On the work of Margulis on Diophantine approximation

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§1. **The Oppenheim conjecture**

Oppenheim conjecture - proved by Margulis

1.1. Theorem

Let

$$Q(x_1,\dots,x_n)=\sum_{i,j=1}^{n} a_{ij}x_ix_j$$

be a nondegenerate indefinite quadratic form in $n \geq 3$ variables. Suppose that it is not a multiple of a rational form. Then

$$\min\{Q(x) \mid x \in \mathbb{Z}^n \backslash (0)\} = 0.$$

Some history of the problem

The conjecture goes back to a 1929-paper of Oppenheim (for $n \geq 5$).

Extensive work was done by methods of analytic number theory:

In the 1930s (Chowla, Oppenheim), 40s (Davenport-Heilbronn), 50s (Oppenheim, Cassels-Swinnerton-Dyer, Davenport, Birch-Davenport, Davenport-Ridout) ...

Papers of Birch-Davenport, Davenport-Ridout, and one of Ridout in 1968 together confirm the validity of the conjecture for

$n \geq 21$, and

various classes of quadratic forms in fewer variables, satisfying certain conditions on signature, diagonalisability etc..

Partial results continued to be obtained in the 70s and 80s (Iwaniech, Baker-Schlickewei) by number-theoretic methods, ...

Dynamical approach

Margulis proved the following:

1.2. Theorem

Let $G = SL(3, \mathbb{R})$, $\Gamma = SL(3, \mathbb{Z})$.

Let $H =$ the group of elements of G preserving the form $x_1 x_3 - x_2^2$.

If $z \in G/\Gamma$ is such that the orbit Hz is relatively compact in G/Γ, then H/H_z is compact, where H_z is the stabiliser $\{g \in G \mid gz = z\}$ of z;

(the conclusion is equivalent to the orbit Hz being compact).

By the Mahler criterion this implies the OC for $n = 3$,

and then by a simple restriction argument for all $n \geq 3$.

Approach suggested by Raghunathan inspired Margulis in this.

Margulis discovered later that in an implicit form the approach is also there in a paper of Cassels and Swinnerton-Dyer, from 1955.

Stronger forms of OC

Margulis proved also a strengthening of the original OC proposed by Oppenheim in 1952, that for Q as in OC, $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

My collaboration with Margulis begins.

1.3. Theorem (D- and Margulis) As before, let $G = SL(3, \mathbb{R})$, $\Gamma = SL(3, \mathbb{Z})$, and

H the group of elements of G preserving the form $x_1x_3 - x_2^2$.

Then for every $z \in G/\Gamma$ the H-orbit Hz is either closed dense in G/Γ.

1.4. Corollary *Let Q be as in Oppenheim conjecture and $\mathcal{P}$ be the set of all primitive integral n-tuples. Then $Q(\mathcal{P})$ is dense in $\mathbb{R}$.*

Elementary proof

To prove the corollary one does not need the full strength of Theorem 1.3.

Let ν be the matrix of

$$\sum_{i=1}^{3} \xi_i e_i \mapsto \xi_3 e_1.$$

Then it suffices to prove that for $z \in G/\Gamma$ for which Hz is not closed, the closure $\overline{Hz}$ of Hz in G/Γ contains a point y such that either $\{(\exp t\nu)y \mid t \geq 0\}$ or $\{(\exp t\nu)y \mid t \leq 0\}$ is contained in $\overline{Hz}$.

Using this observation we gave an elementary proof of Corollary 1.4, involving only basic knowledge of topological groups and linear algebra.

A crucial idea

A crucial idea in the proof of Theorem 1.2 is the following:
Let be U is a connected unipotent subgroup of G and X be a minimal U-invariant subset, under the U-action on G/Γ.
It is shown that unless X is a (closed) U-orbit there exists a one-parameter subgroup $\{w_t\}$, outside U and normalising it, that leaves X invariant.
This is used together with some specific properties of subgroups of $SL(3, \mathbb{R})$ to show that the compact invariant subset $\overline{Hz}$ has to be a compact orbit of H.
Theorem 1.3 involves an additional issue of existence of compact minimal invariant subsets under actions of unipotent subgroups. This is related to “non-divergence” properties of orbits of these actions, which I will come to later.

