Math 326a, Fall 2006, Problem Set # 5

The Extremal Principle

Here is my favorite example of a problem solved by applying this principle: the Sylvester Problem, posed by Sylvester in 1893 and solved by Gallai in 1933 in a very complicated way.

Theorem. *Suppose that a finite set S of points in the plane has the property that any line through two of them passes through a third. Then all the points lie on a single line.*

Proof (Kelly, 1948). Consider the set of pairs $\{(p, L)\}$ where L is a line passing through two different points of S, and $p \in S \setminus L$. If this set is nonempty, choose a pair which miminizes the distance d from p to L. Let f be the foot of the prependicular from p to L. By assumption there are at least three elements a, b, c of $L \cap S$. Choose two of them, say a, b, on the same side of f. Let b be nearer to f than a. Then the distance from b to the line passing through a and p is less than d. Contradiction. □

0. Draw a picture and convince yourself that the above proof works.

1. Let B and W be two finite sets of black and white points, respectively, in the plane, such that every line segment joining two points of the same color contains a point of the other color. Prove that all the points lie on a single line.

2. In a circle, a finite set C of chords (segments connecting two points on the circle) is given, with a property that each of the chords from C passes through a midpoint of another chord from C. Prove that all these chords are diameters (that is, connect the antipodal points).

3. Prove that it is not possible to find different natural numbers x, y, z, t which are solutions of

$$x^x + y^y = z^z + t^t .$$

4. Each of the $3n$ members of a parliament of some country slapped one of his/her colleagues. Prove that among them it is possible to choose a committee consisting of n members none of whom slapped each other.

5. Every member of a parliament of another country has at most three enemies among the remaining members. Show that one can split the parliament into two houses so that every member has at most one enemy in his/her house.

6. 3009 numbers $a_1, \ldots, a_{3009}$ are written along the circle in such a way that each of them is equal to the absolute value of the difference between the next two in the clockwise direction (that is, $a_1 = |a_2 - a_3|$, $a_2 = |a_3 - a_4|$, ... , $a_{3008} = |a_{3009} - a_1|$, $a_{3009} = |a_1 - a_2|$. The sum of all of the numbers is equal to 2006. What are they?

7. Consider a walk in the plane according to the following rules. From a given point (x, y) we may move in one step to one of the four points $(x, y + 2x)$, $(x, y - 2x)$, $(x-2y, y)$, $(x+2y, y)$, with the restriction that a step just made cannot be retraced. Prove that if we start from $(1, \sqrt{2})$, we cannot return to this point anymore.