Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be a linear transformation. For $v \in V$ and a subspace $W \subset V$ define

\[ A(v, W) \overset{\text{def}}{=} \{ n \in \mathbb{N} \mid T^n(v) \in W \} \]

(the attendance set of $v$ in $W$). In [D], S.G. Dani asked the following question: how big can be the sets $A(v, W)$ for various choices of $v$ and $W$? Here are two examples/problems:

\textbf{Problem 1.} If $W$ contains a nonzero subspace invariant under $T^r$ for some $r \in \mathbb{N}$, then for any $j \in \mathbb{N}$ there exists $v \in V$ such that $A(v, W)$ contains the infinite arithmetic progression $\{ j + kr \mid k \in \mathbb{N} \}$. Conversely, if $A(v, W)$ contains some infinite arithmetic progression, or even an arithmetic progression of length greater or equal to $\dim(W)$, then some nonzero subspace of $W$ is invariant under some power of $T$.

\textbf{Problem 2.} On the other hand, if $T$ is a unipotent transformation (that is, 1 is its only eigenvalue), then for any $v \in V$ whose $T$-orbit is not wholly contained in $W$, the cardinality of $A(v, W)$ is less than $\dim(V)$.

The question asked by Dani is roughly as follows: is the most general situation a combination of the above two cases? Specifically, he states

\textbf{Conjecture 1} (Dani’s Conjecture). \textit{Given $V$ and $T$ as above, there exists $m = m(T) \in \mathbb{N}$ such that for any $v \in V$ and any subspace $W \subset V$,

\[ (1m) \quad \text{if } A(v, W) \text{ has at least } m \text{ elements,} \]

\[ \text{it contains an infinite arithmetic progression.} \]

He also writes that it is plausible that the following stronger version is true:

\textbf{Conjecture 2} (Strengthening of Dani’s Conjecture). \textit{For any $d \in \mathbb{N}$ there exists $m = m_d \in \mathbb{N}$ such that for $V$ of dimension $d$, any $v \in V$, any $T : V \to V$ and any subspace $W \subset V$, $(1m)$ holds.}

Both of the above statements seem to be too hard at the moment. So let us also consider

\textbf{Conjecture 3} (Weakening of Dani’s Conjecture). \textit{Given $V$, $T$, $v$ and $W$ as above,}

\[ (1\infty) \quad \text{if } A(v, W) \text{ is infinite, it contains an infinite arithmetic progression.} \]
We remark that by Szemerédi’s Theorem, see e.g. [F], any counterexample \(A(v, W)\) to Conjecture 3 must have upper density zero. Furthermore, the following can be obtained using the quantitative version of Szemerédi’s Theorem due to Gowers [G]:

**Theorem 0.** Given \(V\) and \(T\) as above and \(\delta > 0\), there exists \(k \in \mathbb{N}\) such that the following holds: if \(v \in V\) and \(W \subset V\) are such that the set \(A(v, W) \cap \{1, \ldots, k\}\) has at least \(\delta k\) elements, then \(A(v, W)\) contains an infinite arithmetic progression.

**Problem 3.** Each of Conjectures 1, 2, 3 can be reduced to the case when \(T\) is nonsingular (this is how Conjectures 1 and 2 were actually stated by Dani).

**Problem 4.** For two subspaces \(U, W \subset V\) define
\[
A(U, W) \overset{\text{def}}{=} \{n \in \mathbb{N} \mid T^n(U) \subset W\}.
\]
Show that Conjectures 1, 2, 3 imply their analogues where \(v \in V\) is replaced by a subspace \(U \subset V\).

**Problem 5.** Conjecture 2 holds for \(d \leq 2\), with \(m_1 = 1\) and \(m_2 = 2\).

Thus the first nontrivial case of the above conjectures is \(\dim(V) = 3\).

**Problem 6.** If \(\dim(W) = 0\) (resp. 1) and \(A(v, W)\) is nonempty (resp. has at least 2 elements), then it contains an infinite arithmetic progression.

Thus to understand the case \(\dim(V) = 3\) one needs to let \(W\) be a plane in \(\mathbb{R}^3\), say
\[
W = \{(x, y, z) \mid z = 0\}.
\]
In this case, in view of Problem 1, to prove Conjecture 3 it is enough to show that whenever \(A(v, W)\) is infinite, it must contain an arithmetic progression of length 2.

**Problem 7.** For each \(k = 0, 1, 2, 3, 4\) construct explicitly a linear transformation \(T : \mathbb{R}^3 \to \mathbb{R}^3\) and \(v \in \mathbb{R}^3\) such that for this \(T\) and \(W\) as in \((2)\), \(A(v, W)\) has exactly \(k\) elements.

**Problem 8.** Prove that, for \(V\) of arbitrary dimension, whenever \(\dim(W) = 2\) and \(v\) is such that \(\{1, 2, 4, 5\} \subset A(v, W)\), it also follows that \(3 \in A(v, W)\), and hence \(A(v, W) = \mathbb{N}\). In fact, the same proof shows: whenever \(\dim(W) = 2\) and \(A(v, W)\) contains a parallelogram, that is, a set of the form \(\{k, k + a, k + b, k + a + b\}\), it contains an arithmetic progression of length 2, and hence an infinite one.

Dani was able to prove his conjecture under an additional condition. His proof is based on the following

**Lemma 1.** Let \(f(t) = \sum_{i=1}^{r} c_i e^{\alpha_i t}\), where \(c_i, \alpha_i \in \mathbb{R}\). If \(f\) is not identically zero, then the number of zeroes of \(f\) is at most \(r - 1\).