§2. **The Raghunathan conjecture**

Statement of the conjecture

The following generalisation of the Raghunathan conjecture was formulated by Margulis in his ICM address at Kyoto, 1990.

Conjecture Let G be a connected Lie group and Γ be a lattice in G. Let H be a subgroup of G which is generated by the unipotent elements contained in it.

Then for any $z \in G/\Gamma$ there exists a closed subgroup F of G such that $\overline{Hz} = Fz$.

(Raghunathan conjecture is the special case where H is a unipotent one-parameter subgroup of G).

Progress on the conjecture

$G = SL(2, \mathbb{R})$, H a unipotent one-parameter subgroup (Hedlund, 1936),
G reductive and H a horospherical subgroup (Dani, 1986),
G a solvable Lie group (Starkov, 1987),
Theorem 1.3 confirmed the conjecture for $G = SL(3, R)$ and H the special orthogonal group of a nondegenerate indefinite quadratic form.
Pursuing the methods in the proof of OC we proved the following special case of Raghunathan conjecture.

2.1. Theorem (D - Margulis) *Let* $G = SL(3, \mathbb{R})$ *and* $\Gamma = SL(3, \mathbb{Z})$.
Let U *be a unipotent one-parameter subgroup of* G *such that* $u - I$ *is of rank 2 for all* $u \in U \setminus \{I\}$. *Then for every* $z \in G/\Gamma$ *there exists a closed subgroup* F *such that* $\overline{Uz} = Fz$.

An application to Diophantine approximation

2.2 Corollary (D - Margulis)

Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^3$, and L be a linear form on $\mathbb{R}^3$. Let

$$C = \{v \in \mathbb{R}^3 \mid Q(v) = 0\} \text{ and } P = \{v \in \mathbb{R}^3 \mid L(v) = 0\}.$$

Suppose that the plane P is tangential to the cone C.
Suppose also that no linear combination $\alpha Q + \beta L^2$, with $(\alpha, \beta) \neq (0,0)$ is a rational quadratic form.
Then $(Q(x), L(x)) \mid x \in \mathcal{P}\}$ (with $\mathcal{P}$ as before) is dense in $\mathbb{R}^2$, viz. given any $a, b \in \mathbb{R}$ and $\epsilon > 0$ there exists $x \in \mathcal{P}$ such that

$$|Q(x) - a| < \epsilon \text{ and } |L(x) - b| < \epsilon.$$

For $n \geq 4$ the question is not fully understood yet; more on this later.

Raghunathan conjecture settled

Margulis nurtured the hope that the overall method of “building up” orbits of larger subgroups, inside a given closed set invariant under the action, should lead to a proof of Raghunathan’s conjecture. This has not materialised.

The Raghunathan conjecture was in the meantime proved by Marina Ratner, in 1990-91, where she also proved the above-mentioned general conjecture under the additional condition that every connected component of H contains a unipotent element.

The general statement of the conjecture was obtained by Nimish Shah, building up on Ratner’s work.

Ratner’s proof of Raghunathan conjecture is based on classifying the invariant measures of actions of unipotent subgroups, proving a conjecture formulated in my paper in connection with the Raghunathan conjecture.

Classification of invariant measures

2.3 Theorem (Ratner, Shah)
Let G be a connected Lie group and Γ be a discrete subgroup of G (not necessarily a lattice).
Let H be a closed subgroup of G which is generated by the unipotent elements contained in it.
Let μ be a finite H-invariant and H-ergodic measure on G/Γ.
Then there exists a closed subgroup F of G and a F-orbit Φ such that μ is F-invariant and supported on Φ.
Though Margulis missed proving Raghunathan conjecture, later he contributed a more transparent proof of the above theorem (jointly with Tomanov), in the crucial case of G a real algebraic group.

Uniform distribution

The proof of Raghunathan conjecture was obtained from this by first proving the following result on uniform distribution.

2.4 Theorem (Ratner)

Let G be a connected Lie group and Γ be a lattice in G.

Let $U = \{u_t\}$ be a unipotent one-parameter subgroup of G.

Let $z \in G/\Gamma$ and suppose that there is no proper closed connected subgroup F of G, with $U \subset F$, such that Fz is closed and admits a finite F-invariant measure.

