MATHEMATICS IN SUDOKU, FALL 2005

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Abstract. Our first task will be to compute the number of 4×4 Sudoku and Sudoku X grids modulo symmetries. We will begin by looking at the number of grids ignoring certain symmetries, and then we will try to find an intelligent way to show equivalencies. Finally, we will consider Sudoku X grids as matrices and look at their determinants.

1. Introduction

The game of Sudoku was created in 1974 as "Number Place" by retired architect Howard Garns for Dell Magazines. The rules have lasted these 30 years [EP]. The puzzle is to fill a 9×9 grid that contains some given entries so that every column, row, and 3×3 box contains the digits 1 through 9. The game has been generalized to any such $n^2 \times n^2$ grid that is filled so that every column, row, and $n \times n$ box contains the digits 1 through n^2. Each grid is a type of Latin square that illustrates a coloring of a graph of n^2 vertices where each vertex has $n - 1$ neighbors. There are variants on Sudoku that impose rules in addition to the classical ones; one such variation is Sudoku X, which requires the main diagonals of the grid contain the digits 1 through n^2 as well.

2. Counting Grids

With the recent surge of popularity that Sudoku is experiencing, much work is being done on the enumeration of Sudoku grids. In May 2005, the number of ways of filling a blank 9×9 Sudoku grid was calculated by Felgenhauer and Jarvis to be approximately 6.67×10^{21} [FJ]. Here, let us look at ways to count the number of 4×4 grids.

1	2	4	3
3	4	2	1
2	3	1	4
4	1	3	2

(2.1)

Any 4×4 Sudoku grid is contains the numbers 1 through 4 placed in a box as in figure 2.1, where each row, column, and quadrant contains the set $\{1, 2, 3, 4\}$. Were there no such restrictive edges in this graph, the number of colorings would be $\dfrac{16!}{4! \times 4! \times 4! \times 4!} = 63063000$. So how many of these are valid Sudoku grids?

We can start by coloring the upper-left quadrant of a grid without restrictions and include in our computation that there are 4! ways of coloring the one quadrant:

1	2	*	*
3	4	*	*
*	*	*	*
*	*	*	*

Now there are 12 valid ways of coloring the rest of the square:

1	**2**	4	3
3	**4**	2	1
2	3	**1**	4
4	1	3	**2**

1	**2**	3	4
3	**4**	2	1
4	3	**1**	2
2	1	4	**3**

1	**2**	4	3
3	**4**	2	1
4	3	**1**	2
2	1	3	**4**

1	**2**	3	4
3	**4**	1	2
4	1	**2**	3
2	3	4	**1**

1	**2**	3	4
3	**4**	1	2
4	3	**2**	1
2	1	4	**3**

1	**2**	4	3
3	**4**	1	2
4	3	**2**	1
2	1	3	**4**

1	**2**	4	3
3	**4**	1	2
2	1	**3**	4
4	3	2	**1**

1	**2**	4	3
3	**4**	2	1
2	1	**3**	4
4	3	1	**2**

1	**2**	4	3
3	**4**	2	1
4	1	**3**	2
2	3	1	**4**

1	**2**	3	4
3	**4**	1	2
2	1	**4**	3
4	3	2	**1**

1	**2**	3	4
3	**4**	2	1
2	1	**4**	3
4	3	1	**2**

1	**2**	3	4
3	**4**	1	2
2	3	**4**	1
4	1	2	**3**

The colorings are completely determined by the remaining two values on the main diagonal, and there are $4 \times 3 = 12$ ways of choosing those two values. Thus, there are $4! \times 12 = 288$ different valid Sudoku grids, ignoring symmetries.

3. Counting Grids Modulo Symmetries

So let us consider operations (or sequence of such operations) on the vertices of the Sudoku graph that do not change the validity of the graph. These operations are:

(a) a relabelling the numbers by a permutation, $p \in S_4$
(b) a swap of two rows or two columns common to a box
(c) a swap of two rows or two columns of boxes
(d) a rotation or flip the entire grid

So we can already see that there are $\frac{288}{4!} = 12$ colorings modulo relabelling. To more efficiently discuss the grids, let us label each one by a pair (a, b) corresponding to the values on the main diagonal in the lower-right quadrant. Now, by the equivalence relation of (b), we can see that $(1,2) \equiv (4,3)$, $(1,3) \equiv (2,4)$, $(1,4) \equiv (2,3)$, $(2,1) \equiv (3,4)$, $(3,1) \equiv (4,2)$, and $(3,2) \equiv (4,1)$. So now we have narrowed our grid count to only 6 equivalence classes. Finally, by combining operations (d) and (a), we find more equivalences: $(1,2) \equiv (1,3)$, $(3,1) \equiv (1,2)$, $(1,2) \equiv (3,4)$, and $(3,2) \equiv (1,4)$. We end up with two classes of grids, represented by $(1,2)$ and $(1,4)$ that are not equivalent by any sequence of our operations.

