Math 326a, Fall 2006, Problem Set # 7

Elementary Number Theory

1. Prove that if p and p^2+2 are primes, then p^3+2 is also prime.

2. Find all integral solutions of $x+y=x^2-xy+y^2$.

3. For positive integers a, b, c, d, n, show that if $ab=cd$, then $a^n+b^n+c^n+d^n$ is not a prime.

4. Prove that it is possible to choose 2^k different numbers from $0, 1, \ldots, 3^k-1$ so that three numbers in arithmetic progression will not occur.

5. Consider the sequence $A=\{31, 331, 3331, \ldots\}$. Note that the first few elements of A are prime. Prove that there are infinitely many composite numbers in A, and that in fact composite numbers form a subset of A of positive lower density.

6. The powers 2^n and 5^n start with the same digit d. What are possible values of d?

7. Show that $y^2=x^3+7$ has no integral solutions.

8. Show that no prime can be written as a sum of two squares in two different ways.

9. Can the product of (a) three (b) four (c) more than four consecutive integers be equal to a (more than first) power of an integer?

10. Show that if 4^n+2^n+1 is a prime, then n is a power of 3.

11. Prove that there are infinitely many powers of 2 in the sequence $\lfloor n\sqrt{2}\rfloor$.

12. Do there exist positive integers x, y such that $x+y$, $2x+y$ and $x+2y$ are perfect squares?