Chapter 9

Refinements of multiple recurrence

9.1. Sets of topological recurrence

We have already showed that an arbitrary system (X, T) contains recurrent points and then improved this to show that the system contains uniformly recurrent points. In a different direction, we showed that any system contains multiply recurrent points. We now proceed in another direction, showing that an arbitrary system contains points that recur with restrictions on the return times, meaning restrictions on the iterates that demonstrate the recurrence.

A set $R \subseteq \mathbb{N}$ is said to be a *set of (topological) recurrence* if given any dynamical system (X, T) and $\varepsilon > 0$, there exist $x \in X$ and $n \in R$ such that $d(T^{-n}x, x) < \varepsilon$.

A trivial example of a set of recurrence is the set of all integers. The multiples of any integer $p \in \mathbb{N}$ is also a set of topological recurrence, as can be seen by considering the system (X, T^p) instead of (X, T). Other more interesting examples require more work.

Proposition 9.1. *If $R \subseteq \mathbb{N}$, the following are equivalent:*

(1) R is a set of topological recurrence.

(2) Given any minimal dynamical system (X, T) and any nonempty open set $U \subseteq X$, there exists $n \in R$ such that $U \cap T^{-n}U \neq \varnothing$.

(3) *Given any minimal dynamical system (X, T) and any $\varepsilon > 0$, there is a dense set $Y \subseteq X$ such that for each $y \in X$, there exists $n \in R$ such that $d(y, T^{-n}y) < \varepsilon$.*

(4) *For any finite partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$, there is some $j \in \{1, 2, \ldots, r\}$ such that the difference set $C_j - C_j \cap R \neq \varnothing$.*

(5) *If $A \subseteq \mathbb{N}$ is any syndetic set, then $A - A \cap R \neq \varnothing$.*

(6) *If $A \subseteq \mathbb{N}$ is any piecewise syndetic set, then $A - A \cap R \neq \varnothing$.*

A set that satisfies condition (4) is also known as a *chromatically intersective* set.

Proof. (1) $\iff$ (2) $\iff$ (3) This follows from an argument similar to that used in Lemma 8.9 (Exercise 9.2).

(1) $\iff$ (4) Assume that $R \subseteq \mathbb{N}$ is a set of topological recurrence. Let $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ be a finite partition. Set $\Lambda = \{1, 2, \ldots, r\}$ and consider the system $(\Lambda^{\mathbb{N}}, T)$, where $T\colon \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ is the shift. Define the point $x \in \Lambda^{\mathbb{N}}$ by $x_n = j$ if and only if $n \in C_j$ and let $X = \overline{\{T^n x\colon n \in \mathbb{N}\}}$. Let $y \in X$ be a uniformly recurrent point (under T) and so $\overline{\{T^n y\colon n \in \mathbb{N}\}}$ is minimal. Set $U = \{w \in X\colon w_0 = y_0\}$. Then U is an open neighborhood of y and so there exists $n \in R$ such that $U \cap T^n U \neq \varnothing$. This means that there exists $w \in U$ such that $w_0 = w_n$. But since $w \in X$, there is some $m \in \mathbb{N}$ such that $x_m = x_{m+n}$, meaning that m and $m + n$ lie in the same piece of the partition that as x_m. Therefore R is a set of chromatic recurrence.

Conversely, assume that for any finite partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$, there is some $j \in \{1, 2, \ldots, r\}$ such that $C_j - C_j \cap R \neq \varnothing$. Let (X, T) be a minimal dynamical system and assume that $U \subseteq X$ is open. By Proposition 4.9, there exists $r \in \mathbb{N}$ such that $\bigcup_{j=1}^{r} T^j U = X$. Fix some $x \in X$ and define a partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ by setting $n \in C_j$ when $T^n x \in T^j U$. (If some iterate $T^n x$ lies in more than one set, then choose one arbitrarily.) By assumption, for some $n \in R$ and $j \in \{1, 2, \ldots, r\}$, there exists $m \in C_j$ such that we also have $m + n \in C_j$. Then $T^m x$ and $T^{m+n} x$ lie in the same set $T^j U$ and so in particular, $U \cap T^r U \neq \varnothing$.

(4) $\iff$ (5) The set $A \subseteq \mathbb{N}$ is syndetic if and only if there exists $r \in \mathbb{N}$ such that r translates of A cover all of $\mathbb{N}$. Thus we can define a partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$, by setting $C_1 = A$, $C_2 = (A + 1) \setminus C_1, C_2 = (A + 2) \setminus (C_1 \cup C_2), \ldots, C_r = (A + r) \setminus (C_1 \cup \ldots \cup C_{r-1})$. Then for some $j \in \{1, 2, \ldots, r\}$, $C_j - C_j \cap A \neq \varnothing$ if and only if there exist $a_1, a_1 \in A$ such that $(a_1 + j) - (a_2 + j) \in A$. But this means exactly that $a_1 - a_2 \in A$.

(5) $\iff$ (6) Exercise 9.3. □

Sets of recurrence satisfy the Ramsey Property:

Proposition 9.2. *If $R \subseteq \mathbb{N}$ is a set of topological recurrence and $R = R_1 \cup R_2 \cup \ldots \cup R_r$, then for some $j \in \{1, 2, \ldots, r\}$, R_j is a set of topological recurrence.*

Proof. It suffices to prove the statement for $r = 2$. We show the contrapositive. Namely, we assume that $R_1, R_2 \subseteq \mathbb{N}$ are not sets of topological recurrence and show that $R = R_1 \cup R_2$ is not a set of topological recurrence.

By condition (2) of Proposition 9.1, there exist dynamical systems (X_1, T_1) and (X_2, T_2) and nonempty open sets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ such that $U_1 \cap T_1^{-n} U_1 = \varnothing$ for all $n \in R_1$ and $U_2 \cap T_2^{-n} U_2 = \varnothing$ for all $n \in R_2$. Set $X = X_1 \times X_2$ and $T = T_1 \times T_2$. Then $U_1 \times U_2$ is a nonempty open set in the dynamical system (X, T), and $U_1 \times U_2 \cap T^{-n}(U_1 \times U_2) = \varnothing$ for all $n \in R$. Thus R is not a set of topological recurrence. □

In order to determine if $A \subseteq \mathbb{Z}$ is a set of recurrence, it suffices to consider minimal dynamical systems, since any system contains minimal subsystems. In this case, we can say more about the recurrent points:

Proposition 9.3. *IF A is a set of recurrence, then for any minimal dynamical system (X, T) and any $\varepsilon > 0$, the set of x such that $d(T^n x, x) < \varepsilon$ is dense and open.*

Proof. Let $U \subset X$ be open. By minimality, there exists $N \in \mathbb{N}$ such that for all $x \in X$, there is some $0 \leq n \leq N$ such that $T^n x \in U$. Since T^n is uniformly continuous, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(T^n x, T^n y) < \varepsilon$ for all $0 \leq n \leq N$.

Choose $x_0 \in X$ and $n \in A$ such that $d(T^n x_0, x_0), \delta$. Then for some $0 \leq n_0 \leq N$, $T^{n_0} x_0 = y_0 \in U$ and so $d(T^n y_0, y_0) = d(T^{n+n_0} x_0, T^{n_0} x_0) < \varepsilon$. Therefore, points that return to themselves within ε are dense and open. By intersecting over a sequence of $\varepsilon \to 0$, we have a dense and open set of points in X such that $\inf_{n \in A} d(T^n x, x) = 0$. □

We can also define a notion of higher order recurrence for restricted iterates:

Definition 9.4. A set $R \subseteq \mathbb{N}$ is a *set of topological k-recurrence* if for any dynamical system (X, T) and $\varepsilon > 0$, there exist $x \in U$ and $n \in R$ such that $d(T^n x, x) < \varepsilon, d(T^{2n} x, x) < \varepsilon, \ldots, d(T^{kn} x, x) < \varepsilon$.

Analogous to Proposition 9.1, there are many equivalent formulations for sets of topological k-recurrence:

Proposition 9.5. *If $R \subseteq \mathbb{N}$, the following are equivalent:*

(1) R is a set of topological k-recurrence.

(2) *Given any minimal dynamical system* (X, T) *and any nonempty open set* $U \subseteq X$*, there exists* $n \in R$ *such that* $U \cap T^{-n}U \cap T^{-2n}U \cap \ldots \cap T^{-kn}U \neq \varnothing$.

(3) *Given any minimal dynamical system* (X, T) *and* $\varepsilon > 0$*, there is a dense set* $Y \subseteq X$ *such that for each* $y \in X$*, there exists* $n \in R$ *such that* $d(T^n y, y) < \varepsilon, d(T^{2n}y, y) < \varepsilon, \ldots, d(T^{kn}y, y) < \varepsilon$.

(4) *For any finite partition* $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$*, there is some* $j \in \{1, 2, \ldots, r\}$ *that contains an arithmetic progression of length* k *with common difference in* R.

(5) *If* $A \subseteq \mathbb{N}$ *is any syndetic set, then* A *contains an arithmetic progression of length* k *with difference in* R.

(6) *If* $A \subseteq \mathbb{N}$ *is any piecewise syndetic set, then* A *contains an arithmetic progression of length* k *with common difference in* R.

Proof. Exercise 9.12. □

9.2. IP sets

Starting with an arbitrary dynamical system, we find recurrent points in the product of this system and some other system. When the second system has particular properties, we then gain information about recurrence properties in the original system. We make this precise.

Proposition 9.6. *Let* (X, T) *and* (Y, S) *be dynamical systems. Let* $y_0 \in Y$ *be a recurrent point and for each* $\varepsilon > 0$*, set*

$$R_\varepsilon = \{n \in \mathbb{N} \colon d(S^n y_0, y_0) < \varepsilon\}.$$

Then for all $\varepsilon > 0$*, there exists* $x \in X$ *and* $n_j \in R$ *such that* $d(T^{n_j}x, x) < \varepsilon$.

Proof. Since any dynamical system contains a minimal system, we can assume without loss that (X, T) itself is minimal. Define $\tilde{X} = X \times Y$ and $\tilde{T} \colon \tilde{X} \to \tilde{X}$ by $\tilde{T}(x, y) = (Tx, Sy)$. Since X is minimal and $y_0 \in Y$ is recurrent, for any $\varepsilon > 0$ and any point $(x, y_0) \in X \times y_0$, there exists $(x', y_0) \in X \times y_0$ and $n \in \mathbb{N}$ such that $d\big(\tilde{T}^n(x', y_0), (x, y_0)\big) < \varepsilon$. Let $\tilde{S} \colon \tilde{X} \to \tilde{X}$ be defined by $\tilde{S}(x, y) = (Tx, y)$. Then $\tilde{S}$ commutes with $\tilde{T}$ and $(X \times y_0, \tilde{S})$ is minimal. Thus every point $(x, y_0) \in X \times y_0$ is recurrent. But then $X \times y_0$ contains a recurrent point under $\tilde{T}$ and the statement of the proposition follows. □

The dynamical version of van der Waerden's Theorem (Theorem 8.6) is a recurrence statement for iterates $T, T^2, \ldots, T^k$. This can now be done with restrictions on the iterates. The proof is left to Exercise 9.14.

Theorem 9.7. *Assume that* (X,T) *and* (Y,S) *are dynamical systems and* $y_0 \in Y$ *is recurrent. For* $\varepsilon > 0$, *set*

$$R_\varepsilon = \{n \in \mathbb{N} \colon d(T^n y_0, y_0) < \varepsilon\}.$$

Then for all $\varepsilon > 0$, *there exists* $x \in X$ *and and* $n_j \in R_\varepsilon$ *such that* $d(T^{in_j}x, x) < \varepsilon$ *for* $i = 1, 2, \ldots, k$.

