

MATH 47A PROBLEM SETS, FALL 2011

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ABSTRACT. This paper contains the first five homework assignments written in a research paper format. It also includes some theoretical material written to provide background for problems, as well as to illustrate the theorem-proof environment.

1. INVARIANTS AND SEMI-INVARIANTS

Reading material: [E, pp. 1–23].

Due date: September 26, 2011.

1.1. Four grasshoppers sit at four vertices of a square. Every second one of the grasshoppers jumps over another one and lands at the symmetric point (that is, if a grasshopper jumps from point A over B and lands at C , then vectors \overrightarrow{AB} and \overrightarrow{BC} are equal).

Prove that:

- (a) no three of them will ever sit on a straight line parallel to one of the sides of the original square (more generally – on any straight line);
- (b) the four of them will never form a square bigger than the original one (more generally – a parallelogram with area bigger than the area of the original one).

1.2. The set of numbers $\{a, b, c\}$ every second gets replaced with $\{a + b - c, b + c - a, c + a - b\}$. In the beginning $a = 2010, b = 2011, c = 2012$. Can we get $\{2010, 2012, 2013\}$ after a sequence of these operations?

1.3. On a board the 100 numbers $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}$ are written down. Every second we choose two numbers a and b from those on the board, erase them and write the number $a + b + ab$. This operation is performed 99 times until there is just one number. What is this number? Find it and prove that it does not depend on the sequence of choices.

Date: October 26, 2011.

1.4. Initially there is a pile of 637 chips on a table. We are allowed to throw away one chip and divide the pile into two (not necessarily equal) piles. Then the same can be done with any pile containing more than two chips, and so on. Is it possible to get only the piles consisting of three chips?

1.5. Stones are arranged in three piles: in one – 51 stones, in another one – 49 stones, and in the third one – 5 stones. It is allowed to combine any two piles into one, and also to divide a pile with even number of stones into two equal piles. Is it possible to obtain 105 piles with a single stone in each?

1.6. On a circle there are several blue points and several red points. It is allowed to add a red point and change colors of its two neighbors, and also, if there are more than two points, to remove a red point and change colors of its former neighbors. Suppose that initially there were 2 red points and no blue points. Prove that it is impossible to obtain a configuration consisting of:

- (a) (easy) 3 blue points and no red points;
- (b) (difficult!) 2 blue points and no red points.

Now let me practice writing mathematics in the (usual for mathematical papers) ‘Theorem–Proof’ format.

Theorem 1.1. *A circle is divided into six sectors, and the numbers 1, 0, 1, 0, 0, 0 are written into the sectors counterclockwise. Then, no matter how many times one increases two neighboring numbers by 1, it is impossible to equalize all the numbers.*

Proof. Label the numbers written in the sectors at any given time by letters

$$(1.1) \quad a, b, c, d, e, f,$$

and consider

$$(1.2) \quad S \stackrel{\text{def}}{=} a - b + c - d + e - f.$$

Claim 1.2. *S is an invariant; that is, it does not change when any two neighboring numbers are increased by 1.*

Proof. This is obvious. □

Since in the initial state $S = 2$, and in the equalized state $S = 0$, it is impossible to make all of the letters (1.1) equal. □

This was just an elementary example. Let me now take on a more serious challenge, and write up my lecture on

1.7. Rotation numbers of circle homeomorphisms. The circle S^1 can be considered as the quotient space \mathbb{R}/\mathbb{Z} , with the quotient map $\pi : \mathbb{R} \rightarrow S^1$, defined by $\pi(x) = x \bmod 1$; or, if S^1 is identified with the segment $[0, 1]$ with the boundary points glued together, one can say that $\pi(x)$ is the fractional part of a real number x .

Now let $f : S^1 \rightarrow S^1$ be an orientation-preserving homeomorphism (bijective and continuous map). We can *lift* f to an increasing homeomorphism $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $\pi \circ F = f \circ \pi$. Furthermore, for each $x \in \mathbb{R}$ such that $\pi(x) = f(0)$ there is a unique lift F of f such that $F(0) = x$, and any two lifts differ by an integer translation. Also, those lifts F have the periodicity property: for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, one has

$$(1.3) \quad F(x + n) = F(x) + n.$$

Here are two main theorems about invariants of circle homeomorphisms. The exposition mostly follows [BS].

Theorem 1.3. *Let $f : S^1 \rightarrow S^1$ be an orientation-preserving homeomorphism and $F : \mathbb{R} \rightarrow \mathbb{R}$ a lift of f . Then for every $x \in \mathbb{R}$, the limit*

$$(1.4) \quad \rho(F) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n}$$

exists and is independent of the point x . The number

$$\rho(f) \stackrel{\text{def}}{=} \pi(\rho(F))$$

is independent of the lift F .

This quantity $\rho(f)$ is called the *rotation number* of f . It will follow from the proof that $\rho(f)$ is rational if f has a periodic point; converse is also true but is not proved here. Clearly if f is the rotation by α , then for any $k \in \mathbb{Z}$,

$$F_k : x \mapsto x + \alpha + k$$

is a lift of f , and formula (1.4) gives $\rho(F_k) = \alpha + k$, so that $\rho(f) = \alpha$.

