Math 326a, Fall 2006, Problem Set # 1

Invariants and Semi-invariants

0. Recall your favorite theorem whose proof involves using some sort of an invariant. State this theorem and describe its proof.

1. Four grasshoppers sit at four vertices of a square. Every second one of the grasshoppers jumps over another one and lands at the symmetric point (that is, if a grasshopper jumps from point A over B and lands at C, then vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are equal).
Prove that:

(a) no three of them will ever sit on a straight line parallel to one of the sides of the original square (more generally – on any straight line);

(b) the four of them will never form a square bigger than the original one (more generally – a parallelogram with area bigger than the area of the original one).

2. On a board the 100 numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$ are written down. Every second we choose two numbers a and b from those on the board, erase them and write the number $a + b + ab$. This operation is performed 99 times until there is just one number. What is this number? Find it and prove that it does not depend on the sequence of choices.

3. Initially there is a pile of 2006 chips on a table. We are allowed to throw away one chip and divide the pile into two (not necessarily equal) piles. Then the same can be done with any pile containing more than two chips, and so on. Is it possible to get only the piles consisting of three chips?

4. 23 people, each with integral weight, decide to play soccer, separating into 2 teams of 11 players each, plus a referee. To keep things fair, the teams chosen must have equal total weight. It turns out that no matter who is chosen to be the referee, this can always be done. Prove that the 23 people must all have the same weight.

5. The vertices of an n-gon are labeled by real numbers. Let a, b, c, d be four successive labels. If $(a - d)(b - c) < 0$, then it is allowed to switch b with c. Prove that the switching operation can be performed at most finitely many times.

6. (Kontsevich 1981) Decompose the upper-right quadrant into squares with integer vertices, assign to each its lower-left corner as a label, and shade the squares labeled by $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, $(1, 1)$, $(0, 2)$. Some of the squares of the quadrant are occupied by chips. A position may be transformed to another position according to the following rule: if there is a chip at (x, y), and squares $(x+1, y)$ and $(x, y+1)$ are both free, it is possible to remove the chip and replace it with chips at both of these free squares. The goal is to have all the shaded squares free of chips. Is it possible to reach this goal if (a) there is a chip on each of the six shaded squares? (b) there is only one chip at $(0, 0)$?