

Math 326a, Fall 2006, Problem Set # 6

(C, α)-good functions

Notation: For a metric space X , $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball $B(x, r) \stackrel{\text{def}}{=} \{y \in X \mid \text{dist}(x, y) < r\}$ of radius r centered at x . For $B \subset X$ and a function $f : B \rightarrow \mathbb{R}$, we let $\|f\|_B \stackrel{\text{def}}{=} \sup_{x \in B} |f(x)|$. If μ is a locally finite Borel measure on X and B is a subset of X with $\mu(B) > 0$, we define $\|f\|_{\mu, B}$ to be equal to $\|f\|_{B \cap \text{supp } \mu}$.

Problem 1. Show that if f is continuous and B is open, $\|f\|_{\mu, B}$ is the same as the $L^\infty(\mu)$ norm of $f|_B$, i.e. $\|f\|_{\mu, B} = \sup \{c \mid \mu(\{x \in B : |f(x)| > c\}) > 0\}$. Are these additional assumptions on f and B necessary?

Roughly speaking, a function is said to be *good* if the set of points where it takes small values has small measure. To simplify notation, it will be convenient to define

$$B^{f, \varepsilon} \stackrel{\text{def}}{=} \{x \in B \mid |f(x)| < \varepsilon\}$$

for any $f : B \rightarrow \mathbb{R}$ and $\varepsilon > 0$. Then for a subset U of X and $C, \alpha > 0$, say that a Borel measurable function $f : U \rightarrow \mathbb{R}$ is *(C, α)-good on U with respect to μ* if for any open ball $B \subset U$ centered in $\text{supp } \mu$ one has

$$(1) \quad \forall \varepsilon > 0 \quad \mu(B^{f, \varepsilon}) \leq C \left(\frac{\varepsilon}{\|f\|_{\mu, B}} \right)^\alpha \mu(B).$$

If X is Euclidean and $\mu = \lambda$ is Lebesgue, we will omit the reference to the measure and will simply say “(C, α)-good on U ”. In particular, λ is positive on open sets, so in that case one can replace $\|f\|_{\mu, B}$ by $\|f\|_B$ and not pay attention to the restriction of the center of B lying in $\text{supp } \mu$.

Lemma 1. *Let a metric space X , a measure μ on X , $U \subset X$ and $C, \alpha > 0$ be given.*

- (a) f is (C, α)-good on U with respect to $\mu \iff$ so is $|f|$;
- (b) f is (C, α)-good on U with respect to $\mu \implies$ so is $cf \ \forall c \in \mathbb{R}$;
- (c) f is (C, α)-good on U with respect to $\mu \implies$ it is (C', α)-good on U' with respect to μ for any $C' \geq C$ and $U' \subset U$.
- (d) $f_i, i \in I$, are (C, α)-good on U with respect to μ and $f = \sup_{i \in I} |f_i|$ is measurable \implies it is also (C, α)-good on U with respect to μ ;
- (e) f is (C, α)-good on U with respect to μ , and $c_1 \leq \frac{|f(x)|}{|h(x)|} \leq c_2$ for all $x \in U \implies h$ is $(C(c_2/c_1)^\alpha, \alpha)$ -good on U with respect to μ .

Problem 2. Prove parts (d) and (e) of the Lemma (note that in (d) the set I may be uncountable).

Problem 3. Does the assumption in part (c) imply that f is (C, α')-good on U with respect to μ for any $\alpha' \leq \alpha$? Prove it or find a counterexample.

The next proposition describes a model example of good functions:

Proposition 1. For any $n \in \mathbb{N}$, any real polynomial of degree not greater than n is $(C_n, 1/n)$ -good on \mathbb{R} , where $C_n = n(n+1)^{1/n}$.

Proof. Fix an open interval $B \subset \mathbb{R}$, a real polynomial f of degree not exceeding n , and a positive ε . We need to show that

$$(2) \quad \lambda(B^{f,\varepsilon}) \leq n(n+1)^{1/n} \left(\frac{\varepsilon}{\|f\|_B} \right)^{1/n} \lambda(B).$$

Lemma 2. Let $A \subset \mathbb{R}$ be such that $\lambda(A) > m$. Then there exist $x_1, \dots, x_{n+1} \in A$ with $|x_i - x_j| \geq m/n$ for each $1 \leq i \neq j \leq n+1$.

Problem 4. Prove the Lemma.