Definition 9.8. An IP-set A is a subset of $\mathbb{N}$ defined by an infinite set of generators $P = \{p_1 < p_2 < \ldots\}$ such that

$$A = \{\sum_{j=1}^{k} p_{i_j} : k \in \mathbb{N}, p_{i_j} \in P\}.$$

The return times that arise in this theorem are essentially IP sets, and this is made precise in the next two propositions:

Proposition 9.9. *Let* (Y,S) *be a dynamical system and assume that* $y_0 \in Y$ *is recurrent. Then for all* $\varepsilon > 0$, *the set*

$$R_\varepsilon = \{n \in \mathbb{N} \colon d(T^n y_0, y_0) < \varepsilon\}$$

contains an IP *set.*

Proof. Fix $\varepsilon > 0$ and assume that y_0 is recurrent in (Y,S). Choose $p_1 \in \mathbb{N}$ such that $d(T^{p_1}y_0, y_0) < \varepsilon$. Choose ε_2 with $0 < \varepsilon_2 \le \varepsilon$ such that if $d(y, y_0) < \varepsilon_2$, then $d(T^{p_1}y, y_0) < \varepsilon$. Now choose $p_2 \in \mathbb{N}$ such that $d(T^{p_2}y_0, y_0) < \varepsilon_2$. Then $d(T^{p_1}y_0, y_0)$, $d(T^{p_2}y_0, y_0)$ and $d(T^{p_1+p_2}y_0, y_0)$ are all bounded by ε.

We continue inductively. Assume that $p_1, p_2, \ldots, p_n$ have been defined such that

$$d(T^m y_0, y_0) < \varepsilon \tag{9.1}$$

for all $m = p_{i_1} + p_{i_2} + \ldots + p_{i_k}$ with $i_1 < i_2 < \ldots < i_k \le n$. Choose $\varepsilon_{n+1} < \varepsilon$ such that if $d(y, y_0) < \varepsilon_{n+1}$, then $d(T^m y, y_0) < \varepsilon$ for all $m = p_{i_1} + p_{i_2} + \ldots + p_{i_k}$ with $i_1 < i_2 < \ldots < i_k \le n$. By recurrence of y_0, we can choose $p_{n+1} \in \mathbb{N}$ such that $d(T^{p_{n+1}}y_0, y_0) < \varepsilon_{n+1}$. Then Equation (9.1) still holds for p_{n+1} and $m + p_{n+1}$, where m is of the form given in this equation. Thus the IP set

$$R = \{p_{i_1} + p_{i_2} + \ldots + p_{i_k} : i_1 < i_2 < \ldots < i_k\}$$

is contained in R_ε. □

For the converse, we have:

Proposition 9.10. *If* $R \subseteq \mathbb{N}$ *is an* IP *set, there exist a system* (Y,S) *and a recurrent point* $y_0 \in Y$ *such that* $R \supset R_1$, *where*

$$R_1 = \{n \in \mathbb{N} \colon d(T^n y_0, y_0) < 1\}.$$

Proof. Assume that $R \subseteq \mathbb{N}$ is an IP set with generators $p_1 < p_2 < \ldots$. Thus

$$R = \{p_{i_1} + p_{i_2} + \ldots + p_{i_k} : i_1 < i_2 < \ldots < i_k\}.$$

If $A_1, A_2, \ldots$ is a sequence of disjoint finite subsets of $\mathbb{N}$ and $p'_n = \sum_{i \in A_n} p_i$, then

$$\{p'_{i_1} + p'_{i_2} + \ldots + p'_{i_k} : i_1 < i_2 < \ldots < i_k\}$$

is an IP set and it is a sub-IP set of R. (Note that we are grouping generators of the original IP set R together.) Thus we can define subsets of natural numbers A_n inductively such that p'_n can be taken as large as we want. In particular, without loss of generality, we can assume that $p'_{n+1} > p'_1 + p'_2 + \ldots + p'_n$. Call this IP set R'.

Define $Y = \{0, 1, \}^{\mathbb{N} \cup \{0\}}$ and choose a metric on Y such that $d(y, y') = 1$ if and only $y_0 \neq y'_0$. Let $T \colon Y \to Y$ be the shift. Define $y \in Y$ by

$$y_n = \begin{cases} 1 & \text{if } n \in R' \cup \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that y is recurrent. If $m = p'_{i+1} + p'_{i+2} + \ldots + p'_k$ for $k > j$, then $T^m y_n = 1$ if and only if $y_{m+n} = 1$, which is equivalent to $m + n \in R'$. By construction, $p'_{n+1} > p'_1 + p'_2 + \ldots + p'_n$ and so $p'_{i_1} + \ldots + p'_{i_k}$ and $p'_{j_1} + \ldots + p'_{j_\ell}$ in R' are equal if and only if $k = \ell$ and $i_1 = j_1, \ldots, i_k = j_k$. Thus for $n < p'_{j+1}$, we have $m + n \in R'$ if $n \in R'$. Thus $d(T^m y, y) \leq 1/p'_{j+1}$ and so y is recurrent in (Y, T). Furthermore, $d(T^n y, y) < 1$ if and only if $(T^n y)_0 = 1$, which happens if and only if $y_n = 1$, meaning that $n \in R'$. □

Using this, one can derive an IP-multidimensional van der Waerden Theorem (Exercise 9.15).

Theorem 9.11. *If $\mathbb{N}^m = B_1 \cup B_2 \cup \ldots \cup B_r$ is a finite partition and $R \subseteq \mathbb{N}$ is an* IP*-set, then some B_j has the property that if F is any finite subset of $\mathbb{N}^m$, there exsts $a \in \mathbb{N}^m$ and $b \in R$ such that $bF + a \subseteq B_j$.*

We give another direct (and slicker) proof of this in Section 9.5.

9.3. van der Corput collections

Definition 9.12. A set $P \subset \mathbb{Z}$ is called a *van der Corput collection* if for any sequence $\{u_n\}_{n \in \mathbb{N}}$, if the sequence $\{v_n = u_{n+h} - u_n\}_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 for each $h \in P$, then the sequence $\{u_n\}_{n \in \mathbb{N}}$ is also uniformly distributed modulo 1.

There are several equivalent formulations for a set being a van der Corput collection:

Theorem 9.13. *The following conditions are equivalent for a set $P \subseteq \mathbb{Z}$:*

(1) *P is a van der Corput collection*

(2) *For any positive measure μ on the circle with*

$$\widehat{\mu}(n) = \int e^{-in\theta}\mu(d\theta) = 0$$

for all $n \in P$, μ is continuous.

(3) *Whenever $\{y_n\}$ is a sequence of complex numbers satisfying*

$$\sum_{n\le x} |y_n|^2 = O(x)$$

and

$$\sum_{n\le x} y_{n+k}\overline{y_n} = o(x) \text{ for } k \in P,$$

then

$$\sum_{n\le x} y_n = o(x).$$

(4) *For all $\varepsilon > 0$, there is a polynomial*

$$P(x) = \sum_{n\in P\cup\{0\}} a_n \cos nx$$

with $a_n \in \mathbb{R}$ satisfying

$$P(x) \ge 0,\ \ P(0) = 1,\ \ a_0 \le \varepsilon.$$

9.4. Bohr recurrence

Definition 9.14. A set $R \subseteq \mathbb{N}$ is a set of *Bohr recurrence* if for every minimal rotation system (X, T) and open set $U \subset X$, there exists $r \in R$ such that $U \cap T^{-r}U \neq \varnothing$.

It is clear that a set of topological recurrence is a set of Bohr recurrence, but the converse statement is an open question:

Question 9.15. *If $R \subseteq \mathbb{N}$ is a set of Bohr recurrence, is R also a set of topological recurrence?*

One can easily check that a set R of Bohr recurrence is also a set of multiple Bohr recurrence, meaning that for any open set $U \subseteq X$ and any $k \geq 1$, there exists $r \in R$ such that $U \cap T^{-r}U \cap \ldots \cap T^{-kr}U \neq \varnothing$. One can ask the analogous question: does multiple Bohr recurrence imply multiple recurrence in general? However, the answer to this is no, as one can construct a set of Bohr recurrence that is not a set of double recurrence, even for some system that are group extensions of rotations.

9.5. IP-multiple recurrence for commuting maps

We generalize the notion of an IP set for an arbitrary semigroup.

Let $\mathcal{F}$ denote the collection of finite nonempty subsets of $\mathbb{N}$. A partial ordering on $\mathcal{F}$ is defined as follows: if $\alpha, \beta \in \mathcal{F}$, we say that $\alpha < \beta$ if $\max\{j : j \in \alpha\} < \min\{j : j \in \beta\}$. (Note that α and β are finite sets, and so this makes sense.)

Definition 9.16. An $\mathcal{F}$-*sequence* is a collection σ_α indexed by $\alpha \in \mathcal{F}$.

The simplest example of an $\mathcal{F}$-sequence is $\mathbb{N}$. A more interesting is given by an IP set.

Definition 9.17. If S is a semigroup, an IP-*system in* S is an $\mathcal{F}$-sequence $\{\sigma_\alpha\}_{\alpha \in \mathcal{F}}$ with values in S such that

$$\sigma_{\alpha \cup \beta} = \sigma_\alpha \sigma_\beta$$

if $\alpha < \beta$. Thus if $j_1 < j_2 < \ldots < j_k$ are integers, then $\sigma_{\{j_1, j_2, \ldots, j_k\}} = \sigma_{j_1} \sigma_{j_2} \ldots \sigma_{j_k}$.

This means that an arbitrary IP-system is generated by some sequence $\{\sigma_{(n)}\}_{n \in \mathbb{N}}$. For example, if we start with an $\mathcal{F}$-sequence given by an IP-set, then we can perform the operation $\sigma_{\alpha \cup \beta}$ if α and β have distinct generators.

We study IP-systems for a commutative group G that acts minimally on a compact space X: G is said to be a commutative group acting on the space X if G is a group such that for all $T \in G$, $T : X \to X$ is continuous and furthermore, for all $T, S \in G$, $T \circ S = S \circ T$. Recall that minimality means that $\{gx : g \in G\}$ is dense in X for all $x \in X$.

Theorem 9.18. *Assume that G is a commutative group of transformations acting minimally on a compact space X, let $k \in \mathbb{N}$, and let $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(k)}$ be* IP-*systems in G. If $U \subseteq X$ is a nonempty open set and $\alpha \in \mathcal{F}$, then there exists $\beta \in \mathcal{F}$ with $\alpha < \beta$ such that*

$$U \cap \sigma_\beta^{(1)} U \cap \sigma_\beta^{(2)} U \ldots \cap \sigma_\beta^{(k)} U \neq \varnothing.$$

This theorem includes the dynamical version of van der Waerden's Theorem given in Theorem 8.6 and the IP version given in Theorem 9.11 (Exercise 9.1).

Proof. Let U be a nonempty open set and let $\alpha \in \mathcal{F}$. Since G acts minimally on X, by Exercise 4.20, there exist $g_1, g_2, \ldots, g_\ell \in G$ such that

$$X = \bigcup_{j=1}^{\ell} g_j(U). \tag{9.2}$$

We proceed by induction on the number of IP-systems. Consider a single IP-system σ_α. We define a sequence $W_0, W_1, \ldots$ of nonempty open sets in X inductively. Set $W_0 = U$ and assume that we have already defined W_n. Using the partition of X given in Equation (9.2), we can choose i_{n+1} with $1 \leq i_{n+1} \leq \ell$ such that $\sigma_{n+1}W_n \cap g_{i_{n+1}}U \neq \varnothing$. Define $W_{n+1} = \sigma_{n+1}W_n \cap g_{i_{n+1}}U$. This sequence satisfies:

(1) $W_0 = U$;

(2) $\sigma_n^{-1}W_n \subseteq W_{n-1}$ for all $n \geq 1$;

(3) For each $n \in$, W_n is contained in one of the sets $g_{i_n}(U)$, where $i_n \in \{1, 2, \ldots, \ell\}$.

Since each W_n is contained in one of the ℓ sets $g_1(U), g_2(U), \ldots, g_\ell(U)$, by the Pigeon Hole Principle, for some $m \in \{1, 2, \ldots, \ell\}$, we have that g_m contains infinitely many integers. Thus we can choose arbitrarily large integers $i < j$ such that

$$W_i \cap W_j \subseteq g_m(U).$$

Without loss, we can assume that $\alpha < \{i\}$. Let $V = g_m^{-1}W_j$ and let $\beta = \{i+1, i+2, \ldots, j\}$. Then using the construction of the sequence W_n, we have:

$$\begin{aligned}
\sigma_\beta^{-1}V &= \sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\ldots\sigma_j^{-1}g_m^{-1}W_j \\
&= g_m^{-1}\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\ldots\sigma_j^{-1}W_j \\
&\subseteq g_m^{-1}\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\ldots\sigma_{j-1}^{-1}W_{j-1} \\
&\quad \vdots \\
&\subseteq g_m^{-1}\sigma_{i+1}^{-1}W_{i+1} \\
&\subseteq g_m^{-1}W_i \subseteq U.
\end{aligned}$$

Therefore $V \subseteq \sigma_\beta(U)$ and $V \subseteq U$. In particular, this means that $U \cap \sigma_\beta(U) \neq \varnothing$, proving the statement for $k = 1$.

