Homework 1: Invariants and Semi-invariants

Due date: September 26, 2011.

1. Four grasshoppers sit at four vertices of a square. Every second one of the grasshoppers jumps over another one and lands at the symmetric point (that is, if a grasshopper jumps from point $A$ over $B$ and lands at $C$, then vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are equal). Prove that:
   (a) no three of them will ever sit on a straight line parallel to one of the sides of the original square (more generally – on any straight line);
   (b) the four of them will never form a square bigger than the original one (more generally – a parallelogram with area bigger than the area of the original one).

2. The set of numbers $\{a, b, c\}$ every second gets replaced with $\{a + b - c, b + c - a, c + a - b\}$. In the beginning $a = 2010, b = 2011, c = 2012$. Can we get $\{2010, 2012, 2013\}$ after a sequence of these operations?

3. On a board the 100 numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$ are written down. Every second we choose two numbers $a$ and $b$ from those on the board, erase them and write the number $a + b + ab$. This operation is performed 99 times until there is just one number. What is this number? Find it and prove that it does not depend on the sequence of choices.

4. Initially there is a pile of 637 chips on a table. We are allowed to throw away one chip and divide the pile into two (not necessarily equal) piles. Then the same can be done with any pile containing more than two chips, and so on. Is it possible to get only the piles consisting of three chips?

5. Stones are arranged in three piles: in one – 51 stones, in another one – 49 stones, and in the third one – 5 stones. It is allowed to combine any two piles into one, and also to divide a pile with even number of stones into two equal piles. Is it possible to obtain 105 piles with a single stone in each?

6. On a circle there are several blue points and several red points. It is allowed to add a red point and change colors of its two neighbors, and also, if there are more than two points, to remove a red point and change colors of its former neighbors. Suppose that initially there were 2 red points and no blue points. Prove that it is impossible to obtain a configuration consisting of:
   (a) (easy) 3 blue points and no red points;
   (b) (difficult!) 2 blue points and no red points.