Then the U-orbit of z is uniformly distributed in G/Γ, viz. for every bounded continuous function g on G/Γ

$$\frac{1}{T}\int_0^T f(u_t z)dt \to \int_{G/\Gamma} f(g\Gamma)dm(g\Gamma),$$

where m is the normalised G-invariant measure on G/Γ.

Simultaneous approximation with quadratic and linear forms

2.5 Corollary (Gorodnik) *Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^n$, $n \geq 4$, and L a linear form on $\mathbb{R}^n$. Suppose that*
(i) the restriction of Q to the subspace $\{v \in \mathbb{R}^n \mid L(v) = 0\}$ is an indefinite quadratic form, and
(ii) no linear combination $\alpha Q + \beta L^2$, with $(\alpha, \beta) \neq (0, 0)$ is a rational quadratic form.
Then $\{(Q(x), L(x)) \mid x \in \mathcal{P}\}$ (with $\mathcal{P}$ as before) is dense in $\mathbb{R}^2$.
The analogue of this is not true for $n = 3$. On the other hand Corollary 2.5 does not complete the picture for $n \geq 4$ since condition (i) can not be expected to be a necessary condition. Gorodnik has conjectured that it may be replaced by the (necessary) condition $\{(Q(v), L(v)) \mid v \in \mathbb{R}^n\} = \mathbb{R}^2$.

§3. **Quantitative versions of Oppenheim conjecture**

Uniform versions of uniform distribution

Margulis and I proved certain uniform versions of Ratner's uniform distribution theorem. Roughly speaking uniformity of distribution is shown to hold even if one varies the initial point, as well as the unipotent one-parameter subgroup being acted upon, provided one is away from the lower dimensional homogeneous subspaces invariant under the limit one-parameter subgroup. (I will not go into the details here). From these results we obtained lower asymptotic estimates for the number of solutions in large balls.

Asymptotics of the solutions

Let ω be a continuous function on $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$, and $\Omega = \{v \in \mathbb{R}^n \mid 0 < \|v\| < \omega(v/\|v\|)\}$. For $T > 0$ let $T\Omega = \{Tv \mid v \in \Omega\}$.

Let $p \geq 2$, $q \geq 1$, $p \geq q$ and $p + q = n$ and $\mathcal{O}(p, q)$ denote the space of quadratic forms on $\mathbb{R}^n$ with discriminant ± 1 and signature (p, q).

3.1 Theorem *Let $\mathcal{K}$ be a compact subset of $\mathcal{O}(p, q)$; $a, b \in \mathbb{R}$, $a < b$.*

Then for any $\theta > 0$ there exists a finite subset $\mathcal{E}$ of $\mathcal{K}$ such that each Q in $\mathcal{E}$ is a scalar multiple of a rational quadratic form, and for all $Q \in \mathcal{K} \backslash \mathcal{E}$ the following holds, for all large T:

(I) (Dani-Margulis)

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\} \geq (1-\theta)\mathrm{vol}\,\{v \in T\Omega \mid a < Q(v) < b\}.$$

(II) (Eskin-Margulis-Mozes) if $p \geq 3$ then

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\} \leq (1+\theta)\mathrm{vol}\,\{v \in T\Omega \mid a < Q(v) < b\}.$$

More about the asymptotics

Moreover, for any compact subset $\mathcal{F}$ of $\mathcal{K} \backslash \mathcal{E}$ there exists a common T_0 such that the statements (I) and (II) as above hold for all $T \geq T_0$ and all $Q \in \mathcal{F}$.

The volumes involved are asymptotic to cT^{n-2} for a constant $c > 0$, depending on a, b, Ω and the quadratic form. (c is determined by the volumes, and can be described in closed form.)

Thus when $p \geq 3$

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\} \sim cT^{n-2}.$$

Eskin-Margulis-Mozes show also that the left hand side is bounded *effectively* by CT^{n-2} if $p \geq 3$ and by $CT^{n-2} \log T$ if $p = 2$, for some constant $C > 0$.