Theorem 1.4. *Rotation number is an invariant of topological conjugacy; that is, if f is an orientation-preserving homeomorphism of S^1 and h is another homeomorphism (used to change variables in S^1), then*

$$(1.5) \quad \rho(f) = \rho(h \circ f \circ h^{-1}).$$

In particular, rotations by different angles are never conjugate to one another.

Proof of Theorem 1.3. We will do it in several steps.

Step 1. Suppose for a moment that the limit (1.4) exists for some $x \in \mathbb{R}$, and take another point $y \in \mathbb{R}$. In view of (1.3), it suffices to consider the case $0 \leq x, y < 1$. Then for any n , it holds that $|F^n(x) - F^n(y)| \leq 1$ (note that F^n is also a lift of a circle homeomorphism, namely of f^n). Therefore

$$|F^n(x) - x - (F^n(y) - y)| \leq |F^n(x) - F^n(y)| + |x - y| \leq 2,$$

which implies

$$\frac{|F^n(x) - x - (F^n(y) - y)|}{n} \leq \frac{2}{n} \Rightarrow \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n} = \lim_{n \rightarrow \infty} \frac{F^n(y) - y}{n}.$$

Step 2. Now let us prove the existence of the limit in the special case when f has a periodic point, that is, $f^q(x) = x$ for some $x \in S^1$ and $q \in \mathbb{N}$, or, equivalently, $F^q(x) = x + p$ for some $x \in \mathbb{R}$ and $p, q \in \mathbb{N}$. Now take an arbitrary $n \in \mathbb{N}$ and write $n = kq + p$, where $0 \leq r < q$. Then

$$F^n(x) = F^r(F^{kq}(x)) = F^r(x + kp) = F^r(x) + kp,$$

therefore

$$\frac{F^n(x) - x}{n} = \frac{F^r(x) - x + kp}{n} = \frac{F^r(x) - x}{n} + \frac{kp}{kq + r}.$$

The first summand in the right hand side of the above equality tends to 0 as $n \rightarrow \infty$, and the second one tends to p/q , showing that $\rho(F) = p/q$ in this case.

Step 3. Suppose now that $F^q(x) \neq x + p$ for all $x \in \mathbb{R}$ and $p, q \in \mathbb{N}$. By continuity (and the Intermediate Value Theorem), for each pair $p, n \in \mathbb{N}$, either $F^n(x) - x < p$ or $F^n(x) - x > p$ for all $x \in \mathbb{R}$ (that is, the graph $y = F^n(x) - x$ cannot cross horizontal lines $y = p$, therefore is stuck in the strip $p - 1 < y < p$ for some $p \in \mathbb{N}$.) In other words, for any $n \in \mathbb{N}$ one can choose $p = p_n \in \mathbb{N}$ such that

$$(1.6) \quad p_n - 1 < F^n(x) - x < p_n \quad \text{for all } x \in \mathbb{R}.$$

Now take $m \in \mathbb{N}$ and write $F^{mn}(x) - x$ in the form

$$F^n(x) - x + F^n(F^n(x)) - F^n(x) + \cdots + F^n(F^{(m-1)n}(x)) - F^{(m-1)n}(x),$$

from which, in view of (1.6), it follows that

$$m(p_n - 1) < F^{mn}(x) - x < mp_n \quad \text{for all } x \in \mathbb{R}.$$

Now let us divide both sides by mn , getting

$$\frac{p_n - 1}{n} < \frac{F^{mn}(x) - x}{mn} < \frac{p_n}{n} \quad \text{for all } x \in \mathbb{R}.$$

Interchanging the roles of m and n , we also have

$$\frac{p_m - 1}{m} < \frac{F^{mn}(x) - x}{mn} < \frac{p_m}{m} \quad \text{for all } x \in \mathbb{R}.$$

Thus $|p_m/m - p_n/n| < 1/m + 1/n$, so $\{p_n/n\}$ is a Cauchy (hence convergent) sequence. But it follows from (1.6) that the limit of p_n/n is equal to the limit (1.4).

Step 4. Finally, let us prove the independence of the rotation number of f on the choice of its lift. If F and G are two lifts of f , then $F(x) = G(x) + k$ for some $k \in \mathbb{Z}$ and all x . Using induction and (1.3) one easily shows that $F^n(x) = G^n(x) + nk$ for any $n \in \mathbb{N}$; hence

$$\frac{G^n(x) - x}{n} = \frac{F^n(x) - x}{n} + k,$$

which implies that the π -images of $\rho(F)$ and $\rho(G)$ are the same. \square

Proof of Theorem 1.4. Let F and H be lifts of f and h respectively. Then $H \circ F \circ H^{-1}$ is a lift of $h \circ f \circ h^{-1}$, and for $x \in \mathbb{R}$ one has

$$\begin{aligned} & \frac{(H \circ F \circ H^{-1})^n(x) - x}{n} = \frac{H \circ F^n \circ H^{-1}(x) - x}{n} \\ &= \frac{H \circ F^n \circ H^{-1}(x) - F^n \circ H^{-1}(x)}{n} + \frac{F^n \circ H^{-1}(x) - H^{-1}(x)}{n} + \frac{H^{-1}(x) - x}{n}. \end{aligned}$$