Now assume that the left hand side of (2) is strictly bigger than some number m , and choose x_1, \dots, x_{n+1} as in Lemma 2 applied to $A = B^{f,\varepsilon}$. Using Lagrange's interpolation formula one can write down the exact expression for f :

$$f(x) = \sum_{i=1}^{n+1} f(x_i) \frac{\prod_{j=1, j \neq i}^{n+1} (x - x_j)}{\prod_{j=1, j \neq i}^{n+1} (x_i - x_j)}.$$

Note that $|f(x_i)| < \varepsilon$ for each i , $|x - x_j| \leq \lambda(B)$ for each j and $x \in B$, and also $|x_i - x_j| \geq m/n$. Therefore

$$\|f\|_B < (n+1)\varepsilon \frac{\lambda(B)^n}{(m/n)^n}.$$

which can be rewritten as

$$m < n(n+1)^{1/n} \left(\frac{\varepsilon}{\|f\|_B} \right)^{1/n} \lambda(B),$$

proving (2). \square

Problem 5. Improve Proposition 1 by observing that the estimate $|x_i - x_j| \geq m/n$ can be strengthened to $|x_i - x_j| \geq m|i - j|/n$. Try to obtain the best possible value of C_n . What is its asymptotics as $n \rightarrow \infty$?

Problem 6. Find a lower bound for C_n by considering 'worst possible' polynomials. Hopefully this will prove that the upper bound obtained in the previous problem is optimal.

Problem 7. State and prove an analogue of Proposition 1 with \mathbb{R} replaced by \mathbb{Q}_p , p a prime. It should be helpful that \mathbb{Q}_p is an *ultrametric* space, that is, the non-Archimedean triangle inequality $\text{dist}(x_1, x_2) \leq \max_{i=1,2} \text{dist}(x, x_i)$ holds for all $x, x_1, x_2 \in \mathbb{Q}_p$, and that Lagrange Interpolation is valid for any ground field. What is C_n in this case? Is the estimate sharp? what are worst-case polynomials?

Problem 8. A 'big picture' question: what if instead of polynomials we take functions with uniform estimates on their n th derivatives? for simplicity start with

$n = 2$, fix $\delta > 0$, and consider twice continuously differentiable functions f on an interval U such that for some $c > 0$ one has

$$c \leq |f''(x)| \leq c(1 + \delta) \quad \text{for all } x \in U.$$

Can you prove that they are all (C, α) -good on U with C dependent only on δ (preferably with $\alpha = 1/2$)? This is clearly a problem harder than the one about polynomials (for example, the set of functions with the above property is an infinite-dimensional space). Does Lagrange Interpolation work? Certainly not precisely, but maybe it is possible to estimate the value of f from above if one knows that it takes small values at three points far away from each other? Is it possible to obtain a result which will tend to Proposition 1 as $\delta \rightarrow 0$?

The next lemma is useful in dealing with functions on products of metric spaces. Here the product of two metric spaces (X, dist_X) and (Y, dist_Y) will be supplied with the *product metric*

$$\text{dist}((x_1, y_1), (x_2, y_2)) = \max(\text{dist}_X(x_1, x_2), \text{dist}_Y(y_1, y_2)),$$

so that balls in $X \times Y$ are products of balls in X and in Y of the same radius.

Lemma 3. *Let metric spaces X, Y with measures μ, ν be given. Suppose f is a continuous function on $U \times V$, where $U \subset X$ and $V \subset Y$ are open subsets, and suppose C, D, α, β are positive constants such that*

$$(3a) \quad \begin{aligned} &\text{for all } y \in V \cap \text{supp } \nu, \text{ the function } x \mapsto f(x, y) \\ &\text{is } (C, \alpha)\text{-good on } U \text{ with respect to } \mu, \end{aligned}$$

and

$$(3b) \quad \begin{aligned} &\text{for all } x \in U \cap \text{supp } \mu, \text{ the function } y \mapsto f(x, y) \\ &\text{is } (D, \beta)\text{-good on } V \text{ with respect to } \nu. \end{aligned}$$

Then f is (E, γ) -good on $U \times V$ with respect to $\mu \times \nu$, where

$$(4) \quad \gamma = \frac{\alpha\beta}{\alpha + \beta} \quad \text{and} \quad E = (\alpha + \beta) \left(\left(\frac{C}{\beta} \right)^\beta \left(\frac{D}{\alpha} \right)^\alpha \right)^{\frac{1}{\alpha + \beta}}.$$

Proof. Fix a ball in $U \times V$ of the form $A \times B$, where A and B are balls in X and Y intersecting the supports of μ and ν respectively. Without loss of generality let us rescale $\mu|_A$, $\nu|_B$ and f so that $\mu(A) = \nu(B) = \|f\|_{\mu \times \nu, A \times B} = 1$. Take an arbitrary $\varepsilon > 0$; we need to demonstrate that

$$(5) \quad (\mu \times \nu)((A \times B)^{f, \varepsilon}) \leq E\varepsilon^\gamma.$$

For $y \in B$ let us denote by f_y the function $x \mapsto f(x, y)$. Also denote by φ the function defined on B by $\varphi(y) \stackrel{\text{def}}{=} \|f_y\|_{\mu, A}$.