Now assume that the theorem holds for k IP-systems and let $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(k+1)}$ be $k+1$ IP-systems in G. The general proof mimics that of the base case, but has significantly more technicalities.

We construct a sequence of nonempty open subsets $W_n \subseteq X$ and an increasing sequence $\alpha_n \in \mathcal{F}$ with $\alpha_n > \alpha$ satisfying

(1) $W_0 = U$;

(2) $\bigcup_{j=1}^{k+1}(\sigma_{\alpha_n}^{(j)})^{-1}W_n \subseteq W_{n-1}$;

(3) For each $n \geq 1$, W_n is contained in one of the sets $g_{i_n}(U)$ with $i_n \in \{1, 2, \ldots, \ell\}$.

We construct this sequence inductively.

Consider the k IP-systems $(\sigma^{(k+1)})^{-1}\sigma^{(j)}$ for $j = 1, 2, \ldots, k$. By the inductive hypothesis applied to W_0, there exists some $\alpha_1 > \alpha$ such that

$$W_0 \cap (\sigma_{\alpha_1}^{(k+1)})^{-1}\sigma_{\alpha_1}^{(1)}(W_0) \cap (\sigma_{\alpha_1}^{(k+1)})^{-1}\sigma_{\alpha_1}^{(2)}(W_0) \cap \ldots \cap (\sigma_{\alpha_1}^{(k+1)})^{-1}\sigma_{\alpha_1}^{(k)}(W_0)$$

is not empty. Inductively, assume that we have defined W_{n-1} and α_{n-1}. Apply the inductive assumption to W_{n-1} with the k IP-systems $(\sigma^{(k+1)})^{-1}\sigma^{(j)}$, for $j = 1, 2, \ldots, k$ to obtain $\alpha_n > \alpha_{n-1}$ such that

$$W_{n-1} \cap (\sigma_{\alpha_n}^{(k+1)})^{-1}\sigma_{\alpha_n}^{(1)}(W_{n-1}) \cap (\sigma_{\alpha_n}^{(k+1)})^{-1}\sigma_{\alpha_n}^{(2)}(W_{n-1}) \cap \ldots$$
$$\cap (\sigma_{\alpha_n}^{(k+1)})^{-1}\sigma_{\alpha_n}^{(k)}(W_{n-1})$$

is nonempty. Applying $\sigma_{\alpha_n}^{(k+1)}$ to this, we have that $\sigma_{\alpha_n}^{(1)}(W_{n-1}) \cap \sigma_{\alpha_n}^{(2)}(W_{n-1}) \cap \ldots \cap \sigma_{\alpha_n}^{(k+1)}(W_{n-1})$ is a nonempty open set. Thus we can choose i_n with $1 \leq i_n \leq m$ such that

$$W_n = g_{i_n}(W_0) \cap \sigma_{\alpha_n}^{(1)}(W_{n-1}) \cap \sigma_{\alpha_n}^{(2)}(W_{n-1}) \cap \ldots \cap \sigma_{\alpha_n}^{(k+1)}(W_{n-1})$$

is not empty, completing the construction of the sequence.

By the Pigeon Hole Principle, there exists some $m \in \{1, 2, \ldots, \ell\}$ such that $W_n \subseteq g_m(U)$ for infinitely many n. Therefore there exist arbitrarily large $i < j$ such that $W_i \cap W_j \subseteq g_m(U)$. Define $W = g_m^{-1}(W_j) \subseteq U$ and set $\beta = \alpha_{i+1} \cup \alpha_{i+2} \cup \ldots \cup \alpha_j$.

Then $W \neq \varnothing$ and for $1 \leq s \leq k+1$,

$$\begin{aligned} (\sigma_\beta^{(s)})^{-1}(W) &= g_m^{-1}(\sigma_\beta^{(s)})^{-1}(W_j) \\ &\subseteq g_m^{-1}(\sigma_{\alpha_{i+1}}^{(s)})^{-1}\ldots(\sigma_{\alpha_{j-1}}^{(s)})^{-1}W_{j-1} \\ &\subseteq \ldots \\ &\subseteq g_m^{-1}W_i \subseteq U. \end{aligned}$$

Therefore, $\bigcup_{n=1}^{k+1}(\sigma_\beta^{(n)})^{-1}W \subseteq U$ and so

$$U \cap \sigma_\beta^{(1)}U \cap \sigma_\beta^{(2)}U \cap \ldots \cap \sigma_\beta^{(k+1)}U \neq \varnothing.$$

□

9.6. Applications of multiple recurrence

We apply Theorem 9.18 to a commutative group of transformations on a compact metric space:

Corollary 9.19. *Assume that (X, G) is a dynamical system with G a commutative group of transformations. Let $k \in \mathbb{N}$, let $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(k)}$ be* IP-*systems in G and let $\alpha \in \mathcal{F}$. For all $\varepsilon > 0$, there exist $\beta > \alpha$ and $x \in X$ such that*

$$\{x, \sigma_\beta^{(1)}x, \sigma_\beta^{(2)}x, \ldots, \sigma_\beta^{(n)}x\}$$

are all within ε of each other.

Proof. Since any system contains a minimal nonempty closed G-invariant subset (see Theorem 3.1), applying Theorem 9.18 to this subset, we have the statement. □

We can use this corollary to provide a proof of van der Waerden's Theorem. By Theorem 3.1, any dynamical system contains a minimal subsystem and so without loss of generality, we can assume that (X, T) is minimal. Fix $k \in \mathbb{N}$ and $\alpha \in \mathcal{F}$. Let $G = \{T^n : n \in \mathbb{N}\}$. For $\alpha \in \mathcal{F}$, define $\|\alpha\| = \sum_{j\in\alpha} j$. Consider the IP-systems $T^{(j)}_\alpha = T^{j\|\alpha\|}$ for $j = 1, 2, \ldots, k$. Note that each of these is an IP-system, since if $\alpha_1, \alpha_2 \in \mathcal{F}$, then

$$T^{(j)}_{\alpha_1\cup\alpha_2} = T^{j(\sum_{i\in\alpha_1} i + \sum_{\ell\in\alpha_2} \ell)} = T^{j\sum_{i\in\alpha_1} i} T^{j\sum_{\ell\in\alpha_2} \ell} = T^{(j)}_{\alpha_1} T^{(j)}_{\alpha_2}.$$

By Corollary 9.19, for all $\varepsilon > 0$, there exist $\beta > \alpha$ and $x \in X$ such that

$$x, T^{\|\alpha\|}x, T^{2\|\alpha\|}x, \ldots, T^{k\|\alpha\|}x$$

all lie within the same ε ball. By definition of the IP-systems, this means that

$$x, T^n x, T^{2n} x, \ldots, T^{kn} x$$

all lie within ε of each other, where $n = \|\alpha\|$.

Although this proof only produces one particular x that is multiply recurrent, in fact this holds for a dense set of x in a minimal system. Indeed, if y is sufficiently close to x (with the distance depending on n), then

$$y, T^n y, T^{2n} y, \ldots, T^{kn} y$$

all lie within ε of each other. By minimality, x lies in the orbit closure of the point y and so we can apply this to $T^m y$ for a sufficiently large choice of m. This means that

$$T^m y, T^{m+n} y, T^{m+2n} y, \ldots, T^{m+kn} y$$

all lie within ε of each other.

Notes

Propositions 9.9 and 9.10 are proven in [**27**]. Theorem 9.18 was proven by Furstenberg and Weiss [**30**], and the proof given here follows Bergelson [**2**].

The question of the equivalence of Bohr recurrence and topological recurrence was raised in Katznelson [**40**], where other equivalent formulations are given for this question. The problem is further discuss in Weiss [**62**] and in Glasner and Boshernitzan [**7**], and was raised in the combinatorial setting in Ruzsa [**51**].

An example of a set of 1-recurrence that is not a set of 2-recurrence is given in Furstenberg [**27**], and examples of sets of k-recurrence that are not sets of $(k+1)$-recurrence are given in Frantzikinakis, Lesigne and Wierdl [**22**].

The equivalences of Theorem 9.13 were proven in Kamae-Mendès France [**39**] and Ruzsa [**52**].

Exercises

Exercise 9.1. Show that Theorem 9.18 implies Theorems 8.6 and 9.11.

Exercise 9.2. Prove (1) $\iff$ (2) $\iff$ (3) of Proposition 9.1.

Exercise 9.3. (5) $\iff$ (6) of Proposition 9.1.

Exercise 9.4. Show that R is a set of topological recurrence if and only if for all dynamical systems (X,T), there exists $x \in X$ and $n_j \to \infty$ with $n_j \in R$ for all $j \in \mathbb{N}$ such that $T^{n_j}x \to x$ as $j \to \infty$.

Exercise 9.5. If $R \subseteq \mathbb{N}$ is a set of topological recurrence, show that for any $k \in \mathbb{N}$, $R \setminus \{1,2,\ldots,k\}$ is a set of topological recurrence.

Exercise 9.6. Show that if $R = \{r_1 < r_2 < \ldots\}$ is lacunary, then it is not a set of topological recurrence.

Exercise 9.7. Show that a set of topological recurrence can be partitioned into infinitely many pieces such that each piece is a set of topological recurrence.

Exercise 9.8. If $R \subseteq \mathbb{N}$ is a set of topological recurrence, show that for any $k \in \mathbb{N}$, $\{kn \colon n \in R\}$ is a set of topological recurrence.

Exercise 9.9. If $R \subseteq \mathbb{N}$ is a set of topological recurrence, show that for any $k \in \mathbb{N}$, $\{n \in \mathbb{N} \colon kn \in R\}$ is a set of topological recurrence.

Exercise 9.10. If $A \subseteq \mathbb{N}$ is infinite, show that $\{a - a' \colon a, a' \in A\}$ is set of topological recurrence.

Exercise 9.11. If $\{a_n\} \subseteq \mathbb{N}$ is infinite, show that $\bigcup_{n=1}^{\infty}\{a_n, 2a_n, \ldots, na_n$ is set of topological recurrence.

Exercise 9.12. Prove Proposition 9.5.

Exercise 9.13. Which of the sets in Exercises 9.8–9.11 are sets of k-topological recurrence?

Exercise 9.14. Prove Theorem 9.7.

Exercise 9.15. Prove Theorem 9.11.

Exercise 9.16. If $p_1, p_2, \ldots, p_k$ are real polynomials and $\varepsilon > 0$, show that

$$\{n \in \mathbb{N} \colon \|p_1(n) - p_1(0)\| < \varepsilon, \|p_2(n) - p_2(0)\| < \varepsilon, \ldots, \|p_k(n) - p_k(0)\| < \varepsilon\}$$

intersects every IP set in $\mathbb{N}$. (The distance $\|\cdot\|$ is defined to be the distance to the closest integer.)

Exercise 9.17. Show that there exist dynamical systems (X, T) and (Y, S), points $x \in X$, $y \in Y$ and neighborhoods U of x and V of y such that the return times of x to U and the return times of y to V do not intersect.

Exercise 9.18. Prove the multidimensional van der Waerden Theorem (Gallai's Theorem): if $\mathbb{N}^m = C_1 \cup C_2 \cup \ldots \cup C_r$ is a finite partition and F is any finite subset of $\mathbb{N}^m$, then for some $j \in \{1, 2, \ldots, r\}$, there exists $z \in \mathbb{N}^m$ and $b \in \mathbb{N}$ such that $bF + a \subseteq C_j$ for some $j \in \{1, 2, \ldots, r\}$.

Exercise 9.19. Show that if S is a syndetic subset of $\mathbb{N}^m$ and $F \subseteq \mathbb{N}^m$ is a finite set, then there exist $a \in \mathbb{N}^m$ and $b \in \mathbb{N}$ such that $bF + a \subseteq S$.

