ON QUANDLES AND KNOT INVARIANTS

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Abstract. This paper explores quandles, a recently rediscovered algebraic structure and its application to the knot recognition problem. More specifically, we demonstrate the resemblance of quandles to Reidemeister moves and show they are general enough to give rise to such strong knot invariants as the knot group.

1. Introduction

The theory of knots has been studied in a systematical way for over a century now but the main problem of telling two knots apart has remained elusive. Here we consider two knots to be equivalent if one of them can be mapped to the other by a homeomorphism of the ambient space. Two such knots are called *isotopic* and will consider an isotopy as the main equivalence class on knots. This is the knot recognition problem whose resolution has become more promising in light of new techniques associated with the theory of quandles. We begin with some basic definitions from knot theory that will be useful to us.

Definition 1.1. *A knot K is a subset of $\mathbb{R}^3$ which is homeomorphic to S^1. More generally we can define a link K as a subset of $\mathbb{R}^3$ which is homeomorphic to the disjoint union of n copies of S^1.*

We will talk primarily about knot projections onto a plane and deal with the singular points, the points where the projected knot intersects itself, by considering the z-coordinates of the preimage of the projection. We will call any part of a knot a *branch* and at the intersection point in the projection the branch with the greater z-coordinate will be called an *over-crossing* and the branch with the lower z-coordinate will be called an *under-crossing*.

It was proved in 1926 by Reidemeister that certain elementary moves on knot projections preserve isotopy classes and this has given the subject a combinatorial flavor.

Definition 1.2. *A Reidemeister move is any one of the moves in figure 1.*

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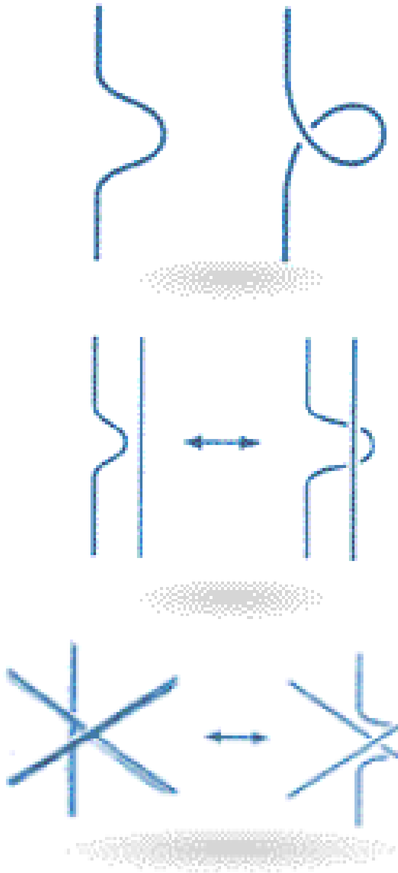

FIGURE 1. Reidemeister Moves

Theorem 1.3. *Two knots (links) are isotopic if and only if the projection of one knot (link) can be transformed by Reidemeister moves to the projection of the other knot (link).*

Proof. The proof which can be found in [Reid] is rather technical and is beyond the scope of this paper. □

2. QUANDLES

Before we provide the connection of quandles to knot theory we give some necessary background information. For additional information see [Man].

Definition 2.1. *A quandle is a set* Λ *with a binary operation*

$$\circ \colon \Lambda \times \Lambda \to \Lambda$$

such that the following conditions hold:

(i) $\forall a \in \Lambda \ \ a \circ a = a$
(ii) $\forall b, c \in \Lambda$ *there exists a unique* $a \in \Lambda$ *such that* $a \circ b = c$
(iii) $\forall a, b, c \in \Lambda \ \ (a \circ b) \circ c = (a \circ c) \circ (b \circ c)$

As we can see quandles are not commutative or associative, however, they are intimately tied to groups. To see this, given any group G we define the quandle operation $a \circ b$ to be bab^{-1}. The three quandle axioms are immediately satisfied: (i) $a \circ a = a^{-1}aa = a$. (ii) There exists a unique $a = bcb^{-1}$ such that $c = a \circ b$. (iii) $(a \circ b) \circ c = c^{-1}(b^{-1}ab)c = (c^{-1}b^{-1}c)(c^{-1}ac)(c^{-1}bc) = (c^{-1}ac) \circ (c^{-1}bc) = (a \circ c) \circ (b \circ c)$. There are other interesting special cases of quandles such as the module over the ring of Laurent polynomials in one variable and matrix representations of finite quandles but we will not go into detail here.

3. Quandles and Knots

The relation of the theory of quandles to the theory of knots is based on labelling (coloring) of arcs which we view as the elements of a given Quandle Λ.

