

ON SPLITTING UP PILES OF STONES

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ABSTRACT. In this paper, I describe the rules of a game, and give a complete description of when the game can be won, and when it cannot be won. The first section describes a position from which the game cannot be won, and the second proves that the game can

1. INTRODUCTION

Throughout this paper, I will consider a game described in the following manner.¹ You are given a finite collection of piles of stones, with a (finite) integer number of stones in each pile. In every move you may either take two piles and combine them (replace two piles with a single pile which has a number of stones equal to the sum of the numbers of stones in the two piles), or take a pile with an even number of stones in it, and split it into two equal piles (replace it with two piles that each have half the number of stones that it had before it was replaced)

Definition 1.1. *The game is **won** if every pile has exactly one stone in it.*

Definition 1.2. *The game is **lost** if there exists an odd prime p such that p divides the number of stones in every one of the piles.*

Definition 1.3. *The game is **winnable** if there exists a sequence of moves leading to the game being won.*

2. LOSING THE GAME

Theorem 2.1. *If the game is lost, then it is not winnable.*

Proof. If an odd prime p divides every integer in a set of integers, then it also divides any sum of two of those integers, and it also divides half of any even integer in the set of integers. Hence no sequence of moves can lead to the game not being lost. (It is obvious that the game is not won if it is lost) \square

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¹ See the original problem at [Link](#)

Theorem 2.2. *For any n there exists a set of n piles such that the game is not winnable but it is also not lost.*

Proof. $\forall i, j$ with $1 \leq i < j \leq n$ choose odd primes $p_{i,j}$ such that $p_{i,j} \neq p_{i',j'}$ if $(i, j) \neq (i', j')$. Then, by the Chinese remainder theorem, choose a_k such that:

- (a) $a_k \equiv 1 \pmod{2}$
- (b) $a_k \equiv 1 \pmod{p_{i,j}}$ for $i = k$
- (c) $a_k \equiv -1 \pmod{p_{i,j}}$ for $j = k$
- (d) $a_k \equiv 0 \pmod{p_{i,j}}$ for $i, j \neq k$
- (e) there is no prime which divides all the a_k

Then, choosing piles with number of stones a_1, \dots, a_n , we have that

- (i) the game is not lost (by (e))
- (ii) it is not a legal move to divide a pile into two piles (by (a))
- (iii) if any two piles a_i and a_j are combined, then the game is lost.
(Because then $p_{i,j}$ divides the number of stones in every pile by (b), (c), (d))

Hence, after any first move, the game is lost. \square

3. WINNING THE GAME

Lemma 3.1. *If the game is not lost, and there are 2 piles with the same number of stones, then the game is winnable.*

Proof. The proof is by induction on the size of the largest pile.

The base case is obvious: if the size of the largest pile is 1 (or 0), then the game is won.

Note also that we may assume that all the piles have an odd number of stones. (Dividing piles into more piles if necessary)

Case 1: First, assume there is only one pile of maximal size.

In this case, let the piles have numbers of stones a_0, \dots, a_n with $a_0 = a_1$ and $a_n > a_i \forall i \neq n$

Then, combining the 0th and n th piles and then splitting the combined pile into two piles of size $\frac{a_0+a_n}{2}$ reduces the size of the largest pile (since $\frac{a_0+a_n}{2} < a_n$) Furthermore, the game is not lost after these two moves, because any odd prime p that divides $\frac{a_0+a_n}{2}$ and a_1 must divide a_n and hence $\exists i \neq 0, 1, n$ such that p does not divide a_i . Hence, by induction, the game is winnable.

Case 2: Now, assume there are multiple piles of maximal size

In this case, the proof is by induction on the number of piles of maximal size. (Case 1 provides the base case for the induction)

Then, let the piles have numbers of stones a_0, \dots, a_n with

$$a_0 = a_1 > a_i \quad \forall i \text{ such that } a_i \neq a_0$$

Assume, without loss of generality, that $a_n \neq a_0$ (otherwise all piles have the same size, so they must have one stone each, or else the game is lost) Then, just as in case one, replace the 0th and n th piles with two piles of size $\frac{a_0+a_n}{2}$. This reduces the number of piles with size a_0 , completing the induction. □

Theorem 3.2. *The game is winnable \iff either the game is won, or there exists a first move such that the game is not lost after the first move.*

Proof. The \Rightarrow implication is obvious, since otherwise the game is automatically lost after the first move.