Problem 9. Prove that φ is measurable and $\|\varphi\|_{\nu, B} = 1$.

In view of (3a), for any $y \in B \cap \text{supp } \nu$ one has

$$(6) \quad \mu(A^{f_{y,\varepsilon}}) \leq C \left(\frac{\varepsilon}{\varphi(y)} \right)^\alpha \iff \varphi(y) \leq \left(\frac{C}{\mu(A^{f_{y,\varepsilon}})} \right)^{1/\alpha} \varepsilon.$$

Take an arbitrary $t > 0$ (to be fixed later), and denote

$$B_t \stackrel{\text{def}}{=} \{y \in B \mid \mu(A^{f_{y,\varepsilon}}) \geq t\}.$$

In view of (6), $y \in B_t$ implies that $\varphi(y)$ is not bigger than $(C/t)^{1/\alpha} \varepsilon$. Since it follows from Lemma 1(d) and (3b) that φ is (D, β) -good on V with respect to ν , one can write

$$(7) \quad \nu(B_t) \leq \nu(\{y \in B \mid \varphi(y) \leq (C/t)^{1/\alpha} \varepsilon\}) \leq D \left((C/t)^{1/\alpha} \varepsilon \right)^\beta.$$

Now observe that one has $\mu(\{x \in A \mid (x, y) \in (A \times B)^{f,\varepsilon}\}) < t$ whenever $y \notin B_t$, therefore, by the Fubini Theorem,

$$(8) \quad \begin{aligned} (\mu \times \nu)((A \times B)^{f,\varepsilon}) &< (\mu \times \nu)(A \times B_t) + t \cdot \nu(B \setminus B_t) \\ &\leq \nu(B_t) + t \stackrel{(7)}{\leq} t + (DC^{\beta/\alpha} \varepsilon^\beta) \cdot t^{-\beta/\alpha}. \end{aligned}$$

Problem 10. Show that the function in the right hand side of (8) attains its minimum when

$$t = \left(C^\beta \left(\frac{D\beta}{\alpha} \right)^\alpha \right)^{\frac{1}{\alpha+\beta}} \varepsilon^{\frac{\alpha\beta}{\alpha+\beta}},$$

and use it to obtain (5) with E and γ given by (4), thus finishing the proof. \square

Problem 11. Use Lemma 3 and induction to prove

Corollary. For $j = 1, \dots, d$, let X_j be a metric space, μ_j a measure on X_j , $U_j \subset X_j$ open, $C, \alpha > 0$, and let f be a continuous function on $U_1 \times \dots \times U_d$ such that for any $j = 1, \dots, d$ and any $x_i \in U_i$ with $i \neq j$, the function

$$y \mapsto f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_d)$$

is (C, α) -good on U_j with respect to μ_j . Then

$$f \text{ is } (dC, \alpha/d)\text{-good on } U_1 \times \dots \times U_d \text{ with respect to } \mu_1 \times \dots \times \mu_d.$$

Problem 12. Now consider \mathbb{R}^d with the Euclidean metric, and conclude that any real polynomial f in d variables of degree not greater than n is $(C_{d,n}, 1/dn)$ -good on \mathbb{R}^d for some $C_{d,n}$. Is the estimate sharp? What could be worst-case polynomials, say for $d = 2$?

Problem 13. Another ‘big picture’ question: is it possible to approach the above problem using interpolation in d variables, say for $d = 2$, or even for small n ? In a sense, the analogue of Lemma 2 would be that in a subset $A \subset \mathbb{R}^d$ of not so small Lebesgue measure one can find $\mathbf{x}_1, \dots, \mathbf{x}_N$ such that the evaluation map:

$$\{\text{polynomials of degree } \leq n\} \rightarrow \mathbb{R}^N, \quad f \mapsto (f(\mathbf{x}_1), \dots, f(\mathbf{x}_N))$$

has determinant not very close to 0... Any ideas?