4. A Better Counting Method

Now let us construct non-equivalent grids more methodically to verify our conclusion above. First of all, we can again assume the first quadrant to be fixed as

1	2	*	*
3	4	*	*
*	*	*	*
*	*	*	*

since two grids are considered equivalent under permuting the numbers (1). Now the rest of the first row is either $(3,4)$ or $(4,3)$. By (2), we can assume it is $(3,4)$. Similarly, the rest of the first column is either $(2,4)$ or $(4,2)$, and we can assume it to be $(2,4)$. So far our grid is:

1	2	3	4
3	4	*	*
2	*	*	*
4	*	*	*

The last 4 has only one possible location, so our grid is now:

1	2	3	4
3	4	$*$	$*$
2	$*$	4	$*$
4	$*$	$*$	$*$

In the lower right-hand corner, we have the values $\{1, 2, 3\}$ from which to choose. Suppose this corner has value 3. Then we can permute the numbers by $p = (2, 3)$ to get:

1	3	2	4
2	4	$*$	$*$
3	$*$	4	$*$
4	$*$	$*$	2

This grid we can reflect on its main diagonal (by operation (d)) to get:

1	2	3	4
3	4	$*$	$*$
2	$*$	4	$*$
4	$*$	$*$	2

So choosing 2 or 3 for the lower right-hand corner results in equivalent grids. Thus, we may assume the value is 1 or 2. The two valid non-equivalent grids that result are:

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

5. Sudoku X Enumeration

There are many variants on the game of Sudoku, each of which modifies or adds its own rules. One variant is sometimes called Sudoku X, and this game adds more edges to the Sudoku graph. Now, the main diagonals of the Sudoku grid are groups as well and must be filled by all the values. Again, let us look at the number of 4×4 grids. We can begin by filling the first box without restrictions. Then, we have 2 choices (from $\{3, 4\}$) for the third entry in the first row. And once we have chosen it, the grid is uniquely determined. So there are $4! \times 2 = 48$ different grids. We can limit our count to non-equivalent grids, but for Sudoku X, not all of our Sudoku equivalency operations map valid grids to valid grids. So the operations that define equivalency for Sudoku

X grids here are permutations of the entry values, and rotations and flips of the grid. Now as we saw earlier, $(2,3) \equiv (3,2)$ by rotating $(2,3)$ counterclockwise by 90^o and then relabelling the entry values by the permutation $p = (14)$. So all Sudoku X grids are equivalent under these operations.

6. Sudoku X Determinants

Now let us consider our Sudoku X grids as matrices in the obvious way. A natural function on square matrices is the determinant; so what are the possible values of the determinant of these matrices? The two Sudoku X grids with $((a,b),(c,d))$ in the first box are:

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

a	b	d	c
c	d	b	a
b	a	c	d
d	c	a	b

The determinant of both of these matrices is $a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2b^2c^2-2a^2d^2-2b^2d^2-2c^2d^2+8abcd$, which is symmetric in a,b,c,d, so since all the other grids can be obtained by permuting the entries of these two grids, the determinant is invariant for all the Sudoku X grids. In the standard case, where a,b,c,d are mapped bijectively to $1,2,3,4$, the determinant is always 0.

One might conjecture then that any $n^2 \times n^2$ Sudoku X grid has determinant zero; however, this is not the case:

5	4	7	2	1	6	3	9	8
2	1	6	3	9	8	5	4	7
3	9	8	5	4	7	2	1	6
4	5	2	7	6	1	9	8	3
7	6	9	8	3	4	1	5	2
1	8	3	9	5	2	6	7	4
8	2	5	6	7	9	4	3	1
9	7	4	1	2	3	8	6	5
6	3	1	4	8	5	7	2	9

(6.1)

The determinant of the 9×9 Sudoku X grid in 6.1 is -9484290, and the determinant of the same grid with entries 1 and 2 permuted is 4602420.

References

[FJ] B. Felgenhauer and F. Jarvis, **Enumerating possible Sudoku grids**, http://www.shef.ac.uk/ pm1afj/sudoku/sudoku.pdf, 2005.

[EP] Ed Pegg Jr. **Math Games: Sudoku Variations**, http://www.maa.org/editorial/mathgames/mathgames_09_05_05.html, 2005.

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