Exercise 9.20. Use Theorem 9.18 to prove the multi-dimensional van der Waerden Theorem: show that if $k \in \mathbb{N}$ and $A \subseteq \mathbb{N}^k$ is a finite subset, then for any finite partition $\mathbb{N}^k = C_1 \cup C_2 \cup \ldots \cup C_r$, there exist $j \in \{1, 2, \ldots, r\}$, $z \in \mathbb{N}^k$ and $n \in \mathbb{N}$ such that $z + na \subseteq C_j$ for all $a \in A$.

Exercise 9.21. Prove the Graham-Leeb-Rothschild Theorem: show that if V_F is an infinite dimensional vector space over the finite field F, then in any finite partition $V_F = C_1 \cup C_2 \cup \ldots \cup C_r$, some C_j, with $j \in \{1, 2, \ldots, r\}$, contains affine subspaces of arbitrary large dimension.

Exercise 9.22. Show that if $T_1, T_2, \ldots, T_k$ are commuting continuous transformations acting minimally on a compact space X, $k \in \mathbb{N}$, and $\varepsilon > 0$, then there is a dense set $X_0 \subseteq X$ such that for each $x \in X$, there exists $n \in \mathbb{N}$ with $d(T_j^n x, x) < \varepsilon$ for $j = 1, 2, \ldots, k$.

Chapter 10

Polynomial recurrence

Somewhat surprisingly, it turns out that some sequences with sparse growth suffice for recurrence and coloring results. For example, any finite coloring of $\mathbb{N}$ contains monochromatic a, b such that $b - a = n^2$ for some $n \in \mathbb{N}$.

This was generalized to show:

Theorem 10.1. *Let $k, r \in \mathbb{N}$, let $p_1(n), p_2(n), \ldots, p_k(n)$ be polynomials with integer coefficients such that $p_j(0) = 0$ for $j = 1, 2, \ldots, k$, and let $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ be a finite partition. Then there exists some $a \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $a, a + p_1(n), a + p_2(n), \ldots, a + p_k(n) \in C_j$ for some $j \in \{1, 2, \ldots, r\}$.*

As van der Waerden's Theorem follows from a dynamical statement, so does this (Exercise 10.1):

Theorem 10.2. *Assume that (X, T) is a dynamical system, $\varepsilon > 0$, $k \in \mathbb{N}$ and $p_1(n), p_2(n), \ldots, p_k(n)$ are polynomials with integer coefficients such that $p_j(0) = 0$ for $j = 1, 2, \ldots, k$. Then there exists $x \in X$ and $n \in \mathbb{N}$ such that*

$$x, T^{p_1(n)}x, T^{p_2(n)}x, \ldots, T^{p_k(n)}x$$

all lie within ε of each other.

We only give the proof of this theorem in a particular case, namely with the single polynomial $p(n) = n^2$. The general proof requires a complicated induction.

Proof. (of Theorem 10.2, with $k = 1$ and $p_1(n) = p(n) = n^2$) Without loss, it suffices to prove this in a minimal system (X, T). By Exercise 4.2, this implies that T, and therefore T^n for all $n \in \mathbb{N}$ is onto.

We construct a sequence $x_0, x_1, \ldots \in X$ and $n_1, n_2, \in \mathbb{N}$ such that

$$d(T^{(n_m+n_{m-1}+\ldots+n_{p+1})^2}x_m, x_p) < \varepsilon/2$$

for all $p, m \in \mathbb{N} \cup \{0\}$ with $p < m$. Once we have this sequence, since X is compact, for some $p < m$, $d(x_m, x_p) < \varepsilon/2$. But then

$$d(T^{(n_m+n_{m-1}+\ldots+n_{p+1})^2}x_m, x_m) \leq \\ d(T^{(n_m+n_{m-1}+\ldots+n_{p+1})^2}x_m, x_p) + d(x_p, x_m) < \varepsilon$$

and so taking $n = n_m + n_{m-1} + \ldots + n_{p+1}$, we have that x returns within ε of itself under T^{n^2}.

We construct the appropriate sequences inductively. Fix some $x_0 \in X$ and set $n_1 = 1$. Since T is onto, there exists $x_1 \in X$ such that $T^{n_1^2}x_1 = x_0$. Choose $\varepsilon_1 < \varepsilon/2$ such that

$$d(y, x_1) < \varepsilon_1 \Rightarrow d(T^{n_1^2}y, x_0) < \varepsilon/2. \tag{10.1}$$

By Exercise 8.12, applied with $\varepsilon = \varepsilon_1/2, k = 1$ and $a_0 = 2n_1$, there exists $y_1 \in X$ and $n_2 \in \mathbb{N}$ such that $d(y_1, x_1) < \varepsilon_1/2$ and $d(T^{2n_1n_2}y_1, y_1) < \varepsilon_1/2$.

Since T is onto, we can find x_2 such that $T^{n_2^2}x_2 = y_1$. Then

$$d(T^{n_2^2}x_2, x_1) = d(y_1, x_1) < \varepsilon_1/2 < \varepsilon/2$$

and

$$d(T^{2n_1n_2+n_2^2}x_2, x_1) \leq d(T^{2n_1n_2}y_1, y_1) + d(y_1, x_1) < \varepsilon_1. \tag{10.2}$$

Combining Equations 10.1 and 10.2, we have that

$$d(T^{(n_1+n_2)^2}x_2, x_0) = d(T^{n_1^2}T^{2n_1n_2+n_2^2}x_2, x_0) < \varepsilon/2.$$

Assume that we have found $x_m \in X$ and $n_m \in \mathbb{N}$ satisfying the conditions and we need to extend the sequences to $x_{m+1} \in X$ and $n_{m+1} \in \mathbb{N}$. Choose $\varepsilon_m < \varepsilon/2$ such that

(10.3)

$$d(y, x_m) < \varepsilon_m \Rightarrow d(T^{(n_m+n_{m-1}+\ldots+n_{p+1})^2}y, x_p) < \varepsilon/2 \text{ for } p = 0, 1, \ldots, m-1.$$

Again we apply Exercise 8.12, with $\varepsilon = \varepsilon_m/2, k = m, c_p = 2(n_m + n_{m-1} + \ldots, n_{p+1})$ for $p = 1, 2, \ldots, p-1$, and we find $y_m \in X$ and $n_{m+1} \in \mathbb{N}$ such that

$$d(T^{2(n_m+n_{m-1}+\ldots+n_{p+1})n_{m+1}}y_m, y_m) < \varepsilon_m/2$$

for $p = 0, 1, \ldots, m-1$. Since T is onto, we can choose x_{m+1} such that $T^{n_{m+1}^2}x_{m+1} = y_m$. Then

$$d(T^{2(n_m+n_{m-1}+\ldots+n_{p+1})n_{m+1}+n_{m+1}^2}x_{m+1}, x_m) < \\ d(T^{2(n_m+n_{m-1}+\ldots+n_{p+1})n_{m+1}}x_{m+1}y_m, y_m) + d(y_m, x_m) < \varepsilon_m$$

for $p = 0, 1, \ldots, m-1$. Thus, by choice of ε_m,

$$d(T^{n_{m+1}^2} x_{m+1}, x_m) < \varepsilon/2$$

and

$$d(T^{(n_{m+1}+n_m+\ldots+n_{p+1})^2} x_{m+1}, x_p) < \varepsilon/2$$

for $p = 0, 1, \ldots, m-1$. This completes the construction of the sequence and so completes the proof. □

Notes

Furtsenberg [**25**] and Sárközy [**56**] independently proved that a finite coloring of $\mathbb{N}$ contains differences that are squares. Bergelson and Leibman [**3**] generalized this to prove the multidimensional version of Theorem 10.1 and the dynamical version in Theorem 10.2, as well as the more general version for commuting transformations.

Exercises

Exercise 10.1. Show that Theorems 10.1 and 10.2 are equivalent.

Exercise 10.2. Show that Theorem 10.1 fails for the polynomial $n^2 + 1$.

Chapter 11

Proximal and distal

11.1. The proximal relation

Definition 11.1. In a dynamical system (X, G), the points $x, y \in X$ are *proximal* if there exists a sequence $g_n \in G$ such that

$$\liminf_{n\to\infty} d(g_n x, g_n y) = 0.$$

In particular, for a dynamical system (X, T), we have that

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0.$$

It is immediate that any $x \in X$ is proximal to itself.

This condition can be reformulated in terms of the topology on X (Exercise 11.18).

Definition 11.2. The set of all proximal pairs in a system (X, G) is a subset of $X \times X$ called the *proximal relation* and is denoted by P_X. The system is *proximal* if every pair of points is proximal. Thus $P_X = X \times X$.

We leave checking that the proximal relation is a relation to Exercise 11.20.

All points in a minimal system may be proximal:

Example 11.3. Consider the system $(\mathbb{T}, G)$, where $\mathbb{T}$ as usual denotes the unit circle and G is the group of all homeomorphisms from $\mathbb{T}$ to $\mathbb{T}$. This is minimal, since for any $x, y \in \mathbb{T}$, there exists $g \in G$ such that $gx = y$. All pairs of points are proximal, since for any $x, y \in \mathbb{T}$ it is easy to find a sequence $g_n \in G$ with $g_n x \to x$ and $g_n y \to x$.

Proposition 11.4. *Let (X,T) be a dynamical system. Then $x, y \in X$ are proximal if and only if for all $\varepsilon > 0$,*

$$\{n \colon d(T^n x, T^n y) < \varepsilon\}$$

is thick.

Proof. Fix $\varepsilon > 0$. For all $N \in \mathbb{N}$, there exists $\delta = \delta(N)$ such that if $d(x,y) < \delta$, then $d(T^j x, T^j y) < \varepsilon$ for $j = 1, 2, \ldots, N$. Since x and y are proximal, there exists $m \in \mathbb{N}$ such that $d(T^m x, T^m y) < \delta$. By choice of δ,

$$m+1, m+2, \ldots, m+N \in \{n \colon d(T^n x, T^n y) < \varepsilon\}.$$

Conversely, taking a sequence of ε tending to 0, we immediately have that x and y are proximal. □

In a symbolic system, this means that two points are proximal if they have arbitrarily long sequences on which they agree in the same positions:

Corollary 11.5. *Assume that (X,T) is a symbolic system. Two points in X are proximal if and only if they agree on arbitrarily long intervals.*

We have already shown that any dynamical system (X,T) contains a minimal set. Consider the case that there is a unique minimal set $Y \subseteq X$ and furthermore, that Y consists of a single point $\{y\}$. It follows immediately that y is a fixed point. Moreover, for any $x \in X$, $(\overline{\mathcal{O}_T^+(x)}, T)$ is a dynamical system and so itself contains a minimal set. Thus the orbit of x comes arbitrarily close to y, as it is the only minimal set. Furthermore, by continuity, the orbit of x stays close to the orbit of y for arbitrarily long intervals. This means that given $\varepsilon > 0$, there is a thick set $A \subset \mathbb{N}$ so that $d(T^n x, y) < \varepsilon$ for all $n \in A$. Our goal is to show that this phenomenon is general: in an arbitrary dynamical system, the orbit of any point comes arbitrarily close to a uniformly recurrent point, meaning that any point is proximal to some uniformly recurrent point. We prove this theorem in Section 11.3.

11.2. Distal and equicontinuous

Definition 11.6. A point x in a dynamical system (X,G) is *distal* if it is only proximal to itself. The system is *distal* if every pair of points in X is distal.

Thus in a dynamical system (X,T), this means that for all $x, y \in X$ with $x \neq y$, $\inf_{n\in\mathbb{N}} d(T^n x, T^n y) > 0$. As for proximality, this can be reformulated solely in terms of the topology (Exercise 11.25).

A rotation on the circle is distal, since any two points stay the same distance apart under iteration. A rational circle rotation is an example of a distal system that is not minimal. Conversely, Example 11.3 shows that a minimal system need not be distal.

A system can contain both proximal pairs and distal points. For example, under the map $x \mapsto x^2 - 9/16$ on the circle $\mathbb{T}$, $3/4$ is the only distal point and all pairs of points not including $3/4$ are proximal.

Definition 11.7. The system (X, G) is *equicontinuous* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(g_n x, g_n y) < \varepsilon$ for all $g \in G$.

