

## Math 326a, Fall 2006, Problem Set # 4

### A. Solitaire Jumps

Suppose some of the lattice points  $\mathbb{Z}^2 \subset \mathbb{R}^2$  are occupied by chips. A *solitaire jump* is a horizontal or vertical jump of any chip over its neighbor to a free point with the chip jumped over removed. For instance, with  $(x, y)$  and  $(x, y + 1)$  occupied and  $(x, y + 2)$  free, a jump consists in removing the two chips on  $(x, y)$  and  $(x, y + 1)$  and placing a chip onto  $(x, y + 2)$ .

1. (J. Conway) Some (or all) of the lattice points on or below the  $x$ -axis are occupied by chips. Prove that it is possible to get a chip to  $(0, 4)$  but not to  $(0, 5)$ . [Hint: here one needs an invariant similar to the one which helped us in Kontsevich's Problem 1.6, but not quite the same.]

2. (IMO 1993) Initially each cell of an  $m \times n$  rectangle is occupied by a chip. Find all values of  $m, n$  for which the game can end with just one chip left over. [The invariant discussed in class can help prove the impossibility of a reduction to one chip; however you would also need to create a strategy of achieving this goal in the cases when it is possible.]

### B. Jumps in $GL_n(\mathbb{Z})$

In response to Apratim's question regarding a possible strengthening of Problem 2.8, let us prove the following

**Theorem.** *Let  $A$  and  $B$  be two  $n \times n$  matrices with integer entries such that*

$$\{A \pm k_i B \mid i = 1, \dots, 2n\} \subset GL_n(\mathbb{Z}),$$

where  $\{k_1, \dots, k_{2n}\}$  are  $2n$  different integers. Then  $A + kB \in GL_n(\mathbb{Z}) \forall k \in \mathbb{Z}$ .

*Proof.* The only way this could happen, as we learned by solving the aforementioned problem, is when  $\det(A + k_i B)$  is equal to 1 for  $n$  out of  $2n$  given values of  $k_i$ , and to  $-1$  for the remaining ones. Denote  $I = \{i = 1, \dots, 2n \mid \det(A + k_i B) = 1\}$  and  $J = \{j = 1, \dots, 2n \mid \det(A + k_j B) = -1\} = \{1, \dots, 2n\} \setminus I$ . Then we have

$$c \prod_{i \in I} (x - k_i) = 2 + c \prod_{j \in J} (x - k_j)$$

for all real  $x$ , where  $c = \det(B)$ . Plugging in  $x = k_i$  or  $k_j$  we conclude that 2 is divisible by  $k_i - k_j$  for all  $(i, j) \in I \times J$ .  $\square$

3. Finish the proof of the theorem.

4. Is it possible to improve the result even further? say instead of  $2n$  consider  $2n - 1$  different integers? when  $n = 2$  we had a proof in class (or so I think). Write it down and see if it can be generalized.