The $p = 2$ case

The asymptotics as above do not hold for $p = 2$; given $q = 1$ or 2, for every $\epsilon > 0$ and interval (a, b) in $\mathbb{R}$ there exists a quadratic form Q of signature $(2, q)$, a constant $\delta > 0$ and a sequence $T_i \to \infty$ such that

$$\#\{x \in \mathbb{Z}^n \cap T_i\Omega \mid a < Q(x) < b\} \geq \delta T_i^q (\log T_i)^{1-\epsilon} \text{ for all } i.$$

The examples, first noticed by P. Sarnak, arise as irrational forms which are very well approximable by split rational forms. Sarnak showed however that the asymptotics do hold for almost all quadratic forms from the two-parameter family $(x_1^2 + \alpha x_1 x_2 + \beta x_2^2) - (x_3^2 + \alpha x_3 x_4 + \beta x_4^2)$.
This family arises in problems related to quantum chaos.

A recent theorem of Eskin, Margulis and Mozes

The constant c as above depends linearly on $(b-a)$. On the other hand, whenever a quadratic form of signature $(2,2)$ has a rational isotropic subspace, say L, then for any $\epsilon > 0$,

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid -\epsilon < Q(x) < \epsilon\} \geq \#\{x \in \mathbb{Z}^n \cap T\Omega \mid x \in L\} \geq \sigma T^2,$$

where $\sigma > 0$ is a constant independent of ϵ.

3.2 Theorem (Eskin, Margulis, Mozes) *Notation as before. Let $Q \in \mathcal{O}(2,2)$, and suppose that it is not extremely well approximable, in the sense that there exists $N > 0$ such that for all split integral forms Q' and $k \geq 2$, $\|Q - \frac{1}{k}Q'\| > k^{-N}$. Let X be the set of points in $\mathbb{Z}^4$ which are not contained in any isotropic subspace of Q. Then, as $T \to \infty$,*

$$\#\{x \in X \cap T\Omega \mid a < Q(x) < b\} \sim \mathrm{vol}\,\{v \in T\Omega \mid a < Q(v) < b\}.$$

§4. **View of orbits from infinity**

“Homecoming” of orbits of unipotent flows

4.1 Theorem (Margulis) *Let* $G = SL(n, \mathbb{R})$ *and* $\Gamma = SL(n, \mathbb{Z})$*. Let* $\{u_t\}$ *be a unipotent one-parameter subgroup of G. Then for every* $x \in G/\Gamma$ *there exists a compact set* K *of* G/Γ *such that* $\{t \geq 0 \mid u_t x \in K\}$ *is unbounded; (in other words, the trajectory* $\{u_t x\}_{t \geq 0}$ *does not “tend to infinity”).*

Via closer analysis of Margulis’s original proof I strengthened the result to the following:

4.2 Theorem (D-) *Let G and* Γ *be as in Theorem 4.1. Then for every* $\epsilon > 0$ *there exists a compact subset* K *of* G/Γ *such that for any* $x = g\Gamma \in G/\Gamma$ *and any unipotent one-parameter subgroup* $\{u_t\}$ *of G one of the following holds:*

i) $l(\{t \geq 0 \mid u_t x \notin K\}) < \epsilon T$ *for all large* T*, or*

ii) $\{g^{-1} u_t g\}$ *leaves invariant a proper nonzero rational subspace of* $\mathbb{R}^n$*.*

Relation with invariant measures and minimal sets

Analogous results hold also for general Lie groups G and lattices Γ.

Consequences:

i) Every ergodic invariant measure of a unipotent flow is necessarily finite; this turned out to be useful in Ratner's work on Raghunathan's conjecture.

ii) Every closed nonempty subset invariant under a unipotent one-parameter subgroup contains a minimal closed invariant subset, and the minimal sets are compact; this was used in our proofs of Theorems 1.3 and 2.1. The result was extended by Margulis to actions of general connected unipotent Lie subgroups acting on G/Γ.

Margulis also used the ideas involved to give a new proof of the theorem of Borel and Harish-Chandra on arithmetic subgroups of semisimple groups being lattices.

Quantitative version of Theorem 4.1:

4.3 Theorem (Kleinbock and Margulis): *Let* Λ *be a lattice in* $\mathbb{R}^n$*,* $n \geq 2$*. Then there exists* $\rho > 0$ *such that for any unipotent one-parameter subgroup* $\{u_t\}$ *of* $SL(n, \mathbb{R})$*,* $T > 0$ *and* $\epsilon \in (0, \rho)$*,*

$$l(\{t \in [0, T] \mid u_t\Lambda \cap B(\epsilon) \neq 0\}) \leq c_n(\epsilon/\rho)^{1/n^2} T,$$

where $B(\epsilon)$ *denotes the open ball of radius* ϵ *with center at* 0*, and*

c_n *is an explicitly described constant depending only on* n*.*

Their results apply also to a large class of curves, and also higher dimensional submanifolds, in the place of orbits of unipotent groups involved in the above theorem. These results are involved in their work on Diophantine approximation on manifolds (next section).

