(x) = \omega_n\left(\frac{ax+b}{cx+d}\right)$, where $a, b, c, d \in \mathbb{Q}$ with $ad - bc \neq 0$.

7. Use the previous problem and Lebesgue's Density Theorem to conclude that for any n, $\omega_n(\cdot)$ is constant on a set of full Lebesgue measure.

The last three problems constitute a sketch of proof of the '$n = 2$' case of Mahler's Conjecture, following a 1958 paper of Friedrich Kasch.

8. Let $P(X) = aX^2 + bX + c \in \mathbb{Z}_2[X]$ be a polynomial with distinct roots $\alpha, \beta \in \mathbb{C}$, and let $\mathrm{Disc}(P) \stackrel{\text{def}}{=} b^2 - 4ac$ denote the discriminant of P. Prove that for any complex number z one has

$$\min\big(|z-\alpha|, |z-\beta|\big) \leq \frac{2|P(z)|}{\sqrt{|\mathrm{Disc}(P)|}}\,.$$

9. Let s be a real number with $0 < s < 1$. Prove that there exists a positive constant c, depending only on s, such that for any positive integer T the inequality

$$\sum |\mathrm{Disc}(P)|^{-s} \leq cT^{2(1-s)}$$

holds, where the summation is taken over all $P \in \mathbb{Z}_2[X]$ of height T with non-zero discriminant.

10. Combine Problem 7, the '$s = 1/2$' case of Problem 8, and the Borel-Cantelli Lemma to conclude that $\omega_2(x) = 2$ for Lebesgue-a.e. $x \in \mathbb{R}$.