1. Invariants and Semi-invariants (*due Wednedsay 9/21*)

1.1. Four grasshoppers sit at four vertices of a square. Every second one of the grasshoppers jumps over another one and lands at the symmetric point (that is, if a grasshopper jumps from point A over B and lands at C, then vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are equal). Prove that:

(a) no three of them will ever sit on a straight line parallel to one of the sides of the original square (more generally – on any straight line);

(b) the four of them will never form a square bigger than the original one (more generally – a parallelogram with area bigger than the area of the original one).

1.2. The set of numbers $\{a, b, c\}$ every second gets replaced with $\{a+b-c, b+c-a, c+a-b\}$. In the beginning $a = 2004$, $b = 2005$, $c = 2006$. Can we get $\{2004, 2006, 2007\}$ after a sequence of these operations?

1.3. On a board the 100 numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$ are written down. Every second we choose two numbers a and b from those on the board, erase them and write the number $a+b+ab$. This operation is performed 99 times until there is just one number. What is this number? Find it and prove that it does not depend on the sequence of choices.

1.4. Initially there is a pile of 637 chips on a table. We are allowed to throw away one chip and divide the pile into two (not necessarily equal) piles. Then the same can be done with any pile containing more than two chips, and so on. Is it possible to get only the piles consisting of three chips?

1.5. Stones are arranged in three piles: in one – 51 stones, in another one – 49 stones, and in the third one – 5 stones. It is allowed to combine any two piles into one, and also to divide a pile with even number of stones into two equal piles. Is it possible to obtain 105 piles with a simgle stone in each?

1.6. On a circle there are several blue points and several red points. It is allowed to add a red point and change colors of its two neighbors, and also, if there are more than two points, to remove a red point and change colors of its former neighbors. Suppose that initially there were 2 red points and no blue points. Prove that it is impossible to obtain a configuration consisting of:

(a) (easy) 3 blue points and no red points;

(b) (difficult!) 2 blue points and no red points.

2. Coloring Proofs (*due "Tuesday" 10/3*)

2.1. 9 chips occupy the lower left 3×3 square of an 8×8 chessboard. For one move any chip can jump over any other (not necessarily adjacent) chip and land into the cell symmetric to the first cell around the second one. It is possible after a number of moves to arrange all the chips in the form of the upper right 3×3 square?

2.2. A piece of cheese has the form of a $3 \times 3 \times 3$ cube from which the central cube is cut out. A mouse starts to gnaw this piece of cheese. First she eats some $1 \times 1 \times 1$ cube. After the mouse finishes up each little $1 \times 1 \times 1$ cube, she goes on to one of the adjacent (on a side) cubes. Can she eat all the cheese?

2.3. A king, moving according to the rules of chess, visited all the cells of the standard chessboard exactly once and returned to its original location. Prove that it made an even number of diagonal moves.

2.4. A convex n-gon is divided into triangles by non-intersecting diagonals so that an odd number of triangles meet in each of its vertices. Prove that n is divisible by 3.

2.5. 99 squares of size 2×2 are cut from a 29×29 square sheet of graph paper. Prove that it is possible to cut out another 2×2 square from the remaining part of the big square.

2.6. The points on the boundary of a regular triangle are colored in two colors. Prove that among them there exist three points of the same color forming a right triangle.

3. The Extremal Principle (*due Wednedsay 10/12*)

3.1. In a circle, a finite set S of chords (segments connecting two points on the circle) is given, with a property that each of the chords from S passes through a midpoint of another chords from S. Prove that all these chords are diameters (that is, connect the antipodal points).
[**Hint**: consider the chord of minimal length.]

3.2. Prove that it is not possible to find different natural numbers x, y, z, t which are solutions of

$$x^x + y^y = z^z + t^t \,.$$

[**Hint**: look at the maximal number among x, y, z, t.]

3.3. Each of the $3n$ members of a parliament slapped one of his/her colleagues. Prove that among them it is possible to choose a committee consisting of n members none of whom slapped each other.
[**Hint**: consider those lucky ones who got slapped the least number of times.]

3.4. 30 numbers $a_1, \dots, a_{30}$ are written along the circle in such a way that each of them is equal to the absolute value of the difference between the next two in the clockwise direction (that is, $a_1 = |a_2 - a_3|$, $a_2 = |a_3 - a_4|$, $\dots$, $a_{29} = |a_{30} - a_1|$, $a_{30} = |a_1 - a_2|$. The sum of all of the numbers is equal to 20. What are they?
[**Hint**: start by looking at the largest number.]

3.5. Natural numbers are placed at each of the 8 vertices of a cube in such a way that numbers at adjacent vertices (i.e. those sharing an edge) differ by no more than 1. Prove that one can find two opposite vertices of the cube such that the numbers placed there also differ by no more than 1.
[**Hint**: consider a vertex containing the smallest number.]