Since the numerators in the first and third terms of the last expression are uniformly bounded independently of n , we conclude that

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{(H \circ F \circ H^{-1})^n(x) - x}{n} = \lim_{n \rightarrow \infty} \frac{F^n(x) - (x)}{n} = \rho(h \circ f \circ h^{-1}).$$

\square

This is just the beginning of the study of circle homeomorphisms, a branch of the theory of dynamical systems which was actively developing in the first half of the 20th century. In particular, it can be proved that two homeomorphisms with the same irrational rotation number are conjugate if they are twice differentiable (and there are counterexample if only the first derivative exists). Depending on your interest, I may try to state some exercises on this topic which could be transformed into research projects.

2. COLORING PROOFS

Reading material: [E, pp. 25–37].

Due date: October 5, 2011.

2.1. 9 chips occupy the lower left 3×3 square of an 8×8 chessboard. For one move any chip can jump over any other (not necessarily adjacent) chip and land into the cell symmetric to the first cell around the second one. It is possible after a number of moves to arrange all the chips in the form of the upper right 3×3 square?

2.2. A piece of cheese has the form of a $3 \times 3 \times 3$ cube from which the central cube is cut out. A mouse starts to gnaw this piece of cheese. First she eats some $1 \times 1 \times 1$ cube. After the mouse finishes up each little $1 \times 1 \times 1$ cube, she goes on to one of the adjacent (on a side) cubes. Can she eat all the cheese?

2.3. A king, moving according to the rules of chess, visited all the cells of the standard chessboard exactly once and returned to its original location. Prove that it made an even number of diagonal moves.

2.4. A convex n -gon is divided into triangles by non-intersecting diagonals so that an odd number of triangles meet in each of its vertices. Prove that n is divisible by 3.

2.5. 99 squares of size 2×2 are cut from a 29×29 square sheet of graph paper. Prove that it is possible to cut out another 2×2 square from the remaining part of the big square.

2.6. The points on the boundary of a regular triangle are colored in two colors. Prove that among them there exist three points of the same color forming a right triangle.

A nice elementary exposition of an example of use of colors in serious mathematics (tricolorability of knots) can be found at

www.math.utah.edu/~bestvina/HS07/knots-hs07.pdf

(by Mladen Bestvina, University of Utah). The list of exercises there might work as a source for research projects.

3. SCHMIDT GAMES AND SCHMIDT DIAGRAMS

Reading material: [S] (online at the course web page).

Due date: October 17, 2011.

Denote by I the open unit square:

$$I \stackrel{\text{def}}{=} \{(\alpha, \beta) : 0 < \alpha, \beta < 1\} = (0, 1) \times (0, 1).$$

Then pick $(\alpha, \beta) \in I$ and consider the following game, commonly referred to as *Schmidt's game*, played by two players, whom we will call Alice and Bob. The game starts with Bob choosing a closed interval $B_1 \subset \mathbb{R}$. Alice may now choose a subinterval $A_1 \subset B_1$ provided that $|A_1| = \alpha|B_1|$. Next, Bob chooses a subinterval $B_2 \subset A_1$ satisfying $|B_2| = \beta|A_1|$, and so on. Continuing in the same manner, one obtains a nested sequence of shrinking intervals:

$$(3.1) \quad B_1 \supset A_1 \supset B_2 \supset \dots$$

A subset S of \mathbb{R} is called (α, β) -*winning* if Alice can play in such a way that the unique point of intersection

$$(3.2) \quad x_\infty \stackrel{\text{def}}{=} \bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} A_k$$

(its existence follows from the completeness of \mathbb{R}) lies in S , no matter how Bob plays. In other words, for any $k \in \mathbb{N}$ and for any choice of intervals B_k made by Bob, Alice can choose A_k in such a way that $x_\infty \in S$. We will say that S is (α, β) -*losing* if it is not (α, β) -winning.

Clearly a lot in this game depends on the choice of the *target set* S , as well as on the parameters α and β . Still there exist sets S for which being winning can be proved (or disproved) regardless of the choice of α and β . Those constitute rather trivial examples:

Theorem 3.1. (a) $S = \mathbb{R}$ is (α, β) -winning for any $(\alpha, \beta) \in I$;
 (b) if S is not dense, then it is (α, β) -losing for any $(\alpha, \beta) \in I$.