The \Leftarrow is as follows:

If there exists a first move such that the game is not lost, then the game is not lost, and either there is a pile with an even number of stones in it or there isn't.

If there is one, then dividing that pile into two equal piles will result in a position with two piles of equal size such that the game is still not lost, and hence by the lemma, the game is winnable by that position.

On the other hand, if there is no pile with an even number of stones, then there exist two piles such that combining them does not lose the game (since the first move must involve combining two piles) This reduces the problem to the first case, since then dividing this pile in half still does not lose the game. □

Thus, we have that the game is winnable unless it is obviously not winnable in one of the two ways described in section 2.

4. THE HARDER GAME

At this point, I define a new game (**the n -game**) in which two piles can still be combined, but instead of being able to divide an even pile into two piles, you are able to divide a pile with a multiple of n stones in it into n equal piles. The definitions of **won** and **winnable** remain the same, although now we say that the game is **lost** if there is a prime p not dividing n such that p divides the number of stones in every pile.

First, I extend Lemma 3.1 to a new theorem for the n -game

Theorem 4.1. *If the game is not lost, and there are n piles all with the same number of stones, then the n -game is winnable.*

Proof. The proof is almost exactly as for the 2-game, by induction on the size of the largest pile and by secondary induction on the number of times that the largest pile is repeated.

Case 1: Assume that the pile that is repeated n times has a number of stones that is prime to n

In this case, just as in the proof of Lemma 3.1, add copies of this pile to the largest pile until that pile has a number of stones divisible by n and then divide that pile into n piles. (If the pile that is repeated n times is the largest pile, add copies of it to any other pile until you can divide that pile into n piles, reducing the number of times that the largest pile is repeated)

Case 2: Now, assume that $(a_0, n) = m$ where $m > 1$ and a_0 is the number of stones in the pile that is repeated n times

In this case, combine $\frac{n}{m}$ copies of the repeated pile together, and then divide this pile into n equal piles. Repeat if the n new piles still have size that is not prime to n . (This process obviously terminates eventually, since it decreases the size of the repeated pile, also it does not lose the game, since the set of all sizes of piles after the process contains the set of all sizes of piles before the process)

□

Corollary 4.2. *If there exist n piles that can all be combined without losing the game, then the game is winnable.*

Proof. This follows immediately from the theorem, since there is a subset of the n piles such that combining the piles of this subset gives a pile with number of stones divisible by n

□

One might hope that Theorem 3.2 would generalize to the n game as a theorem of the form "if there exist m moves such that the game is not lost after them, then the game is winnable" where m is a constant depending only on n but this turns out not to be the case.

Theorem 4.3. *For $n > 2$, for any $m > 0$ there exists a position in the n -game such that there exist m moves such that the game is not lost after those moves, but the game is not winnable.*

Proof. The proof is by iterated application of the Chinese remainder theorem.

$\forall i, j$ with $1 \leq i \neq j \leq k$ choose distinct odd primes $p_{i,j}, q_{i,j}, r_{i,j}$ such that none of the chosen primes divide n . Then, by the Chinese Remainder Theorem, choose a_k, b_k $1 \leq k \leq m - 1$ such that:

- (a1) $a_k \equiv 1 \pmod n$
- (a2) $b_k \equiv 1 \pmod n$
- (b1) $a_k \equiv 1 \pmod{p_{i,j}}$ for $i = k$
- (b2) $a_k \equiv -1 \pmod{p_{i,j}}$ for $j = k$
- (b3) $a_k \equiv 0 \pmod{p_{i,j}}$ for $i, j \neq k$
- (b4) $b_k \equiv 0 \pmod{p_{i,j}}$ for all k
- (c1) $b_k \equiv 1 \pmod{q_{i,j}}$ for $i = k$
- (c2) $b_k \equiv -1 \pmod{q_{i,j}}$ for $j = k$
- (c3) $b_k \equiv 0 \pmod{q_{i,j}}$ for $i, j \neq k$
- (c4) $a_k \equiv 0 \pmod{p_{i,j}}$ for all k
- (d1) $a_k \equiv 1 \pmod{r_{i,j}}$ for $i = k$
- (d2) $b_k \equiv -1 \pmod{r_{i,j}}$ for $j = k$
- (d3) $a_k \equiv 0 \pmod{r_{i,j}}$ for $i, j \neq k$
- (d4) $b_k \equiv 0 \pmod{r_{i,j}}$ for $i, j \neq k$

Finally, having chosen $a_1, b_1, \dots, a_{m-1}, b_{m-1}$, let S be the set of all primes p such that $p \leq \max\{a_k + b_k\}$, and such that p is not one of the $p_{i,j}, q_{i,j}, r_{i,j}$. Then apply the Chinese Remainder Theorem one more time to choose a_m, b_m such that:

- (a1') $a_m \equiv 1 \pmod n$
- (a2') $b_m \equiv 1 \pmod n$
- (b1') $a_m \equiv 1 \pmod{p_{i,j}}$ for $i = m$
- (b2') $a_m \equiv -1 \pmod{p_{i,j}}$ for $j = m$
- (b3') $a_m \equiv 0 \pmod{p_{i,j}}$ for $i, j \neq m$
- (b4') $b_m \equiv 0 \pmod{p_{i,j}}$ for all m
- (c1') $b_m \equiv 1 \pmod{q_{i,j}}$ for $i = m$
- (c2') $b_m \equiv -1 \pmod{q_{i,j}}$ for $j = m$
- (c3') $b_m \equiv 0 \pmod{q_{i,j}}$ for $i, j \neq m$
- (c4') $a_m \equiv 0 \pmod{p_{i,j}}$ for all m
- (d1') $a_m \equiv 1 \pmod{r_{i,j}}$ for $i = m$
- (d2') $b_m \equiv -1 \pmod{r_{i,j}}$ for $j = m$
- (d3') $a_m \equiv 0 \pmod{r_{i,j}}$ for $i, j \neq m$
- (d4') $b_m \equiv 0 \pmod{r_{i,j}}$ for $i, j \neq m$
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- (d4') $b_m \equiv 0 \pmod{r_{i,j}}$ for $i, j \neq m$
- (e1) $a_m \equiv 1 \pmod p$ for $p \in S$

(e2) $b_m \equiv 0 \pmod p$ for $p \in S$

Then, choosing $2m$ piles with sizes a_k, b_k , we have that:

- (i) There exist m moves such that the game is not lost after those m moves
- (ii) The game is not winnable

Condition (e), along with the other conditions, ensure that the game is not lost after combining a_k with b_k for all k . ($\forall i, a_i + b_i \equiv 1 \pmod{p_{i,j}}, q_{i,j}$, and $r_{i,j}$. Also $\forall p \in S, a_m + b_m \equiv 1 \pmod p$. Any other prime is too large to divide the sizes of any of the piles except for the m th pile.) Conditions (b) - (d) ensure that combining any two piles with different indices will result in losing the game.

Finally, condition (a) ensures that none of the created piles have size divisible by n (since $n > 2$). So therefore the only legal move throughout the game is to combine piles until the game is lost.

□

Note also that the same kind of situation can be created in which piles can be combined $n-1$ at a time without the game being winnable, but it requires double indices on the constants and quadruple indices on the primes, so I chose to omit this, for the sake of clarity.

This proves that Corollary 4.2 is more or less the best result along the lines of Theorem 3.2 that can be obtained for the n -game, although Theorem 4.1 is probably the easiest way to determine whether a position is winnable or not.

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