Definition 3.1. *(intuitive) An orientation μ on a knot (knot projection) K is a choice of direction along the knot (knot projection).*

Definition 3.2. *Let an orientation be given on a knot K. An arc c is a piece of the knot projection which begins at one under-crossing and ends at the next one (moving along the orientation of the knot) and is always an over-crossing in between these two under-crossings.*

For a knot projection K a *knot quandle* is a set Γ whose elements correspond to the labels on the arcs of K according to the scheme in figure 2.

To show that the knot quandle is an invariant for isotopic knots we need to check that the condition given in figure 2. remains invariant under Reidemeister moves. We can see this below in figures 3,4,5:

So each quandle generates a rule for a proper coloring of the knot projection where a proper coloring is simply a way of associating a color (label) with each arc of the knot projection so that the coloring rule illustrated in figure 2 is satisfied. This gives us the following:

Proposition 3.3. *The number of proper colorings by elements of any quandle is a knot invariant.*

To demonstrate the generalizing power of knot quandles we will show how knot quandles can generate well known knot invariants. Our example will be the knot group. As before some basic definitions are in order.

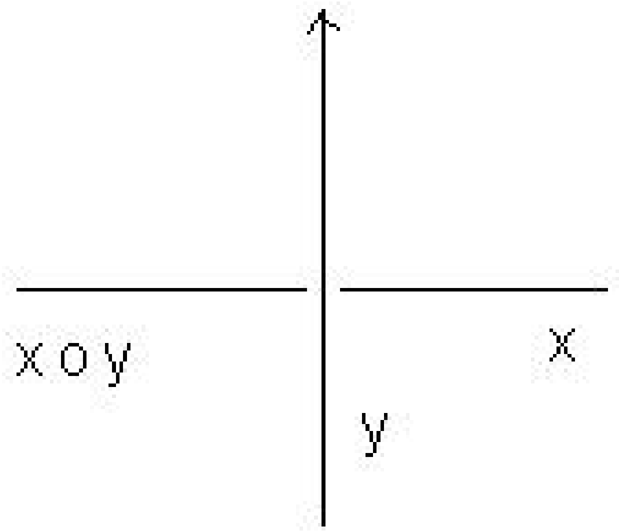

FIGURE 2. Quandle product

Definition 3.4. *Let X be a topological space. We say that two paths α and β in X, both of which start at x_1 and end at x_2, are path-homotopic if there exists a continuous function $F\colon I \times I \to X$ such that*

(i) $\forall s \in I\ F(s,0) = \alpha(s)$
(ii) $\forall s \in I\ F(s,1) = \beta(s)$
(iii) $\forall t \in I\ F(0,t) = x_1$
(iii) $\forall t \in I\ F(1,t) = x_2$

Two loops (paths with equal initial and endpoints) are path-homotopic if they share the same initial point and if they are homotopic as paths. When we consider path-homotopy as an equivalence relation on paths this leads to the definition of a topological invariant called the fundamental group.

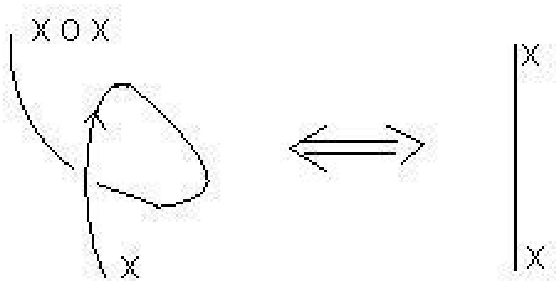

FIGURE 3. Quandle Product for Reidemeister Move of Type I

Definition 3.5. *Let x_0 be a point in a topological space X. Let $\pi_1(X, x_0)$ the set of all path-homotopy classes of loops based at x_0. Define a binary operation $*$ on $\pi_1(X, x_0)$ by $[\alpha] * [\beta] = \bar{\alpha}\bar{\beta}$ where $\bar{\alpha}$ and $\bar{\beta}$ are representative paths and $\bar{\alpha}\bar{\beta}$ means going along the path α and then going along the path β. We call $\pi_1(X, x_0)$ the fundamental group of X.*

It is not hard to check that $\pi_1(X, x_0)$ with the product $*$ defines a group.

In the theory of knots the fundamental group gives rise to a knot invariant called the knot group.

Definition 3.6. *If K is an oriented knot (link) then the knot group of K is defined as $\pi_1(\mathbb{R}^3 - K, x_0)$, the fundamental group of the complement of the knot K in $\mathbb{R}^3$.*

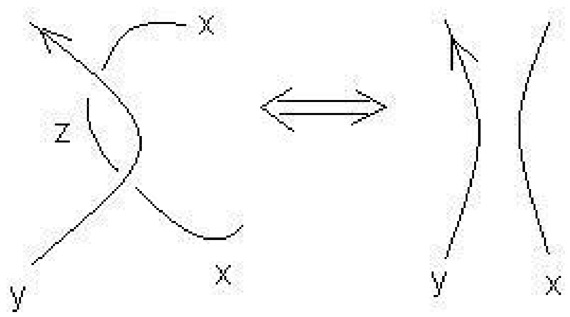

FIGURE 4. Quandle Product for Reidemeister Move of Type II

$\pi_1(\mathbb{R}^3 - K)$ (we omit the base point to simplify notation) can be given a presentation called the *Wirtinger Presentation*. We will present the algorithm for the construction of this group presentation here but will omit the details of its proof which employs Van Kampen's theorem and can be found in [Rolf].