Proof. In case (a), $x_\infty \in S$ will be true regardless of the moves of the players. On the other hand, if S is not dense in \mathbb{R} , it is possible to choose a nonempty open ball U disjoint from S . Then Bob can decide to place his first move inside that ball, and, regardless of the subsequent moves and the values of α and β , guarantee that x_∞ belongs to U , hence is not contained in S . \square

Let us now introduce the following definition: if $S \subset \mathbb{R}$, define the *Schmidt diagram* $D(S)$ to be the set of pairs $(\alpha, \beta) \in I$ such that S is (α, β) -winning. One of our goals will be understanding Schmidt

diagrams of sets more complicated than those considered in the above theorem. The latter, incidentally, can be rephrased in the language of Schmidt diagrams as follows: part (a) states that

$$(3.3) \quad S = \mathbb{R} \Rightarrow D(S) = I,$$

while part (b) asserts that

$$(3.4) \quad S \text{ is not dense} \Rightarrow D(S) = \emptyset.$$

Another interesting fact, which is rather easy to understand, is that the implications in the two statements above can be reversed. Indeed, the following strengthening of Theorem 3.1 is true:

Theorem 3.2. (a) $S = \mathbb{R} \Leftrightarrow D(S) = I$;

(b) $S \text{ is not dense} \Leftrightarrow D(S) = \emptyset$.

Proof. We did case (a) in class, even though we did not state it this way. Indeed, to prove the converse implication one needs to show that if $S \neq \mathbb{R}$ (that is, if there exists a point not in S), then S is (α, β) -losing for some $(\alpha, \beta) \in I$. And here is what we proved:

Claim 3.3. *For any $x \in \mathbb{R}$, the set $\mathbb{R} \setminus \{x\}$ is (α, β) -losing for $\alpha = 3/4$ and $\beta = 1/2$.*

Proof. Here is what Bob should do to prevent Alice from winning: his first choice would be an interval B_1 centered at x . Let us say that its length is $2r$. Then, no matter how Alice places her interval A_1 inside B_1 , the distance from x to the endpoint of A_1 closest to x is at least $r/2$. Therefore Bob can choose B_2 centered at x : its length is supposed to be $2r \cdot \frac{3}{4} \cdot \frac{1}{2} = 3r/4$, so it fits! This basically reduces the game to the initial position; adhering to this strategy produces a sequence of nested balls B_k centered at x , which implies that $x_\infty = x$, a point not in S . \square

This clearly implies that $D(S)$ is a proper subset of I whenever S is a proper subset of \mathbb{R} : indeed, S is contained in $\mathbb{R} \setminus \{x\}$ for some $x \in \mathbb{R}$, hence $D(S) \subset D(\mathbb{R} \setminus \{x\})$; see Problem 3.1.

To do case (b), we need to show that if S is dense (that is, if there exists a point of S in any nonempty interval), then S is (α, β) -winning for some $(\alpha, \beta) \in I$. We did not do it in class, but it can be done by exactly the same argument as the proof of case (a) above, with the roles of Alice and Bob reversed ($\alpha = 1/2$ and $\beta = 3/4$ will work); see Problem 3.2. \square

At this point we can already formulate some homework problems:

3.1. Prove that the correspondence D preserves inclusion: $S_1 \subset S_2$ implies $D(S_1) \subset D(S_2)$. (Note: this is straightforward, but please write it down, just as an exercise in writing proofs.)

3.2. Finish the proof of Theorem 3.2 (b).

3.3. Following the proof of Claim 3.3, find other values of α and β such that the set $\mathbb{R} \setminus \{x\}$ is (α, β) -losing. (The more the better!)

One other thing we did in class is proving that $\mathbb{R} \setminus \{x\}$ is (α, β) -winning for some (α, β) . Namely we proved

Claim 3.4. *For any $x \in \mathbb{R}$, the set $\mathbb{R} \setminus \{x\}$ is (α, β) -winning for any $\alpha < 1/2$ and any $\beta > 0$.*

Proof. It suffices to observe that regardless of the choice of B_1 made by Bob, the distance from x to the endpoint of B_1 furthest to x is at least half of the length of B_1 . Therefore, since α is less than $1/2$, Alice can choose A_1 disjoint from x , and thus no matter what happens afterwards, the limit point x_∞ will belong to S . \square

In other words, the Schmidt diagram $D(\mathbb{R} \setminus \{x\})$ contains the strip $\{(\alpha, \beta) \in I : \alpha < 1/2\}$. Naturally this leads to new questions:

3.4. Find other values of α and β (that is, with $\alpha \geq 1/2$) such that the set $\mathbb{R} \setminus \{x\}$ is (α, β) -winning. For example, consider the case $\alpha = 1/2$; the proof of Claim 3.4 does not apply verbatim, since, assuming that Bob chooses an interval centered at x , Alice's interval will have x as an endpoint. But then you can look at Bob's next possible moves, and think of countermoves for Alice. Generalize, trying to find as many points in $D(\mathbb{R} \setminus \{x\})$ as you can. If you make sufficient progress with this problem and the previous one, perhaps you can summarize your findings and give a precise description of $D(\mathbb{R} \setminus \{x\})$?

3.5. Convince yourself that whatever you have proved for $\mathbb{R} \setminus \{x\}$ also works for $\mathbb{R} \setminus C$ where C is an arbitrary countable set.

3.6. Following the argument of case (b) of Claim 3.3 (Problem 3.2), find other values of α and β such that any dense set is (α, β) -winning. (The more the better!) In other words, find pairs (α, β) which belong to $D(S)$ whenever S is dense (equivalently, whenever $D(S)$ is nonempty.)