**Problem 9.** Prove Lemma 1 (e.g. by induction and using Rolle’s Theorem).

**Theorem 1.** Let \(T : V \to V\) be a linear transformation of \(\mathbb{R}^d\) diagonalizable over \(\mathbb{R}\) with positive eigenvalues. Then for any \(v \in \mathbb{R}^d\) and \(W \subset \mathbb{R}^d\), the set \(A(v, W)\) either is equal to \(\mathbb{N}\) or has at most \(d - 1\) elements.

**Problem 10.** Derive Theorem 1 from Lemma 1.
Problem 11. Show that Theorem 1 implies

Theorem 2. Let \( T : V \rightarrow V \) be a linear transformation of \( \mathbb{R}^d \) diagonalizable over \( \mathbb{R} \). Then for any \( v \in \mathbb{R}^d \) and \( W \subset \mathbb{R}^d \), either \( A(v,W) \) contains all even or all odd natural numbers, or it has at most \( 2(d-1) \) elements.

Problem 12. Generalize Lemma 1 to the class of functions of the form \( f(t) = \sum_{i=1}^{r} Q_i(t)e^{\alpha_i t}, \) where \( Q_i \) are polynomials.

Problem 13. Using the previous problem, show that the assumptions on \( T \) in Theorem 1 (resp. Theorem 2) can be replaced by “all eigenvalues of \( T \) are positive (resp. real),” with the same conclusions.

Problem 14. Theorem 2 in its generalized form as in Problem 13 implies Conjecture 1 (but not Conjecture 2!) in the case when all eigenvalues of \( T \) are roots of real numbers, namely \( z \in \mathbb{C} \) such that \( z^n \in \mathbb{R} \) for some \( n \in \mathbb{N} \).

The above discussion shows that to prove Dani’s Conjecture for \( \dim(V) = 3 \) it remains to analyze the case of \( T \) having one real and two complex conjugate eigenvalues of the form \( e^{2\pi i \theta} \) with \( \theta \notin \mathbb{Q} \).

Problem 15. Investigate this case. Maybe Dani’s Conjecture is not true after all? then this would be a case to try to build a counterexample. For example, note that the proof of Theorem 2, as well as its generalized form as in Problem 13, produces a stronger result: if \( T_t \) is a one-parameter group of linear transformations of \( V \) with real eigenvalues, then for any \( v \in \mathbb{R}^d \) and \( W \subset \mathbb{R}^d \), either the set \( \{ t \in \mathbb{R} | T_t(v) \in W \} \) contains an infinite arithmetic progression, or it has at most \( 2(d-1) \) elements. This however is not true for \( T_t \) of the form

\[
T_t = \begin{pmatrix}
e^{\lambda t} & 0 & 0 \\
0 & \cos t & -\sin t \\
0 & \sin t & \cos t
\end{pmatrix}
\]

So if Dani’s Conjecture holds for \( T = T_1 \), its proof must incorporate some new argument.

Finally let us think of possible further generalizations.

Problem 16. Given a \( d \)-dimensional \( V, T : V \rightarrow V \) and \( \ell < d \) define

\[
m_d(\ell, T) \overset{\text{def}}{=} \inf \{ m : \text{for any } v \in V \text{ and } W \text{ of dimension } \ell, \text{ (1m) holds} \},
\]

and let \( m_d(\ell) \) be the supremum of \( m_d(\ell, T) \) over all \( T : V \rightarrow V \). Clearly one has \( m_d = \sup_{\ell < d} m_d(\ell) \). Problem 6 asserts that \( m_d(0) = 1 \) and \( m_d(1) = 2 \) for all \( d \).

What about the behavior of \( m_d(2) \) as \( d \to \infty \)? or more generally \( m_d(2) \) for fixed \( \ell \)? In the case of \( T \) having real eigenvalues, we know that \( m_d(\ell, T) \leq 2(d-1) \) for all \( \ell \), but what about the growth of \( m_d(\ell, T) \) for fixed \( \ell \), say for \( \ell = 2 \)? The proof of Theorem 2 does not simplify by placing a restriction on \( \dim(W) \). Maybe it is even true that \( m_d(\ell, T) \) for these \( T \), or \( m_d(\ell) \) in general, are uniformly bounded in \( d \) for fixed \( \ell \)? If not, how fast can these numbers grow?
Problem 17. Given any (finite or infinite) subset $A \subset \mathbb{N}$, construct a linear transformation $T$ of an infinite-dimensional (Hilbert) space $V$ such that $A = A(v, W)$ for some $v \in V$ and $W \subset V$. What is the spectrum of this $T$? Can you do it when the spectrum of $T$ is discrete? is a subset of $\mathbb{R}$? or $\mathbb{R}_+$?

Problem 18. Can the previous problem be solved if a restriction is placed on $\dim(W)$? say for $\dim(W) = 2$? Or can we at least prove that, in the terminology introduced in Problem 16, $m_\infty(\ell, T) = \infty$ for some $T$ and $\ell$? if yes, how do these infinite sets $A(v, W)$ look? (Recall that they are still not allowed to have arithmetic progressions of length $\ell$.)

Problem 19. In the formulation of Dani’s Conjectures replace (a) $T$ by an affine transformation of $V$, or (b) $W$ by an affine subspace of $V$. Should we expect the conjectures to still hold, or are there counterexamples?

Problem 20. Any other ideas related to the topic?

References


