Homework 6: The Box Principle

Due date: November 14, 2011.

1. Each of 102 students in a school is a friend of at least 68 others. Prove that among them one can choose four who have the same number of friends. (Assume that friendship is a symmetric relation.)

2. 2011 different prime numbers form an arithmetic progression, and the smallest of the numbers is greater than 2011. Prove that the common difference of the progression is divisible by 2011.

3. Suppose that a quadratic polynomial $ax^2 + bx + c$ is equal to the fourth power of an integer for all positive integers $x$. Prove that $a = b = 0$.

4. 2000 real numbers are written in a row. Prove that it is possible to choose a subset consisting of adjacent numbers (maybe of just one number) such that the sum of numbers from this subset differs from an integer by no more than $1/1000$.

5. An open subset $U$ of $[0, 1]$ has the property that for any $x, y \in U$, the distance between $x$ and $y$ is different from $1/10$. Prove that the measure (length) of $U$ is not greater than $1/2$.

6. 50 segments are given on a straight line. Suppose that no point on the line belongs to more than 7 of the segments. Prove that one can find 8 of them which are pairwise disjoint.