The method involved has been further sharpened in recent years by Kleinbock.

Non "quasi-unipotent" flows

A system of linear forms is "badly approximable systems" if and only if a point on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ associated with it has bounded orbit under a certain one-parameter subgroup of $SL(n,\mathbb{R})$.

4.4 Theorem (Kleinbock and Margulis):

Let G be a connected Lie group and Γ a lattice in G.

Let $\{g_t\}$ be a one-parameter subgroup of G and U the normal subgroup of G generated by the two opposite horospherical subgroups with respect to $\{g_t\}$. Suppose that $\overline{U\Gamma} = G$.

Let B be the set of points x in G/Γ such that the orbit $\{g_t x\}$ of x is bounded (relatively compact).

Then for every nonempty open subset Ω of G/Γ $B \cap \Omega$ is of Hausdorff dimension equal to the dimension of G.

Boundedness of other orbits also has significance in terms of Diophantine approximation, as shown by Kleinbock, who has further developed the topic.

§5. **Diophantine approximation on manifolds**

Some definitions and background

A point $v \in \mathbb{R}^n$ is said to be *very well approximable (VWA)* if for some $\epsilon > 0$ there exist infinitely many nonzero integers k such that $\operatorname{dist}(kv, \mathbb{Z}^n) \leq k^{-(1+\epsilon)}$.
Also v is said to be *very well multiplicatively approximable (VWMA)* if for some $\epsilon > 0$ there exist infinitely many nonzero integers k such that

$$\inf_{p \in \mathbb{Z}^n} \Pi(kv + p) \leq k^{-(1+\epsilon)},$$

where Π is the function on $\mathbb{R}^n$ defined by $\Pi(v) = |v_1 v_2 \cdots v_n|$ for $v = (v_1, v_2, \ldots, v_n)$. It turns out that if a vector is VWA then it is also VWMA.
These notions were studied by number theorists, especially since the sixties (Sprindzuk, A. Baker, Bernik, Beresnevich), inspired by a conjecture of Mahler which goes back to 1932.

Baker and Sprindzuk conjectures

Kleinbock and Margulis (1998) proved the following result, settling a conjecture of Sprindzuk (1980); the conjecture was a generalisation of a conjecture of A. Baker; the latter corresponds to the special case of $d = 1$ and $f_k(t) = t^k$, $k = 1, \ldots, n$ in the statement below.

5.1 Theorem (Kleinbock and Margulis) *Let Ω be a domain in $\mathbb{R}^d$ for some $d \geq 1$, and let $f_1, f_2, \ldots, f_n$ be n real analytic functions on Ω such that $\sum a_i f_i$ is not a constant function for any $a_1, \ldots a_n$ in $\mathbb{R}$, not all zero. Then for almost all v in Ω the vector $(f_1(v), \ldots, f_n(v))$ is not VWMA (and hence not VWA either).*

In fact they proved such a result in greater generality, allowing $f_1, \ldots f_n$ to be C^r functions satisfying a certain “nondegeneracy” condition.

The dynamical method

The question is reduced to one of estimating measures of subsets of the parameter set Ω for which $u_{f_1(v),\dots,f_n(v)}\mathbb{Z}^{n+1}$ belongs to certain neighbourhoods of infinity.

(Here $u_{t_1,\dots,t_n}$ is given by $e_0 \mapsto e_0$ and $e_i \mapsto e_i + t_i e_0$, $i = 1,\dots,n$.)

From this point the ideas are related to Theorems 4.1 and 4.2, but now appear in quantitative and highly intricate form!

A modification of the method was used by Bernik, Kleinbock and Margulis to prove “the convergence part” of the Khintchine-Groshev theorem for nondegenerate smooth submanifolds of $\mathbb{R}^n$.

Kleinbock and Margulis have also other results on Khintchine-Groshev theorem that I will not go into.

Thank you.