3.7. Now let S be a countable dense set (say, \mathbb{Q}). Find points which are NOT in $D(S)$; in fact, try to find as many as you can (you can argue similarly to Problems 3.4 and 3.5, again with the roles of Alice and Bob reversed). Can you give a precise description of $D(S)$ in this case?

An addendum to Homework 3

This is a write-up of what I talked about in class last time, followed by some more problems. Those are more challenging than the previous ones, in my opinion, and might be a beginning of a research project, for either an individual or a group of students working together.

The goal is to study Schmidt diagrams of more interesting sets. Namely, let me define

$$F_{10,0} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} \left| \begin{array}{l} \text{there are at most finitely many 0s} \\ \text{in the decimal expansion of } x \end{array} \right. \right\},$$

and, more generally,

$$F_{b,w} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} \left| \begin{array}{l} \text{there are at most finitely many occurrences} \\ \text{of the word } w \text{ in the base } b \text{ expansion of } x \end{array} \right. \right\},$$

where $b = 2, 3, \dots$ and w is a finite sequence $(a_1 a_2 \dots a_k)$ of digits $0 \leq a_i < b$. We know that these sets all have measure zero (they consist of non-normal numbers) and are dense.

As a warm-up, we are going to prove

Theorem 3.5. $F_{10,0}$ is $(\frac{1}{4}, \frac{2}{5})$ -winning.

Note that $(\frac{1}{4}, \frac{2}{5})$ is not among those pairs for which any dense set is winning (this should be clear to you after looking at Problem 3.7), so this theorem is not a triviality.

Proof. First of all, note that without loss of generality we can assume that the length of the interval B_1 (Bob's original choice) is not bigger than 1. Indeed, if it is bigger than 1, Alice can make arbitrary moves waiting until the length of the interval chosen by Bob at some stage is small enough. Then we can reindex and call that interval B_1 . Let us denote by d its length.

Now, Alice will try to make a move so that no points of A_1 (which by the way is supposed to have length $d/4$) have a zero at their k th decimal place for some suitably chosen k . Note that the set

$$Z_k \stackrel{\text{def}}{=} \{x \text{ with } 0 \text{ at its } k\text{th decimal place}\}$$

is the union of intervals of length 10^{-k} , with gaps of length $9 \cdot 10^{-k}$ between two successive intervals. So it will probably help if k is chosen in such a way that d is close to 10^{-k} . To be precise, let us choose k so that

$$\frac{d}{20} \leq 10^{-k} < \frac{d}{2}.$$

Now consider three cases:

- (a) B_1 does not intersect Z_k : then Alice makes an arbitrary choice and accomplishes her task;
- (b) B_1 intersects exactly one of the intervals comprising Z_k ; since the length of that interval is $10^{-k} < \frac{d}{2}$, the distance between one of the endpoints¹ of B_1 and Z_k is at least

$$(d - 10^{-k})/2 > d/4,$$

therefore Alice can place her interval between that endpoint and Z_k ;

- (c) B_1 intersects more than one of the intervals comprising Z_k ; since the gap between the intervals is

$$9 \cdot 10^{-k} \geq 9d/20,$$

Alice's interval can fit between them.

So after Alice is done, Bob chooses his interval B_2 in some way that Alice has no control over. But regardless of where it is placed, the length of B_2 is equal to

$$\frac{d}{2} \cdot \frac{2}{5} = \frac{d}{10},$$

that is, exactly 10 times less than the length of B_1 . Therefore Alice can perform the same trick with k replaced by $k + 1$ and manage to choose A_2 disjoint from Z_{k+1} . Continuing by induction, her n th move will be disjoint from Z_{n+k-1} for every $n \in \mathbb{N}$, and, as a result, x_∞ will have no zeroes in its decimal expansion except perhaps at the first $k - 1$ places – a clear victory for Alice! \square

Now, this was clearly just an example of what one can prove, and here comes a list of possible extensions and generalizations, which is the optional part of the homework. Feel free to do whatever you can by the deadline, or think about those problems after the deadline; if you find them interesting, they can grow into research projects.

3.8. What happens if we keep $\alpha = 1/4$ but make β bigger? it is supposed to make Bob's life more miserable; however, the above proof does not apply verbatim, since we used the fact that $\alpha\beta = 1/10$. What do you think Alice should do now to win?

¹When reading, it will be helpful to draw pictures corresponding to cases (b) and (c).

3.9. For what other values of α does the same strategy, perhaps with some modifications, work? try to find as many points in $D(F_{10,0})$ as you can. Also, for which pairs can you prove that the set is losing? eventually I'd be curious to see a precise description of $D(F_{10,0})$, not sure whether or not it is realistic, but we can try.

3.10. If we replace 10 by another base b , what would be values of α and β for which a similar set would be winning? This is important because the problems about $F_{b,w}$, where w is a word of length k , can be reduced to $F_{b^k,a}$ where a is a digit between 0 and $b^k - 1$. And presumably the bigger is the base, the smaller values of β can be given to Bob so that he still loses the game.