In particular, the dynamical system (X, T) is equicontinuous if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{N}$ (or for all $n \in \mathbb{Z}$ if T is invertible).

By definition, any isometry is equicontinuous.

Example 11.8. Let $X = \mathbb{T}^2$ and let $T\colon X \to X$ be defined by $T(x, y) = (x + \alpha, x + y)$, where $\alpha \notin \mathbb{Q}$. We have already shown that this system is minimal. It is also distal. Consider $(x, y), (x', y') \in X \times X$. If $x \neq x'$, then they stay a fixed distance apart in the first coordinate. Otherwise, $x = x'$ and $y \neq y'$ and so the two points stay a fixed distance apart in the second coordinate. This system is not equicontinuous. Take a sequence of $n_j \to \infty$ such that $T^{n_j}(0, 0) \to (0, 0)$. Then $T^{n_j}(\frac{1}{2n_j}, 0) = (\frac{1}{2n_j} + n_j\alpha, 1/2 + \frac{n(n-1)}{2})\alpha \to (0, 1/2)$ and so we have a sequence of points coming arbitrarily close, but their images stay a fixed distance apart. This can be generalized (Exercise 11.29).

An equicontinuous system is always distal, but this example shows that the converse is false.

Proposition 11.9. *If (X, T) is an invertible equicontinuous dynamical system, then it is distal.*

Proof. Assume that T is equicontinuous. If T is not distal, then there exist $x, y \in X$ and a sequence $n_j \to \infty$ of natural numbers such that $d(T^{n_j} x, T^{n_j} y) \to 0$ as $j \to \infty$. Set $\varepsilon = d(x, y)$. Since T is equicontinuous, for all $\delta > 0$, there exists n_j such that $d(T^{n_j} x, T^{n_j} y) < \delta$. However, $d(T^{-n_j} T^{n_j} x, T^{-n_j} T^{n_j} y) = d(x, y) = \varepsilon$, giving a contradiction of T being equicontinuous. □

11.3. Semisimple systems

Definition 11.10. A dynamical system (X, T) is *semisimple* if every $x \in X$ belongs to a minimal set. In particular, any minimal system is semisimple.

Although this is a natural generalization of minimality, it is difficult to say much about semisimple systems. We do have:

Proposition 11.11. *If (X,T) is an invertible equicontinuous dynamical system, then it is semisimple.*

Proof. Fix $\varepsilon > 0$. By equicontinuity of T, there exists $\delta > 0$ such that if $d(x,y) < \delta$, then $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{N}$. Let $x \in X$. We show that $\overline{\mathcal{O}_T x}$ is minimal. If $y \in \overline{\mathcal{O}_T x}$, we can pick $n \in \mathbb{N}$ such that $d(T^n x, y) < \delta$. Then $d(T^{m+n}x, T^m y) < \varepsilon/2$ for all $m \in \mathbb{Z}$. In particular, this holds for $n = -m$ and so $d(x, T^{-n}y) < \varepsilon$. Since this holds for all $\varepsilon > 0$, we have that $x \in \overline{\mathcal{O}_T y}$. □

Definition 11.12. If (X,T) is a dynamical system, a set $A \subseteq X$ is said to be *uniformly recurrent* if for all finite choices $x_1, x_2, \ldots, x_n \in A$ and neighborhoods $U_1, U_2, \ldots, U_n$ of $x_1, x_2, \ldots, x_n$, the set

$$\{m \in \mathbb{N} \colon T^m x_j \in U_j, 1 \leq j \leq n\}$$

is syndetic.

Clearly a subset of a uniformly recurrent set is uniformly recurrent. If x is a uniformly recurrent point, then $\{x\}$ is a uniformly recurrent set.

Lemma 11.13. *Every uniformly recurrent set in a dynamical system (X,T) lies in a maximal uniformly recurrent set.*

Proof. Consider the collection of all uniformly recurrent sets in (X,T). Order them by inclusion and take a totally ordered collection. Then their union is a uniformly recurrent set and is an upper bound for the totally ordered collection. By Zorn's Lemma, there is a maximal element. □

Notation 11.14. Given a dynamical system (X,T) and a subset $A \subseteq X$, let X^A denote the set of all transformations (continuous or not) from A to X.

The space X^A is a semigroup, where the product of two transformations is given by composition. It is a topological space, by taking the product topology on it. By Tychonof's Theorem, it is compact. This can also be shown directly (Exercise 11.32). A basis for this topology is given by neighborhoods

$$U_{\varepsilon,x_1,x_2,\ldots,x_n}(f) = \{g \in X^A \colon d\big(f(x_1), g(x_1)\big) < \varepsilon, \\ d\big(f(x_2), g(x_2)\big) < \varepsilon, \ldots, d\big(f(x_n), g(x_n)\big) < \varepsilon\}.$$

Note that a basis element is defined by some $\varepsilon > 0$ and a finite set of points $x_1, x_2, \ldots, x_n$.

The map T induces a map $T_A \colon X^A \to X^A$, by mapping $S \in X^A$ to the map $T \circ S \in X^A$. Note that this this map $S \mapsto T \circ S$ is continuous for every $S \in X^A$, since T is assumed to be continuous (in fact, $S \mapsto T \circ S$ is continuous if and only if T is continuous). We call T_A the *induced map* on X^A. The topology on X^A is not a metric topology. However, if $B \subset X^A$ and $S \in \overline{B}$, then for any finite set of points $x_1, x_2, \ldots, x_n \in X$, there exists a sequence $\{S_n\} \subset B$ such that $S_n(x_j) \to S(x_j)$ for $j = 1, 2, \ldots, n$ (Exercise 11.33).

In this notation, A is uniformly recurrent if every $S \in X^A$ with range A is a uniformly recurrent point in the system (X^A, T_A).

Theorem 11.15. *If (X, T) is a dynamical system, then any point $x \in X$ is proximal to a uniformly recurrent point in its orbit closure.*

Proof. If x itself is uniformly recurrent, it is proximal to itself and we are done. Thus we can assume that x is not uniformly recurrent. Let $A \subset X$ be a maximal uniformly recurrent set (this exists by Lemma 11.13). By choice, $x \notin A$.

Let T_A be the induced map on X^A. We consider the system $(X \times X^A, T \times T_A)$.

Let $S \in X^A$ have range A and consider $(x, S) \in (X \times X^A)$ and the dynamical subsystem it generates, $(\overline{\mathcal{O}^+_{T\times T_A}(x,S)}, T \times T_A)$. Let (x_0, S_0) be a uniformly recurrent point under $T \times T_A$ in $\overline{\mathcal{O}^+_{T\times T_A}(x,S)}$. Since S_0 is uniformly recurrent, we have that $S \in \overline{\mathcal{O}^+_{T_A} S_0}$ (by minimality). Using the iterates that force this relation, we find that there exists $y \in \overline{\mathcal{O}^+_T(x_0)}$ such that (y, S) is uniformly recurrent. By transitivity of inclusion (Exercise 3.18) $(y, S) \in (\overline{\mathcal{O}^+_T(x_0)}, \overline{\mathcal{O}^+_{T_A} S_0}) \subset \overline{\mathcal{O}^+_{T\times T_A}(x,S)}$. Since (y, S) is a uniformly recurrent point, we have that $\{y\} \cup \operatorname{range}(S) = \{y\} \cup A$ is a uniformly recurrent set. Since A is maximal, we must have $y \in A$. In particular, $y \in \operatorname{range}(S)$, meaning it appears as some coordinate of S. Thus $(y, y) \in \overline{\mathcal{O}^+_{T\times T}(x,y)}$ and so the orbit closure of (x, y) under $T \times T$ meets the diagonal of $X \times X$. But this means that x is proximal to y. □

As a corollary of Theorem 11.15, we have the following:

Corollary 11.16. *Any distal system is semisimple.*

Proof. Given $x \in X$, by Theorem 11.15, it is proximal to some uniformly recurrent point $y \in X$. Since (X, T) is distal, there are no nontrivial proximal pairs and so $x = y$. Therefore x itself is minimal. □

We can say more about systems with $(X \times X, T \times T)$ semisimple:

Theorem 11.17. *A dynamical system (X, T) is distal if and only if $(X \times X, T \times T)$ is semisimple.*

Proof. If (X, T) is distal, then $(X \times X, T \times T)$ is distal and so by Corollary 11.16, it is semisimple. Conversely, assume that $(X \times X, T \times T)$ is semisimple. Let $x, y \in X$ be proximal. By hypothesis, the pair (x, y) belongs to a minimal set $E \subset X \times X$, which is invariant under $T \times T$. Since (x, y) is a proximal pair, its orbit closure under $T \times T$ meets the diagonal of $X \times X$. The diagonal is clearly invariant under $T \times T$ and so by minimality, E is a subset of the diagonal. Therefore $x = y$ and there are no nontrivial proximal pairs. □

Notes

Theorem 11.15 was proven by Auslander [**1**] and Ellis [**19**], and the proof given here follows that in We follow the proof in Brin and Stuck [**10**]. Corollary 11.16 and Theorem 11.17 were proven by Ellis [**19**].

Exercises

Exercise 11.18. If (X, T) is a dynamical system, show that $x, y \in X$ are proximal if and only if $\overline{\mathcal{O}^+_{T\times T}(x, y)}$ has nontrivial intersection with the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$ of $X \times X$.

Exercise 11.19. Show that there exists a minimal dynamical system with a nontrivial pair of proximal points, but not all points are proximal.

Exercise 11.20. Show that the proximal relation is symmetric, reflexive and $T \times T$ invariant.

Exercise 11.21. Show that there exists a dynamical system such that the proximal relation is not an equivalence relation .

Exercise 11.22. Show that there exists a minimal system such that the proximal relation is not closed.

Exercise 11.23. Show that any infinite closed invariant subset of the k-shift contains a pair of proximal points.

Exercise 11.24. Show that in a dynamical system (X, T), if $x, y \in X$ are proximal and (x, y) is uniformly recurrent in $(X \times X, T \times T)$, then $x = y$.

Exercise 11.25. If (X, T) is a dynamical system, show that $x, y \in X$ are distal if and only if $\overline{\mathcal{O}^+_{T\times T}(x, y)}$ has empty intersection with the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$ of $X \times X$.

Exercise 11.26. Show that a system (X, T) with metric d is equicontinuous if and only if there is a metric d' equivalent to d that makes T an isometry.

Exercise 11.27. Show that a Kronecker system is equicontinuous.

Exercise 11.28. Show that the adding machine (Exercise 2.15) is equicontinuous.

Exercise 11.29. Assume that (X, T) is a distal dynamical system and let $\phi\colon X \to \mathbb{R}$ be continuous. Define $T_\phi\colon X \times \mathbb{T} \to X \times \mathbb{T}$ by $T_\phi(x, \theta) = (Tx, \theta + \phi(x) \mod 1)$. Show that the system $(X \times \mathbb{T}, T_\phi)$ is distal.

Exercise 11.30. Show that if (X, T) is a dynamical system and there exists some syndetic sequence $n_j \to \infty$ such that $\{T^{n_j}\}$ is equicontinuous, then T is equicontinuous.

Exercise 11.31. Show that a transitive and equicontinuous dynamical system (X, T) is minimal.

Exercise 11.32. Show (without use of Tychonof's Theorem) that if X is a compact metric space and $A \subset X$, then X^A with the product topology is compact.

Exercise 11.33. Show that if X is a compact metric space, $A \subset X$, $B \subset X^A$, and $S \in \overline{B}$, then for any finite set of points $x_1, x_2, \ldots, x_n \in X$, there exists a sequence $\{S_n\} \subset B$ such that $S_n(x_j) \to S(x_j)$ for $j = 1, 2, \ldots, n$.

Exercise 11.34. Show that a factor of a distal system is distal. (Combined with Exercise 11.29, this shows that (X, T) and (Y, S) are distal if and only if $(X \times Y, T \times S)$ is distal.)

Exercise 11.35. Show that a factor of a semisimple system is semisimple.

