Math 326a, Fall 2006, Problem Set # 3

Dani's Conjecture

Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be a linear transformation. For $\mathbf{v} \in V$ and a subspace $W \subset V$ define

$$A(\mathbf{v}, W) \overset{\text{def}}{=} \{n \in \mathbb{N} \mid T^n(\mathbf{v}) \in W\}$$

(the *attendance set* of $\mathbf{v}$ in W). In [D], S.G. Dani asked the following question: how big can be the sets $A(\mathbf{v}, W)$ for various choices of $\mathbf{v}$ and W? Here are two examples/problems:

Problem 1. If W contains a nonzero subspace invariant under T^r for some $r \in \mathbb{N}$, then for any $j \in \mathbb{N}$ there exists $\mathbf{v} \in V$ such that $A(\mathbf{v}, W)$ contains the infinite arithmetic progression $\{j + kr \mid k \in \mathbb{N}\}$. Conversely, if $A(\mathbf{v}, W)$ contains some infinite arithmetic progression, or even an arithmetic progression of length greater or equal to $\dim(W)$, then some nonzero subspace of W is invariant under some power of T.

Problem 2. On the other hand, if T is a unipotent transformation (that is, 1 is its only eigenvalue), then for any $\mathbf{v} \in V$ whose T-orbit is not wholly contained in W, the cardinality of $A(\mathbf{v}, W)$ is less than $\dim(V)$.

The question asked by Dani is roughly as follows: is the most general situation a combination of the above two cases? Specifically, he states

Conjecture 1 (Dani's Conjecture). *Given V and T as above, there exists $m = m(T) \in \mathbb{N}$ such that for any $\mathbf{v} \in V$ and any subspace $W \subset V$,*

$$\text{if } A(\mathbf{v}, W) \text{ has at least } m \text{ elements, it contains an infinite arithmetic progression.} \tag{1m}$$

He also writes that it is plausible that the following stronger version is true:

Conjecture 2 (Strengthening of Dani's Conjecture). *For any $d \in \mathbb{N}$ there exists $m = m_d \in \mathbb{N}$ such that for V of dimension d, any $\mathbf{v} \in V$, any $T : V \to V$ and any subspace $W \subset V$, (1m) holds.*

Both of the above statements seem to be too hard at the moment. So let us also consider

Conjecture 3 (Weakening of Dani's Conjecture). *Given V, T, $\mathbf{v}$ and W as above,*

$$\text{if } A(\mathbf{v}, W) \text{ is infinite, it contains an infinite arithmetic progression.} \tag{1∞}$$

We remark that by Szemeredi's Theorem, see e.g. [F], any counterexample $A(\mathbf{v}, W)$ to Conjecture 3 must have upper density zero. Furthermore, the following can be obtained using the quantitative version of Szemeredi's Theorem due to Gowers [G]:

Theorem 0. *Given V and T as above and $\delta > 0$, there exists $k \in \mathbb{N}$ such that the following holds: if $\mathbf{v} \in V$ and $W \subset V$ are such that the set $A(\mathbf{v}, W) \cap \{1, \dots, k\}$ has at least δk elements, then $A(\mathbf{v}, W)$ contains an infinite arithmetic progression.*

Problem 3. Each of Conjectures 1, 2, 3 can be reduced to the case when T is nonsingular (this is how Conjectures 1 and 2 were actually stated by Dani).

Problem 4. For two subspaces $U, W \subset V$ define

$$A(U, W) \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid T^n(U) \subset W\}.$$

Show that Conjectures 1, 2, 3 imply their analogues where $\mathbf{v} \in V$ is replaced by a subspace $U \subset V$.

Problem 5. Conjecture 2 holds for $d \leq 2$, with $m_1 = 1$ and $m_2 = 2$.

Thus the first nontrivial case of the above conjectures is $\dim(V) = 3$.

Problem 6. If $\dim(W) = 0$ (resp. 1) and $A(\mathbf{v}, W)$ is nonempty (resp. has at least 2 elements), then it contains an infinite arithmetic progression.

Thus to understand the case $\dim(V) = 3$ one needs to let W be a plane in $\mathbb{R}^3$, say

$$W = \{(x, y, z) \mid z = 0\}. \tag{2}$$

In this case, in view of Problem 1, to prove Conjecture 3 it is enough to show that whenever $A(\mathbf{v}, W)$ is infinite, it must contain an arithmetic progression of length 2.

Problem 7. For each $k = 0, 1, 2, 3, 4$ construct explicitly a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{v} \in \mathbb{R}^3$ such that for this T and W as in (2), $A(\mathbf{v}, W)$ has exactly k elements.

Problem 8. Prove that, for V of arbitrary dimension, whenever $\dim(W) = 2$ and $\mathbf{v}$ is such that $\{1, 2, 4, 5\} \subset A(\mathbf{v}, W)$, it also follows that $3 \in A(\mathbf{v}, W)$, and hence $A(\mathbf{v}, W) = \mathbb{N}$. In fact, the same proof shows: whenever $\dim(W) = 2$ and $A(\mathbf{v}, W)$ contains a *parallelogram*, that is, a set of the form $\{k, k + a, k + b, k + a + b\}$, it contains an arithmetic progression of length 2, and hence an infinite one.