3.11. Now, as I did in class, let us define

$$M_{10}^+(0) \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} F_{10^k,0}$$

$$= \left\{ x \in \mathbb{R} \left| \begin{array}{l} \exists N \in \mathbb{N} \text{ such that there are no blocks of} \\ \text{successive } N \text{ zeroes in the decimal expansion of } x \end{array} \right. \right\}.$$

Verify that the equality above holds, and show that there exists α such that $M_{10}^+(0)$ is (α, β) -winning for any $\beta > 0$. Those sets are called α -winning.

3.12. Now take y which is written as $0.y_1y_2y_3\dots$ in base b , and do the same for

$$M_b^+(y) \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} F_{b^k, b^{k-1}y_1 + \dots + by_{k-1} + y_k}$$

$$= \left\{ x \in \mathbb{R} \left| \begin{array}{l} \exists N \in \mathbb{N} \text{ such that there are no occurrences of} \\ \text{the word } y_1 \dots y_N \text{ in the base } b \text{ expansion of } x \end{array} \right. \right\}.$$

Moreover, show that α such that $M_b^+(y)$ is α -winning can be chosen independently of b and y . This is important in view of the intersection properties of winning sets which I mentioned in class; we'll talk about those in more detail later.

Maybe there are other questions that come to your mind? then ask them and try to answer. Chances are that it will evolve into something interesting!

4. THE EXTREMAL PRINCIPLE

Reading material: [E, pp. 39–57].

Due date: October 24, 2011.

4.1. In a circle, a finite set S of chords (segments connecting two points on the circle) is given, with a property that each of the chords from S passes through a midpoint of another chord from S . Prove that all these chords are diameters (that is, connect the antipodal points).

[**Hint:** consider the chord of minimal length.]

4.2. Prove that it is not possible to find different natural numbers x, y, z, t which are solutions of

$$(4.1) \quad x^x + y^y = z^z + t^t.$$

[**Hint:** look at the maximal number among x, y, z, t .]

4.3. Each of the $3n$ members of a parliament slapped one of his/her colleagues. Prove that among them it is possible to choose a committee consisting of n members none of whom slapped each other.

[**Hint:** consider those lucky ones who got slapped the least number of times.]

4.4. 30 numbers a_1, \dots, a_{30} are written along the circle in such a way that each of them is equal to the absolute value of the difference between the next two in the clockwise direction (that is, $a_1 = |a_2 - a_3|$, $a_2 = |a_3 - a_4|$, \dots , $a_{29} = |a_{30} - a_1|$, $a_{30} = |a_1 - a_2|$). The sum of all of the numbers is equal to 20. What are they?

[**Hint:** start by looking at the largest number.]

4.5. Natural numbers are placed at each of the 8 vertices of a cube in such a way that numbers at adjacent vertices (i.e. those sharing an edge) differ by no more than 1. Prove that one can find two opposite vertices of the cube such that the numbers placed there also differ by no more than 1.

[**Hint:** consider a vertex containing the smallest number.]

5. (C, α) -GOOD FUNCTIONS

Due date: November 7, 2011.

Let f be a real-valued function of a real variable. In what follows, for $B \subset \mathbb{R}$ we will denote

$$\|f\|_B \stackrel{\text{def}}{=} \sup_{x \in B} |f(x)|.$$

Lebesgue measure of a set B will be denoted by $|B|$.

For positive numbers C and α , say that f is (C, α) -good if for any open interval $B \subset \mathbb{R}$ and for all $\varepsilon > 0$ one has

$$(5.1) \quad |\{x \in B : |f(x)| < \varepsilon\}| \leq C \left(\frac{\varepsilon}{\|f\|_B} \right)^\alpha |B|.$$

Here we adopt the convention $\frac{1}{0} = \infty$, so that (5.1) holds if $f|_B \equiv 0$.

Informally speaking, this property means the following: if a function attains a large enough value on an interval, then the set of points where the function is very small cannot be too big (relatively to the length of the interval). A motivation comes from linear functions: for those, the measure of $\{x \in B : |f(x)| < \varepsilon\}$ is bounded from above by a constant times ε . Let me prove it here (in a different way than in class):

Proposition 5.1. *Any linear function is $(2, 1)$ -good on \mathbb{R} .*

Proof. Fix an open interval $B \subset \mathbb{R}$, a linear function f and a positive ε , and denote $E \stackrel{\text{def}}{=} \{x \in B : |f(x)| < \varepsilon\}$ and $s \stackrel{\text{def}}{=} \|f\|_B$. We need to show that

$$(5.2) \quad |E| \leq \frac{2\varepsilon}{s} |B|.$$

Since f is monotonic, E is an interval. Assume that its length is strictly bigger than some number ℓ ; then one can choose two points

$$x_1 < x_2 \in E \quad \text{with} \quad x_2 - x_1 > \ell.$$

Let

$$(5.3) \quad y_1 = f(x_1) \quad \text{and} \quad y_2 = f(x_2).$$

Then we can reconstruct $f(x) = kx + m$ by solving a system of two linear equations in two variables and finding k and m . Here is a good way to write down the solution:

$$(5.4) \quad f(x) = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}.$$

It is easy to check, by plugging in $x = x_1$ and $x = x_2$, that (5.3) is satisfied (and we know that the solution is unique). Now apply the triangle inequality to the right hand side:

$$|f(x)| \leq |y_1| \frac{|x - x_2|}{|x_1 - x_2|} + |y_2| \frac{|x - x_1|}{|x_2 - x_1|}.$$