Chapter 12

Hindman's Theorem

12.1. Schur's Lemma

Our goal is to prove Hindman's Theorem (see Section 9 for the definition of an IP set):

Theorem 12.1. *Let $r > 0$ be an integer. If $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ is a finite partition, then for some $j \in \{1, 2, \ldots, r\}$, C_j contains an IP-set.*

We start by proving the simplest finite version of this theorem. Recall that two points in a symbolic system are proximal if and only if arbitrarily long words occur at the same position, arbitrarily far out in the sequence. We use this and the Theorem 11.15 to show:

Proposition 12.2. *Let Λ be a finite alphabet and let $(\Lambda^{\mathbb{N}\cup\{0\}}, T)$ be the shift system. If $x \in \Lambda^{\mathbb{N}\cup\{0\}}$, then there exist $n_1, n_2 \in \mathbb{N}$ such that $x_{n_1+n_2} = x_{n_1} = x_{n_2}$.*

Proof. By Theorem 11.15, there exists a uniformly recurrent point $y \in \Lambda^{\mathbb{N}\cup\{0\}}$ that is proximal to x. Assume that $y_0 = a$. Since y is uniformly recurrent, the word consisting of the symbol a recurs syndetically in y. Since x and y are proximal, they agree on arbitrarily long intervals and so there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} = y_{n_1} = a$.

Now consider the word $y_0 y_1 \ldots y_{n_1}$. Once again, since y is uniformly recurrent, this word recurs syndetically in y and the same word occurs in x at the same position. Thus there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} x_{n_2+1} \ldots x_{n_2+n_1} = y_0 y_1 \ldots y_{n_1}$. Therefore $x_{n_2} = y_0 = y_{n_1} = x_{n_1+n_2}$. □

As a corollary, we have:

Corollary 12.3. *Let $r > 0$ be an integer. If $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ is a finite partition, then for some $j \in \{1, 2, \ldots, r\}$, C_j contains m, n and $m + n$.*

Proof. Let $\Lambda = \{1, 2, \ldots, r\}$ and define $x \in \Lambda^{\mathbb{N}\cup\{0\}}$ by setting $x_n = j$ if and only if $n \in C_j$ for $n \in \mathbb{N}$ and define x_0 arbitrarily. By the proposition, there exist $n_1, n_2 \in \mathbb{N}$ such that $x_{n_1+n_2} = x_{n_1} = x_{n_2}$, meaning that n_1, n_2 and $n_1 + n_2$ all lie in the same piece of the partition of $\mathbb{N}$. □

12.2. Central sets

We show that a larger structure, known as a central set, can always be found at least one piece of a finite partition. Then we show that central sets contain IP sets, proving Theorem 12.1.

Definition 12.4. A subset $S \subseteq \mathbb{N}$ is *central* if there exists a dynamical system (X, T), $x \in X$, a uniformly recurrent point $y \in X$ that is proximal to x and a neighborhood U of y such that

$$S = \{n \colon T^n x \in U\}.$$

In particular, if x is a uniformly recurrent point, then the return times of x to a neighborhood of itself is central. Central sets are piecewise syndetic (Exercise 12.8).

We show that any finite coloring contains a monochromatic central set:

Theorem 12.5. *If $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ is a finite partition, then some C_j contains a central set.*

Proof. Consider the shift system (Ω, T), where $\Omega = \{1, 2, \ldots, r\}^{\mathbb{Z}}$. Let $\omega \in \Omega$ be the point defined by $\omega_n = i$ if and only if $n \in C_i$ for $n \in \mathbb{N}$ and defined arbitrarily for $n \leq 0$. By Theorem 11.15, there exists a uniformly recurrent point η that is proximal to ω. Say that $\eta_0 = j$. Define the neighborhood U of η by $U = \{\nu \in \Omega \colon \nu_0 = j\}$ and let $S = \{n \colon T^n\omega \in U\}$. By definition S is central. Furthermore, if $n \in S$, then $\omega_n = (T^n\omega)_0 = j$ and so $n \in C_j$. Thus $S \subseteq C_j$. □

Note that we have made use of the fact that r is finite in order to prove this: otherwise, the orbit closure of ω might not be compact, making it impossible to use Theorem 11.15 to find a uniformly recurrent point proximal to ω.

We use this to show that central sets contain IP sets, and so in particular obtain Hindman's Theorem. We start with a lemma:

Lemma 12.6. *Let (X, T) be a dynamical system. Assume that $x, y \in X$ are proximal and that y is uniformly recurrent. Let U be a neighborhood of y. Then there exists $n \in \mathbb{N}$ such that $T^n x, T^n y \in U$.*

Proof. Let $U' \subseteq U$ be a neighborhood of y and $\epsilon > 0$ such that any point whose distance is at most ϵ from U' still lies in U. Since y is uniformly recurrent, by Proposition 4.9 there exists $N \in \mathbb{N}$ such that

$$\overline{\mathcal{O}_T^+ y} \subseteq \bigcup_{j=1}^{N} T^{-j} U'.$$

Pick $\delta > 0$ such that $d(x', x'') < \delta$ implies that $d(T^j x', T^j x'') < \epsilon$ for $j = 1, 2, \ldots, N$.

Since x and y are proximal, for some $n \in \mathbb{N}$, $d(T^n x, T^n y) < \delta$. Thus $d(T^{j+n} x, T^{j+n} y) < \epsilon$ for $j = 1, 2, \ldots, N$. By choice of N, we also have that $T^{j+n} x \in U'$ for some $j \in \{1, 2, \ldots, N\}$. Thus $T^{j+n} x \in U$. □

Proposition 12.7. *Any central set contains an* IP *set.*

Proof. Assume that $S = \{n : T^n x \in U\}$ is a central set. Thus (X, T) is a dynamical system, $x \in X$ is proximal to a uniformly recurrent point $y \in Y$ and U is a neighborhood of y. Inductively, we define a sequence of neighborhoods U_n of y and a sequence of integers $\{p_j\}$ such that if $j_1 < j_2 < \ldots < j_k$, then

$$p_{j_1} + p_{j_2} + \ldots + p_{j_k} \in S.$$

Define $U_1 = U$. By Lemma 12.6, there exists $p_1 \in \mathbb{N}$ such that $T^{p_1} x, T^{p_1} y \in U_1$. Let $U_2 = U_1 \cap T^{-p_1} U_1$. (By choice of p_1, this intersection is nonempty.) Again using the lemma, there exists $p_2 \in \mathbb{N}$ such that $T^{p_2} x, T^{p_2} y \in U_2$. Continuing this procees, we choose $U_{k+1} \subseteq U_k$ and $p_k \in \mathbb{N}$ such that $T^{p_k} U_{k+1} \subseteq U_k$ and $T^{p_k} x, T^{p_k} y \in U_k$. Then :

$$T^{p_{j_1} + p_{j_2} + \ldots + p_{j_k}} x \in T^{p_{j_1} + p_{j_2} + \ldots + p_{j_{k-1}}} (U_{j_k}) \subseteq \ldots \subseteq T^{p_{j_1}} (U_{j_2}) \subseteq U_{j_1} \subseteq U_1.$$

Therefore $p_{j_1} + p_{j_2} + \ldots + p_{j_k} \in S$. □

In fact, more holds; central sets contain arbitrarily long arithmetic progressions (Exercise 12.11).

Notes

Theorem 12.1 was originally proven by Neil Hindman [**38**] via combinatorial techniques. Furstenberg and Weiss [**30**] gave a dynamical proof. Furstenberg [**27**] defined central sets and gave the proof of Theorem 12.5, using it to derive Theorem 12.1 and Theorem 1.4. Corollary 12.3 was originally proved by Schur [**57**], and the proof we give here follows Furstenberg [**27**].

Exercises

Exercise 12.8. Show that any central set is piecewise syndetic. Show that the intersection of any syndetic set and the return times of a uniformly recurrent point is central.

Exercise 12.9. If $S \subseteq \mathbb{N}$ is central and $S' \supseteq S$, show that S' is central.

Exercise 12.10. Assume that $A \subseteq \mathbb{N}$ contains an IP-set. Show that if $A = A_1 \cup A_2 \cup \ldots \cup A_r$ is a finite partition, then some A_j, $j \in \{1, 2, \ldots, r\}$ contains an IP-set.

Exercise 12.11. Show that a central set contains arbitrarily long arithmetic progressions.

Exercise 12.1. Show that the Hales-Jewett Theorem implies both van der Waerden's Theorem and the Graham-Leeb-Rothschild Theorem.

Exercise 12.2. State the density versions (see the notes at the end of Chapter 1) of the Hales-Jewett Theorem and of the Graham-Leeb-Rothschild Theorem.

Exercise 12.3. Show that the set theoretic version of Hindman's Theorem, Theorem 1.9, and the number theoretic version, Theorem 1.8 are equivalent.

Exercise 12.4. If $\mathcal{F}$ in Theorem 1.9 is replaced by the collection of all subsets of $\mathbb{N}$, find a counterexample that shows that conclusion of the Theorem no longer holds.

Exercise 12.5. Use Schur's Theorem to show that in any finite coloring of $\mathbb{N}$, one can find infinitely many distinct x and y such that the Schur triple $(x, y, x + y)$ is monochromatic.

Chapter 13

Mixing properties

13.1. Mixing

Definition 13.1. The dynamical system (X, T) is *(topologically) mixing* if for all non-empty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $T^n U \cap V \neq \varnothing$.

It is clear that mixing implies transitivity (use Proposition 4.32), but the converse is false. For example, an irrational rotation on the circle is transitive (it is even minimal), but is not mixing. More generally, an isometry on a space with at least two points cannot be mixing. If the space has only two points, then this is clearly. More generally:

Example 13.2. Isometries are not mixing. Given an isometry $T\colon X \to X$, let $x, y, z \in X$ be distinct points. Let $\varepsilon = \min\bigl(d(x, y), d(y, z), d(x, z)\bigr)$ and let U_x, U_y, U_z be balls around x, y, z, respectively, of radius $\varepsilon/4$. Since T is an isometry, the diameter of T^n applied to any of these balls is still $\varepsilon/2$. Therefore, the distance between any $w \in U_y$ and $v \in U_z$ is at least $\varepsilon/2$ and so given any $n \in \mathbb{N}$, one cannot have both $T^n U_x \cap U_y \neq \varnothing$ and $T^n U_x \cap U_y \neq \varnothing$.

Example 13.3. The two-sided shift (X, T) is mixing. The topology on X has a basis consisting of open balls

$$B(x; 2^{-n}) = \{x' \in X : x'_j = x_j \text{for all } |j| \leq n\}.$$

It suffices to show that for two balls $B(x; 2^{-n})$ and $B(x'; 2^{-n'})$, there exists $N > 0$ such that $T^m B(x; 2^{-n}) \cap B(x'; 2^{-n'}) \neq \varnothing$ for all $m \geq N$. Since $T^m B(x; 2^{-n})$ consists in sequences with fixed values at entries

$-m-n, -m-n+, \ldots, -m+n$ and no resitrictions on the other entires, and so the intersection is non-empty so long as $m > n + n'$.

13.2. Weak mixing

Definition 13.4. The dynamical system (X, T) is *(topologically) weak mixing* if for all non-empty open sets $U, V \subset X$, the set

$$\{n \in \mathbb{N} \colon T^{-n}U \cap V \neq \varnothing\}$$

is thick.

Clearly any mixing system is weakly mixing.

Proposition 13.5. *The dynamical system (X, T) is weakly mixing if and only if every non-empty open set $U \subset X \times X$ is dense in $(X \times X, T \times T)$.*

13.3. Mild mixing

13.4. Invariant functions

Notation 13.6. If (X, T) is a dynamical system, we let $C(X)$ denote the continuous functions from X to itself.

The map T induces a map $U_T \colon C(X) \to C(X)$, given by $f \mapsto f \circ T$. This map U_T is linear, meaning that $U_T(af + bg) = aU_Tf + bU_Tg$ for $f, g \in C(X)$ and $a, b \in \mathbb{R}$. Furthermore, U_T is multiplicative, meaning that $U_T(f \circ g) = (U_Tf)(U_Tg)$.

If T is onto, then U_T is an isometry. If T is a homeomorphism, then U_T is an isometric automorphism, meaning that it is a multiplicative linear isometry of $C(X)$ onto $C(X)$.

Definition 13.7. A function $f \in C(X)$ is said to be *T-invariant* if $f \circ T = f$.