Dani was able to prove his conjecture under an additional condition. His proof is based on the following

Lemma 1. *Let $f(t) = \sum_{i=1}^{r} c_i e^{\alpha_i t}$, where $c_i, \alpha_i \in \mathbb{R}$. If f is not identically zero, then the number of zeroes of f is at most $r - 1$.*

Problem 9. Prove Lemma 1 (e.g. by induction and using Rolle's Theorem).

Theorem 1. *Let $T : V \to V$ be a linear transformation of $\mathbb{R}^d$ diagonalizable over $\mathbb{R}$ with positive eigenvalues. Then for any $\mathbf{v} \in \mathbb{R}^d$ and $W \subset \mathbb{R}^d$, the set $A(\mathbf{v}, W)$ either is equal to $\mathbb{N}$ or has at most $d - 1$ elements.*

Problem 10. Derive Theorem 1 from Lemma 1.

Problem 11. Show that Theorem 1 implies

Theorem 2. *Let $T : V \to V$ be a linear transformation of $\mathbb{R}^d$ diagonalizable over $\mathbb{R}$. Then for any $\mathbf{v} \in \mathbb{R}^d$ and $W \subset \mathbb{R}^d$, either $A(\mathbf{v}, W)$ contains all even or all odd natural numbers, or it has at most $2(d-1)$ elements.*

Problem 12. Generalize Lemma 1 to the class of functions of the form $f(t) = \sum_{i=1}^{r} Q_i(t)e^{\alpha_i t}$, where Q_i are polynomials.

Problem 13. Using the previous problem, show that the assumptions on T in Theorem 1 (resp. Theorem 2) can be replaced by "all eigenvalues of T are positive (resp. real)", with the same conclusions.

Problem 14. Theorem 2 in its generalized form as in Problem 13 implies Conjecture 1 (but not Conjecture 2!) in the case when all eigenvalues of T are roots of real numbers, namely $z \in \mathbb{C}$ such that $z^n \in \mathbb{R}$ for some $n \in \mathbb{N}$.

The above discussion shows that to prove Dani's Conjecture for $\dim(V) = 3$ it remains to analyze the case of T having one real and two complex conjugate eigenvalues of the form $e^{2\pi i\theta}$ with $\theta \notin \mathbb{Q}$.

Problem 15. Investigate this case. Maybe Dani's Conjecture is not true after all? then this would be a case to try to build a counterexample. For example, note that the proof of Theorem 2, as well as its generalized form as in Problem 13, produces a stronger result: if T_t is a one-parameter group of linear transformations of V with real eigenvalues, then for any $\mathbf{v} \in \mathbb{R}^d$ and $W \subset \mathbb{R}^d$, either the set $\{t \in \mathbb{R} \mid T_t(\mathbf{v}) \in W\}$ contains an infinite arithmetic progression, or it has at most $2(d-1)$ elements. This however is not true for T_t of the form

$$T_t = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}!$$

So if Dani's Conjecture holds for $T = T_1$, its proof must incorporate some new argument.

Finally let us think of possible further generalizations.

Problem 16. Given a d-dimensional V, $T : V \to V$ and $\ell < d$ define

$$m_d(\ell, T) \overset{\text{def}}{=} \inf\{m \mid \text{ for any } \mathbf{v} \in V \text{ and } W \text{ of dimension } \ell,\ (1m) \text{ holds.}\},$$

and let $m_d(\ell)$ be the supremum of $m_d(\ell, T)$ over all $T : V \to V$. Clearly one has $m_d = \sup_{\ell<d} m_d(\ell)$. Problem 6 asserts that $m_d(0) = 1$ and $m_d(1) = 2$ for all d. What about the behavior of $m_d(2)$ as $d \to \infty$? or more generally $m_d(2)$ for fixed ℓ? In the case of T having real eigenalues, we know that $m_d(\ell, T) \le 2(d-1)$ for all ℓ, but what about the growth of $m_d(\ell, T)$ for fixed ℓ, say for $\ell = 2$? The proof of Theorem 2 does not simplify by placing a restriction on $\dim(W)$. Maybe it is even true that $m_d(\ell, T)$ for these T, or $m_d(\ell)$ in general, are uniformly bounded in d for fixed ℓ? If not, how fast can these numbers grow?

Problem 17. Given any (finite or infinite) subset $A \subset \mathbb{N}$, construct a linear transformation T of an infinite-dimensional (Hilbert) space V such that $A = A(\mathbf{v}, W)$ for some $\mathbf{v} \in V$ and $W \subset V$. What is the spectrum of this T? Can you do it when the spectrum of T is discrete? is a subset of $\mathbb{R}$? or $\mathbb{R}_+$?

Problem 18. Can the previous problem be solved if a restriction is placed on $\dim(W)$? say for $\dim(W) = 2$? Or can we at least prove that, in the terminology introduced in Problem 16, $m_\infty(\ell, T) = \infty$ for some T and ℓ? if yes, how do these infinite sets $A(\mathbf{v}, W)$ look? (Recall that they are still not allowed to have arithmetic progressions of length ℓ.)

Problem 19. In the formulation of Dani's Conjectures replace (a) T by an affine transformation of V, or (b) W by an affine subspace of V. Should we expect the conjectures to still hold, or are there counterexamples?

Problem 20. Any other ideas related to the topic?

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