Since $|y_i| < \varepsilon$, $|x - x_i| \leq |B|$ when $x \in B$, and $|x_1 - x_2| > \ell$ by construction, we have

$$|f(x)| < 2\varepsilon \frac{|B|}{\ell}$$

for all $x \in B$, that is,

$$s < 2\varepsilon \frac{|B|}{\ell} \iff \ell < \frac{2\varepsilon}{s} |B|.$$

Since ℓ can be taken arbitrary close to $|E|$ (but smaller than $|E|$), (5.2) follows. \square

Note that what we proved is just an estimate and it is not clear a priori whether or not a better estimate can be found. However it turns out that what we did is really optimal – there is no way one can replace $C = 2$ by a smaller number and $\alpha = 1$ by a bigger number and get the same result.

Let us try to understand why. First of all, some general remarks. If $f(x)$ is identically equal to zero, then, as was remarked above, $\|f\|_B = 0$ for any B , therefore (5.1) holds for arbitrary values of C and α . This is really a degenerate case. So assume that f is not identically zero. Then I claim that:

5.1. f cannot be (C, α) -good, no matter what α is, unless $C \geq 1$.
[Hint: assume that f is (C, α) -good with $C < 1$, take B on which f does not vanish, take ε to be just a little bit bigger than $\|f\|_B$, and see what happens.]

So in what follows we will assume that $C \geq 1$. Then (5.1) trivially holds when $\varepsilon > \|f\|_B$. The next problem is

5.2. If f is (C, α) -good and $\beta \leq \alpha$, then f is (C, β) -good as well.
[Hint: consider the cases $\varepsilon/\|f\|_B > 1$ and $\varepsilon/\|f\|_B \leq 1$ separately.]

Also clearly f is (C, α) -good \implies it is (D, α) -good if $D \geq C$. So after proving some functions to be (C, α) -good, the next step would be to try to make C smaller and α bigger, if possible, and find optimal values. In some cases this is easy to do, in some other cases it is still unknown.

Here are several other general properties:

Lemma 5.2. *Let $C, \alpha > 0$ be given. Then*

- (a) f is (C, α) -good \iff so is $|f|$;
- (b) f is (C, α) -good \implies so is $cf \ \forall c \in \mathbb{R}$;
- (c) If f is (C, α) -good and $c_1 \leq \frac{|f(x)|}{|g(x)|} \leq c_2$ for all x , then g is $(C(c_2/c_1)^\alpha, \alpha)$ -good;
- (d) $f_i, i \in I$, are (C, α) -good \implies so is $\sup_{i \in I} |f_i|$ (here I can be finite or infinite).

5.3. Prove parts (c) and (d) of the above lemma (parts (a) and (b) should be completely obvious).

5.4. Prove that the property of being (C, α) -good is NOT additive; that is, find two functions which are (C, α) -good but their sum is not.

Now let us establish the optimality of constants in Proposition 5.1.

5.5. Let f be a nonconstant linear function. Suppose that f is (C, α) -good. Then

- (a) $\alpha \leq 1$ (regardless of the value of C), and
- (b) if $\alpha = 1$, then $C \geq 2$.

[**Hint:** assume that $\alpha > 1$ (respectively, that $\alpha = 1$ and $C < 2$, and construct B and ε so that (5.1) is violated.]

Let us now move from linear functions to more complicated ones. An example of $f(x) = x^2$ shows that one cannot hope to prove that quadratic polynomials are $(C, 1)$ -good. However, $\alpha = 1/2$ can be achieved. Moreover, the following can be proved:

Theorem 5.3. *For any $n \in \mathbb{N}$, any polynomial $f \in \mathbb{R}[x]$ of degree not greater than n is $(C_n, 1/n)$ -good on \mathbb{R} , where $C_n = n(n+1)^{1/n}$.*

Proof. Fix an open interval $B \subset \mathbb{R}$, a polynomial $f \in \mathbb{R}[x]$ of degree not exceeding n , and a positive ε . We need to show that

$$(5.5) \quad |\{x \in B : |f(x)| < \varepsilon\}| \leq n(n+1)^{1/n} \left(\frac{\varepsilon}{\|f\|_B} \right)^{1/n} |B|.$$

The proof will basically follow along the lines of the case $n = 1$ discussed in the proposition above. First we need to show

Claim 5.4. *Let $\ell > 0$, $n \in \mathbb{N}$, and let $E \subset \mathbb{R}$ be such that $|E| > \ell$. Then there exist $x_1, \dots, x_{n+1} \in E$ with*

$$(5.6) \quad |x_i - x_j| \geq \ell/n \quad \text{for each } 1 \leq i \neq j \leq n+1.$$

5.6. Prove Claim 5.4. [**Hint:** you can use the Extremal Principle, letting k be the maximal number of points x_1, \dots, x_k of E such that $|x_i - x_j| \geq \ell/n$ for each $1 \leq i \neq j \leq k$, and figuring out what it implies for the measure of E .]