The only invariant functions in a transitive system are the constant functions:

Proposition 13.8. *Let (X, T) be a dynamical system and assume that $T \colon X \to X$ is a minimal homeomorphism. If $f \in C(X)$ satisfies $f \circ T = f$, then f is constant.*

Proof. Since $f \circ T = f$, by iterating we have that $f \circ T^n = f$ for all $n \in \mathbb{Z}$. This means that for any $x \in X$, f is constant on the orbit $\mathcal{O}_T^+(x)$. Since X is transitive, there exists some $x \in X$ with a dense orbit and so f is constant on this orbit. Since f is continuous, it is constant. $\square$

The converse is false (Exercise 13.15).

13.5. Eigenfunctions and eigenvalues

Definition 13.9. In a dynamical system (X, T), the function f is an *eigenfunction* of T if it is a non-identically 0 complex valued continuous function such that there exists $\lambda \in \mathbb{C}$ with $f(Tx) = \lambda f(x)$ for all $x \in X$. We call λ the *eigenvalue* associated with f.

Note that if T and S are conjugate, then λ is an eigenvalue for S if and only if it is an eigenvalue for T. Assume that $\phi \circ T = S \circ \phi$, we have that $f(S) = \lambda f$ if and only if $f \circ \phi \circ T = \lambda f \circ \phi$.

Theorem 13.10. *Assume that (X, T) is a transitive dynamical system. Then*

(1) If f is an eigenfunction of T with eigenvalue λ, then $|\lambda| = 1$ and $|f|$ is constant.

(2) If f and g are eigenfunctions of T with the same eigenvalue, then $f = cg$ for some constant c.

(3) Any finite collection of eigenfunctions for T corresponding to distinct eigenvalues are linearly independent in $C(X)$.

(4) The eigenvalues form a countable subgroup of $\mathbb{T}$.

Proof. (1). By hypothesis, $|\lambda||f(x)| = |f(Tx)|$ for all $x \in X$. Therefore

$$|\lambda| \sup_{x\in X} |f(x)| = \sup_{x\in X} |f(Tx)| = \sup_{x\in X} |f(x)|,$$

since $TX = X$. Thus $|\lambda| = 1$. This means that $|f(Tx)| = |f(x)|$. Since T is transitive, it has no non-constant invariant continuous functions (Proposition 13.8) and so $|f(x)$ is a constant.

(2). By part (1), $|g(x)| > 0$ for all $x \in X$. Since f/g is T-invariant, it is constant by using part (1) again.

(3). Assume that $f_j(Tx) = \lambda_j f_j(x)$ for $j = 1, 2, \ldots, k$ and assume that that $\lambda_i \neq \lambda_j$ for $i \neq j$. We proceed by contradiction. Assume that there exist $c_1, c_2, \ldots, c_k \in \mathbb{C}$ such that $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_k f_k(x) = 0$ for all $x \in X$. By iterating this n times, we have that $c_1\lambda_1^n f_1(x) + c_2\lambda_2^n f_2(x) + \ldots + c_k\lambda_k^n f_k(x) = 0$ for all $x \in X$. Since the λ_j are distinct, we can solve simultaneously the k equations arising from the iterates $n = 0, 1, \ldots, k-1$, giving that $c_j f_j(x) = 0$ for all $x \in X$. Since f_j is not identically 0, $c_j = 0$ for all j, meaning that $\{f_1, f_2, \ldots, f_k\}$ are linearly independent.

(4). It is easy to check that the eigenvalues form a subgroup. To check that there are countable many, we first show that an eigenfunction $f \colon X \to \mathbb{T}$ with eigenvalue $\lambda \neq 1$ is a fixed positive distance from the constant functions 1. Let $x_0 \in X$ and choose n such that $\lambda^n f(x_0)$ lies in the left side of the unit circle. Then

$$\sup_{x\in X} |f(x) - 1| \geq |f(T^n x_0) - 1| = |\lambda^n f(x_0) - 1| > 1/4.$$

This means that any two eigenfunctions are a distance at least 1/4 apart in $C(X)$. Since $C(X)$ has a dense countable subset, there can only be finitely many eigenvalues. □

Definition 13.11. We say that T has *(topological) discrete spectrum* if the smallest closed linear subspace containing the eigenfunctions of T is $C(X)$.

In particular, this means that the eigenfunctions span $C(X)$.

By the Theorem, if T has discrete spectrum and is transitive, then it has a countable set of eigenvalues and a linearly independent set of eigenfunctions spanning $C(X)$.

Notes

Exercises

Exercise 13.12. Show that a factor of a mixing system is mixing.

Exercise 13.13. Assume that (X, T) is weakly mixing. Show that any finite product $(X \times X \times \ldots \times X, T \times T \times \ldots \times T)$ is transitive. (Thus it is also weakly mixing.)

Exercise 13.14. Show that the Chacon system is weakly mixing.

Exercise 13.15. Construct a system such that all the T-invariant continuous functions are constant, but T is not topologically transitive.

Chapter 14

Theory of compact semigroups

14.1. Compact semigroups

Let E be a compact semigroup. That is, we assume that E is a semigroup endowed with a topology with respect to which E is a compact Hausdorff space. Furthermore, we assume one sided continuity: for all $q \in E$, the map $p \mapsto pq$ is continuous.

Definition 14.1. Let E be a semigroup. An *idempotent* is an element $p \in E$ such that $p^2 = p$. A subset $I \subset E$ is a *left ideal* if it is closed and $EI \subset I$. Similarly, I is a *right ideal* if it is closed and $IE \subset I$ and a *two sided ideal* if both conditions hold.

Note that ideals in a compact semigroup are themselves semigroups.

The following theorem is simple to state and prove, but turns out to be quite useful:

Theorem 14.2. *Any compact semigroup contains an idempotent.*

Proof. Order the compact subsemigroups by inclusion. By compactness, there is a minimal element M. If $p \in M$, then by the assumption of continuity, Mp is closed and so compact. Then $MpMp \subset Mp \subset M$ and so by minimality $Mp = M$. Set $M' = \{x \in M : xp = p\}$. Thus M' is a non-empty semigroup and is closed by the continuity assumption. Therefore by minimality $M' = M$ and so $p^2 = p$. $\square$

If the semigroup E is actually a group, it trivially contains an idempotent. However, in our set up E is not, in general, a group and so contains proper ideals. Since an ideal is itself a semigroup, we have non-trivial idempotents in E and in every ideal.

By Zorn's lemma, minimal left ideals always exist. Also if J is a minimal left ideal, then so is Jx for any x. If $x \in J$, then $Jx = J$. As already noted, by Theorem 14.2, any left ideal contains an idempotent. We can say more:

Theorem 14.3. *Let J be a minimal left ideal in a semigroup and let $p \in J$ be an idempotent. Then for all $x \in J$, $xp = x$.*

Proof. Since $Jp \subset J$, by minimality $Jp = J$. Thus for any $x \in J$, there exists y such that $x = yp$. Therefore $xp = yp^2 = yp = x$. □

Let E^ℓ denote $E \times E \times \ldots \times E$, where there are ℓ terms in the product. We let $\Delta_E \subset E^\ell$ denote the diagonal:

$$\Delta_E = \{(x, x, \ldots, x) \colon x \in E\}.$$

Theorem 14.4. *Let E be a compact semigroup and let Σ be a subsemigroup of E^ℓ containing Δ_E. If J is any two-sided ideal in Σ, then $J \cap \Delta_E \neq \varnothing$.*

Proof. Let I be a minimal left ideal in E and let $p \in I$. Set $\tilde{p} = (p, p, \ldots, p) \in \Delta_E$ and let $\tilde{q}$ be an idempotent in $J\tilde{p}$. Since $\tilde{p} \in \Delta_E$, $\tilde{p} \in \Sigma$ and so $J\tilde{p} \subset J$ and $\tilde{q} \in J$. Writing $\tilde{q} = (q_1, q_2, \ldots, q_\ell)$, we have $q_i = x_i p$ with $(x_1, x_2, \ldots, x_\ell) \in J$ and each q_i is an idempotent in I. Thus $p = pq_i$ and $\tilde{p} = \tilde{p}\tilde{q}$. Since J is a two-sided ideal and $\tilde{p} \in \Sigma$, we have $\tilde{p} \in J \cap \Delta_E$. □

14.2. The enveloping semigroup

Notation 14.5. Throughout this section, we let X denote a compact metric space with metric d. Let X^X denote all functions $X \to X$.

By Tychonoff's Theorem, X^X is a compact space. The topology on X^X is the topology of pointwise convergence.

Definition 14.6. We say that f_α *converges to* f and write $f_\alpha \to f$ if and only if for all $x \in X$, $f_\alpha(x) \to f(x)$.

A neighborhood N of f is defined by specifying a finite set of points and some $\varepsilon > 0$:

$$N(f; x_1, x_2, \ldots, x_k, \varepsilon) = \{g \in X^X \colon d(f(x_i), g(x_i)) < \varepsilon \text{ for } i = 1, 2, \ldots, k\}.$$

We note that X^X has a natural semigroup structure: if $f, g \in X^X$, then $fg \in X^X$ is defined by $(fg)(x) = f(g(x))$. This operation is continuous on one-side: if $f_\alpha \to f$, then $f_\alpha \circ g \to f \circ g$. However, without some extra assumption, we do not necessarily have $g \circ f_\alpha \to g \circ f$. An idempotent in X^X is easy to identify; it is a map that is the identity on its range.

Given a dynamical system (X, T), we can map this into a subset of X^X. Namely, consider $\{T^n : n \in \mathbb{Z}\} \subset X^X$. Noting that the closure of this set is a compact semigroup, we define:

Definition 14.7. If (X, T) is a dynamical system, then the *enveloping semigroup* of (X, T) is the subset of X^X given by $\overline{\{T^n : n \in \mathbb{Z}\}}$.

More generally, we can extend this definition to arbitrary semigroup actions and not just $\mathbb{Z}$-actions.

We start with some examples to illustrate what the enveloping semigroup may be:

Example 14.8. Let $T : [0, 1] \to [0, 1]$ be given by $x \mapsto x^2$. Define

$$T_\infty(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

and

$$T_{-\infty}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then the enveloping semigroup E is

$$\{T_\infty, T_{-\infty}\} \cup \{T^n : n \in \mathbb{Z}\}.$$

Here the semigroup is unusually easy to compute, and yet even in this situation, E is neither a group, nor is it abelian. The only idempotents are Id, T_∞ and $T_{-\infty}$.

Example 14.9. Let T be an irrational rotation of the circle. Then E is a monothetic, compact abelian group. More generally, this holds if T is an equicontinuous map.

In general, the enveloping semigroup is difficult to compute and is complicated, even for maps as simple as $T(x, y) = (x + \alpha, x + y)$.