First we choose an orientation on K which extends to an orientation on its projection. Then under every arc draw an arrow in an East-West direction where the orientation of the arc is taken to be North. Then at every crossing we have the following two possible situations:

The relation (r_i) that must hold in case (i) is $a_i = c_i^{-1} b_i c_i$ and the relation that must hold in case (ii) is $a_i = c_i b_i c_i^{-1}$. The main result is then:

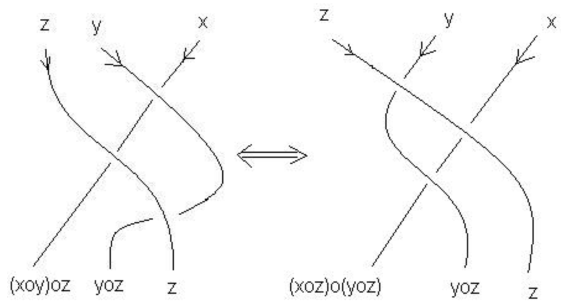

Figure 5. Quandle Product for Reidemeister Move of Type III

Theorem 3.7. *The knot group* $\pi_1(\mathbb{R}^3 - K)$ *has presentation* $< a_i, b_i, c_i \mid r_i >$.

Remark 3.1. *If we have* m *arcs and* n *crossings we get a group presentation on* m *generators and* n *relations.*

Remark 3.2. *To see how the* a_i, b_i, c_i *correspond to generators of the group* $\pi_1(\mathbb{R}^3 - K)$ *imagine a point* x_0 *in* $\mathbb{R}^3 - K$. *Let* γ *be a path from* x_0 *to the initial point of the path (arrow)* a_i *and let* ρ *be a path from the endpoint of* a_i *back to* x_0. *Then* $[\gamma a_i \rho]$ *is a class of path-homotopic loops which is an element of* $\pi_1(\mathbb{R}^3 - K)$.

To see how knot groups arise from quandles we go back to our original definition of quandles. Given a group G we can form a new algebraic

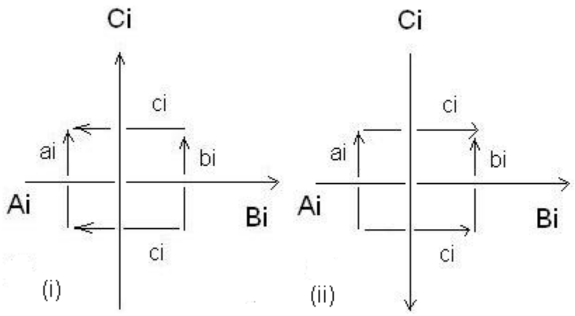

FIGURE 6. Wirtinger Presentation

structure consisting of the underlying set of G and two binary operations which are just the two conjugations, $y^{-1}xy$ and yxy^{-1}. We call the first operation $x \circ y$ and the second operation $x \circ^{-1} y$. The resulting structure, $Conj(G)$, is also called a non-involutory quandle. These two conjugation operations in a group satisfy the following three identities for x, y, and z in $Conj(G)$:

(Q1) $x \circ x = x$
(Q2) $(x \circ y) \circ^{-1} y = x = (x \circ^{-1} y) \circ y$
(Q3) $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$

We can see that the quandle defined earlier, also called an involutory quandle, is actually a special case of the non-involutory quandle where $x \circ y = x \circ^{-1} y$. If we apply y to both sides and then apply (Q2) we get the identity $(x \circ y) \circ y = x$. If we replace (Q2) with this identity we get our previously defined involutory quandle. A more detailed discussion of the algebraic properties of quandles can be found in [Joy]

It becomes apparent immediately that the two relations at every crossing of a knot projection that arise as the result of the construction of the Wirtinger presentation correspond to the two binary operations or conjugations described above. Given a knot projection we can color all the arcs by the a_i, b_i, c_i such that the two relations $a_i = c_i^{-1} b_i c_i$ (which is $b_i \circ c_i$) and $a_i = c_i b_i c_i^{-1}$ (which is $b_i \circ^{-1} c_i$) that arise at each crossing are satisfied. In fact we let the knot quandle of a knot K be given the presentation $Q(K) =< a_i, b_i, c_i \mid r_i >$ where the relations and generators are the same as in the Wirtinger Presentation.

Theorem 3.8. *The knot quandle $Q(K)$ is invariant for isotopic knots.*

Proof. It suffices to show the invariance of the knot quandle $Q(K)$ under Reidemeister moves which we can see in figure 7. □

References

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