Now assume that the left hand side of (5.5) is strictly bigger than some number ℓ , and choose x_1, \dots, x_{n+1} as in Claim 5.4 applied to $E = \{x \in B : |f(x)| < \varepsilon\}$. Using Lagrange's interpolation formula, which is basically a generalization of (5.4), one can write down the exact expression for f :

$$(5.7) \quad f(x) = \sum_{i=1}^{n+1} f(x_i) \frac{\prod_{j=1, j \neq i}^{n+1} (x - x_j)}{\prod_{j=1, j \neq i}^{n+1} (x_i - x_j)}.$$

Now note that:

- $|f(x_i)| < \varepsilon$ for each i ,
- $|x - x_j| \leq |B|$ for each j and every $x \in B$, and
- $|x_i - x_j| \geq \ell/n$, in view of (5.6).

Therefore

$$\|f\|_B < (n+1)\varepsilon \frac{|B|^n}{(\ell/n)^n}.$$

which can be rewritten as

$$\ell < n(n+1)^{1/n} \left(\frac{\varepsilon}{\|f\|_B} \right)^{1/n} |B|,$$

proving (5.5). □

Now a natural question is: are the values of the constants in the above theorem optimal? it is clear that $\alpha = 1/n$ cannot be improved, but I do not know the answer for C , unless $n = 1$.

5.7. Investigate the situation for $n = 2$. Is it possible to improve the value of $C = 2\sqrt{3}$? For example, here is some room for improvement: observe that on top of the estimate $|x_i - x_j| \geq \ell/n$ of Claim 5.4 we have

$$|x_3 - x_1| \geq 2\ell/n$$

(here I assume that $x_1 < x_2 < x_3$), and in general it can be strengthened to

$$|x_i - x_j| \geq |i - j|\ell/n.$$

If you can, identify (with proof) the best value of C_2 .

This concludes the regular portion of this week's homework problems. And a lot is left for you to explore. Here are some examples of research directions which have not been studied at all, so any result you discover here will be new!

5.8. Try to obtain a better upper bound for C_n , if possible. What are the 'worst possible' polynomials of degree n ? Maybe by identifying them it will be possible to show that your upper bound is optimal. A good case to consider is $n = 3$; note that all quadratic polynomials are basically the same (after rescaling, translating and flipping), but this is not the case with cubics.

5.9. It seems to me that a perturbation of Proposition 5.1 can be investigated. Given $0 < c_1 \leq 1 \leq c_2$, consider the set of real-valued differentiable functions f such that

$$c_1 \leq \frac{|f'(x)|}{|f'(y)|} \leq c_2 \quad \forall x, y \in \mathbb{R}.$$

Linear functions satisfy this property with $c_1 = c_2 = 1$. So when these constants are close to 1, the functions are almost linear. Conjecture: all these functions are $(C, 1)$ -good for some C which is a function of c_1, c_2 (in fact, probably of their ratio). Find some bound for C , or maybe even the best possible bound, identifying functions which are 'worst possible' in this sense. Of course in this set-up it won't be possible to reconstruct functions based on their values at two points, but still some estimates can be cooked up. The tool of course will be Taylor's first order approximation, maybe the Mean Value Theorem.

5.10. What about analogues of all this for functions of two (or more) variables? the definition of good functions remains the same, with intervals B replaced by balls or squares. What was done for linear functions is pretty straightforward to generalize, but what about higher degree polynomials? what is the optimal α for quadratic polynomials of two variables? what are 'worst possible' polynomials now? Lagrange interpolation won't work just as easily in several variables, so new tricks and methods are needed!

6. THE BOX PRINCIPLE

Reading material: [E, pp. 59–83].

Due date: November 14, 2011.

6.1. Each of 102 students in a school is a friend of at least 68 others. Prove that among them one can choose four who have the same number of friends. (Assume that friendship is a symmetric relation.)

6.2. 2011 different prime numbers form an arithmetic progression, and the smallest of the numbers is greater than 2011. Prove that the common difference of the progression is divisible by 2011.

6.3. Suppose that a quadratic polynomial $ax^2 + bx + c$ is equal to the fourth power of an integer for all positive integers x . Prove that $a = b = 0$.

6.4. 2000 real numbers are written in a row. Prove that it is possible to choose a subset consisting of adjacent numbers (maybe of just one number) such that the sum of numbers from this subset differs from an integer by no more than $1/1000$.

6.5. An open subset U of $[0, 1]$ has the property that for any $x, y \in U$, the distance between x and y is different from $1/10$. Prove that the measure (length) of U is not greater than $1/2$.

6.6. 50 segments are given on a straight line. Suppose that no point on the line belongs to more than 7 of the segments. Prove that one can find 8 of them which are pairwise disjoint.

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