We use the theorems of Section 14.1 to prove the dynamical version of van der Waerden's Theorem (Theorem 8.6):

Theorem 14.10. *Let (X, T) be a dynamical system, $x_0 \in X$ and $\varepsilon > 0$. Let $\ell > 0$ be an integer. There exist $n, m \in \mathbb{N}$ with $m > 0$ such that $T^n x_0, T^{n+m} x_0, \ldots, T^{n+(\ell-1)m} x_0$ are all within ε of each other.*

Proof. Define

$$S = \{(T^n, T^{n+m}, \ldots, T^{n+(\ell-1)m}) : n, m \in Z, m \geq 0\}$$

and let $\Sigma = \overline{S}$. Then $\Sigma \subset E^\ell$ is a semigroup and clearly $\Delta_E \subset \Sigma$. Set

$$J = \{(T^n, T^{n+m}, \ldots, T^{n+(\ell-1)m}) \colon n, m \in Z, m > 0\}.$$

Then $J \subset S$ and letting $\mathcal{J} = \overline{J}$, $\mathcal{J}$ is a two-sided ideal in Σ. By Theorem 14.4, $\mathcal{J} \cap \Delta_E \neq \varnothing$ and so for some $n, m \in \mathbb{Z}$ with $m > 0$ and for some $f \in X^X$, $(T^n, T^{n+m}, \ldots, T^{n+(\ell-1)m})$ lies in an arbitrarily small neighborhood of $(f, f, \ldots, f)$. In particular, these iterates are all within ε of each other. □

The same proof generalizes for arbitrary patterns and not just arithmetic progressions:

Theorem 14.11. *Let E be a compact semigroup and let Σ be a subsemigroup of S^ℓ containing the diagonal Δ_E. If J is a two-sided ideal in Σ, then J is a van der Waerden collection for E.*

As another application, we prove the Hales-Jewett Theorem. Let $\Lambda = \{a_1, a_2, \ldots, a_\ell\}$ be a finite alphabet and let $W(\Lambda)$ be all finite words in Λ where multiplication is concatenation. Take Σ to be the span of the diagonal

$$\Delta_\ell(\Lambda) = \{(a_1, a_1, \ldots, a_1), (a_2, a_2, \ldots, a_2), \ldots (a_\ell, a_\ell, \ldots, a_\ell)\}$$

and the element $(a_1, a_2, \ldots, a_\ell)$. Taking $J = \Sigma - \Delta_\ell(\Lambda)$, Theorem 14.11 becomes the Hales-Jewett Theorem:

Theorem 14.12. *Let Λ be a finite alphabet not containing the letter x and let $r \in \mathbb{N}$. There exists a positive integer $N(r, |\Lambda|)$ such that if $N \geq N(r, |\Lambda|)$, then for any finite partition $W_N(\Lambda) = C_1 \cup C_2 \cup \ldots \cup C_r$, there exists $f(x) \in W_N^*(\Lambda \cup \{x\})$ and $j \in \{1, 2, \ldots, r\}$ such that the combinatorial line $\{f(\lambda) \colon \lambda \in \Lambda\}$ belongs entirely to C_j.*

As a final application, we prove Hindman's Theorem (Theorem 12.1). The proof contains a proof of Birkhoff's recurrence theorem, reformulated in the language of idempotents.

Theorem 14.13. *Let $r > 0$ be an integer. If $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$, then some C_j contains an IP-set.*

Proof. Assume that $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ and fix $\varepsilon > 0$. Let $X = \{1, 2, \ldots, r\}^{\mathbb{Z}}$, let T be the shift map on X and let d be a metric on X such that $d(x, y) < 1$ if and only if $x(0) = y(0)$. Let $x_0 \in X$ be defined by $x_0(n) = j$ if $n \in C_j$. Thus x_0 corresponds to the given partition of $\mathbb{N}$.

Let $E = \overline{\{T^n \colon n \in \mathbb{N}\}}$, where the closure is taken in X^X and let θ be an idempotent in E. Set $x_1 = \theta x_0$. Then

$$\theta x_1 = \theta^2 x_0 = \theta x_0 = x_1.$$

(Note that this suffices to prove Birkhoff's Theorem.) Thus there exists p_1 such that $d(T^{p_1}x_0, x_1) < \varepsilon$ and $d(T^{p_1}x_1, x_1) < \varepsilon$. Similarly, we can find p_2 such that $d(T^{p_2}x_0, x_1)$ and $d(T^{p_2}x_1, x_1)$ are small enough such that we can iterate each of these p_1 times and have the pairs remain within ε of each other. Inductively, we can define

$$\{p_\alpha = p_{i_1} + p_{i_2} + \ldots + p_{i_k} : i_1 < i_2 < \ldots i_k\}$$

such that $d(T^{p_\alpha}x_0, x_1) < 1$. Thus the two sequences agree on their 0 entries and by choice of x_0, $p_\alpha \in C_j$ for some j. □

14.3. Generalizations of the Hales-Jewett Theorem

Theorem 14.4 states that a two sided ideal J in a compact semigroup E intersects the diagonal. The proof of this theorem actually shows a bit more. Not only do we have a non-empty intersection, but there is an idempotent $(p, p, \ldots, p) \in E$ such that $(p, p \ldots, p) \in J$. We can use this to get a stronger statement than the Hales-Jewett Theorem. We maintain the notations of the previous section.

Assume that $x \notin \Lambda$ and let $W(\Lambda \cup \{x\})$ denote the words in the alphabet $\Lambda \cup \{x\}$. Let $W^*(\Lambda \cup \{x\})$ denote the subset of words of $W(\Lambda \cup \{x\})$ in which x actually occurs. Thus $f \in W^*(\Lambda \cup \{x\})$ defines a function from Λ to $W(\Lambda)$.

A *subspace* (∞-dimensional) of $W(\Lambda)$ consists of all words

$$\mathcal{L}(f_1, f_2, \ldots) = \{w = f_1(t_1)f_2(t_2)\ldots f_n(t_n) : t_1, t_2, \ldots, t_n \in \Lambda, n \in \mathbb{N}\},$$

where $\{f_j\}$ is a sequence with $f_j \in W^*(\Lambda \cup \{x\})$ for all j.

Maintaining the same notations as above, Σ is the closure of the subgroup generated by the diagonal $\Delta_\ell(\Lambda)$ and $J = \Sigma - \Delta_\ell(\Lambda)$. Let p be an idempotent such that $(p, p \ldots, p) \in J$. Then there exists $f \in W^*(\Lambda \cup \{x\})$ such that $(f(a_1), f(a_2), \ldots, f(a_\ell))$ is close to $(p, p, \ldots, p)$. In particular,

$$(f(a_1)x_0, f(a_2)x_0, \ldots, f(a_\ell)x_0) \sim (px_0, px_0, \ldots, px_0)$$

and

$$\begin{aligned}(f(a_1)px_0, f(a_2)px_0, \ldots, f(a_\ell)px_0) &\sim (p^2x_0, p^2x_0, \ldots, p^2x_0)\\ &= (px_0, px_0, \ldots, px_0),\end{aligned}$$

where by $\sim$ we mean that the terms can be chosen arbitrarily close to each other. By choosing the sequence $f_1, f_2, \ldots$ such that the approximations improve substantially at each level, we can iterate and still remain close, while replacing px_0 by $f_n(a_i)x_0$. In this way, we obtain

$$f_1(a_{i_1})f_2(a_{i_2})\ldots f_n(a_{i_n}) \sim px_0$$

for all choices of $a_{i_1}, a_{i_2}, \dots, a_{i_n}$.

We have shown:

Theorem 14.14. *For any coloring of $W(\Lambda)$, there exists a monochromatic subspace.*

14.4. More semigroup theory

There is a partial ordering on the idempotents in a compact semigroup. We use this added structure to strengthen some of our theorems.

Definition 14.15. If p and q are idempotents, we say that p is *less than or equal to* q and write $p \leq q$ if $pq = qp = p$.

Lemma 14.16. *If q is an idempotent in a minimal left ideal and p is an idempotent with $p \leq q$, then $p = q$.*

Proof. Assume that q lies in the minimal left ideal J. Since $p = pq$, $p \in J$. By Theorem 14.3, $xp = x$ for all $x \in J$. Thus $qp = q$ and $qp = p$ and so $p = q$. □

This means that idempotents which belong to minimal left ideals are minimal and distinct idempotents in the same minimal ideal are not comparable. In particular, the partially ordered set of idempotents contains minimal elements.

Lemma 14.17. *If q is an idempotent and J is a left ideal, then Jq contains an idempotent p with $p \leq q$.*

Proof. Since Jq is an ideal, it is semigroup. Thus it contains an idempotent $w = tq$. Set $p = qtq$. Then $pq = p = qp$ and

$$p^2 = qtqqtq = q(tq)^2 = qw^2 = qw = qtq = p$$

and so p is an idempotent. □

Using these lemmas, we obtain a stronger version of Theorem 14.4:

Theorem 14.18. *Let $E \subset F$ be two compact semigroups. Let $\Sigma \subset E^\ell \times F$ be a subsemigroup containing Δ_E and let J be a two-sided ideal in Σ. Then J contains an element of the form $(p, p, \dots, p, q)$ with p and q idempotents and $q \leq p$.*

Proof. Let p be an idempotent in a minimal left ideal of E and set $\tilde{p} = (p, p, \dots, p) \in \Sigma$. By Lemma 14.17 there exists an idempotent $\tilde{q} \in J\tilde{p}$ such that $\tilde{q} \leq \tilde{p}$.

Write $\tilde{q} = (q_1, q_2, \ldots, q_\ell, q_{\ell+1})$. Since $q_i \in E$ for $1 \le i \le \ell$, we have $q_i \le p$. By Lemma 14.16, $q_i = p$. Setting $q_{\ell+1} = q$. we have $\tilde{q} = (p, p, \ldots, p, q) \in J\tilde{p}$ and $q \le p$. □

We apply this result in the context of the Hales-Jewett Theorem. As before, $\Lambda = \{a_1, a_2, \ldots, a_\ell\}$. We can define the action of $W(\Lambda \cup \{x\})$ on $X = \{1, 2, \ldots, r\}^{W(\Lambda \cup \{x\})}$. Let F be the enveloping semigroup of this action and let E be the enveloping semigroup of $W(\Lambda)$ acting on X. Let Σ be the closure of the semigroup generated by the diagonal $\Delta_{\ell+1}(\Lambda)$ and the element $(a_1, a_2, \ldots, a_\ell, x)$, which all lie in $E^\ell \times F$. Let J be the closure of the ideal of these products in which $(a_1, a_2, \ldots, a_\ell, x)$ actually occurs. Thus J is the closure of elements of the form

$$\big\{(f(a_1), f(a_2), \ldots, f(a_\ell), f(x)) : f \in W^*(\Lambda \cup \{x\})\big\}.$$

By Theorem 14.18, there exists $f \in W^*(\Lambda \cup \{x\})$ with

$$\big(f(a_1), f(a_2), \ldots, f(a_\ell), f(x)\big) \sim (p, p, \ldots, p, q),$$

where p and q are idempotents and $pq = qp = q$. We can find $f_1, f_2, \ldots$ such that

$$\big(f_n(a_1), f_n(a_2), \ldots, f_n(a_\ell), f_n(x)\big) \sim (p, p, \ldots, p, q),$$

such that the approximation improves with each f_j. Choosing $t_1, t_2, \ldots, t_n \in \{a_1, a_2, \ldots, a_\ell, x\}$, we have that $f_1(t_1)f_2(t_2)\ldots f_n(t_n)$ is close to q, provided some $t_i = x$. In this case, $f_1(t_1)f_2(t_2)\ldots f_n(t_n) \in W^*(\Lambda \cup \{x\})$. We can identify $W^*(\Lambda \cup \{x\})$ with combinatorial lines, where $f \in W^*(\Lambda \cup \{x\})$ corresponds to the line $\big(f(a_1), f(a_2), \ldots, f(a_n)\big)$.

Now assume that we have a coloring of $W(\Lambda)$ and so $W^*(\Lambda \cup \{x\}) = C_1 \cup C_2 \cup \ldots \cup C_r$. We can extend this to a coloring of $W(\Lambda \cup \{x\})$, thereby obtaining a point $x_0 \in X$. Choosing f_i as above, we have an infinite dimensional subspace that is monochromatic. Namely, all the expressions $f_1(t_1)f_2(t_2)\ldots f_n(t_n)$ for which some $t_i = x$ represent all the combinatorial lines in this subspace. We have shown:

Theorem 14.19. *If the set of combinatorial lines of $W(\Lambda)$ is colored by finitely many colors, then there exists an infinite dimensional subspace all of whose lines are monochromatic.*

Notes

Ellis [**19**] introduced the notion of the enveloping semigroup in order to study the dynamics of a system (X, T). Most of the material in this chapter is based on Furstenberg and Katznelson [**28**], who used this notion to provide new proofs of results in Ramsey theory, including van der Waerden's Theorem and Hindman's Theorem for which we have already given dynamical proofs, as well as the Hales-Jewett Theorem (Theorem 14.12), originally proven in [**34**]) and the Theorems of Carlson and Simpson and of Carlson (Theorems 14.14 and 14.19, originally proven in [**13**] and [**15**]).

Exercises

Exercise 14.20. If J is a minimal left ideal and $p \in J$ is an idempotent, show that $pJ = \{px \colon x \in J\}$ is a subgroup of J and its identity is p. Show that if p' is an idempotent, then the map $x \mapsto p'x$ is a group isomorphism of pJ onto $p'J$.

Exercise 14.21. If J is a minimal left ideal, show that $\{pJ \colon p \text{ is an idempotent of } J\}$ partitions J. Meaning, for all $x \in J$, there exists a unique idempotent p of J such that $p \in xJ$.