Affine Geometry and Hyperbolic Geometry

William Goldman
University of Maryland

Lie Groups: Dynamics, Rigidity, Arithmetic

A conference in honor of Gregory Margulis's 60th birthday

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Discrete groups of affine transformations

Milnor (1977) realized a close relation between the properness of affine actions, amenability, and Tits's alternative for groups of real matrices:

Let $\Gamma \subset \mathsf{Aff}(\mathbb{E}^n)$ be discrete. The following conditions are equivalent:

- Γ is virtually polycyclic (iterated extension of cyclic and finite groups);
- Γ is virtually solvable;
- Γ is amenable;
- Γ does NOT contain $\mathbb{Z} \star \mathbb{Z}$ (Tits).

Milnor asked:

Can a nonamenable group act properly and affinely on $\mathbb{E}^n$?

According to Tits's alternative, this question is equivalent to::

Can $\mathbb{Z} \star \mathbb{Z}$ act properly on $\mathbb{E}^n$ by affine transformations?

In the early 1980's, Margulis showed that the answer is *YES.*

Evidence??

- *A connected Lie group admits a proper affine action $\iff$ it is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

Schottky (1907):
Proper actions of free groups on $\mathbb{E}^n$
(isometries of hyperbolic space).
Not affine!

Milnor suggested:

"Start with a free discrete subgroup of $\mathsf{O}(2,1)$ and and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous."

Auslander's "Conjecture"

If $\Gamma \subset \mathsf{Aff}(\mathbb{E}^n)$ *is discrete and acts properly with* M *compact, then* Γ *is virtually polycyclic.*

Then $M = \mathbb{E}^n/\Gamma$ finitely covered by *solvmanifold* G/Γ, for solvable Lie group G.

Proved in dimension 3 (Fried-Goldman 1983). Now known up to dimension ≤ 6, and under conditions on *the Zariski closure of* $\mathbb{L}(\Gamma) \subset \mathsf{GL}(n, \mathbb{R})$.

- $\mathsf{O}(n-1, 1)$(Goldman-Kamishima);
- $\mathsf{O}(3, 1)$ (Fried);
- $\mathsf{O}(2, 2)$ (Wang);
- $\mathbb{R}$-rank one subgroup (Grunewald-Margulis);
- Product of $\mathbb{R}$-rank one subgroups (Tomanov);
- $\mathsf{O}(p, n-p)$, unless $n = 2p + 1$, and p is odd. (Abels-Margulis-Soifer)

(Abels-Margulis-Soifer): $\exists$ proper actions, with $\mathbb{L}(\Gamma)$ Zariski-dense in $\mathsf{O}(2k, 2k-1)$ (but $\mathbb{E}^{4k-1}/\Gamma$ is not compact).

Hyperbolic geometry

Milnor's idea is the *only way* to construct interesting proper affine actions on $\mathbb{E}^3$.

Suppose $\Gamma \subset \mathsf{Aff}(\mathbb{E}^3)$ is discrete, infinite and non-solvable, acting properly and freely.

Linear holonomy

$$\mathsf{Aff}(\mathbb{E}^3) \xrightarrow{\mathbb{L}} \mathsf{GL}(3, \mathbb{R})$$

maps Γ isomorphically onto a discrete subgroup of (a conjugate of) $\mathsf{O}(2,1) \subset \mathsf{GL}(3, \mathbb{R})$. (Fried-Goldman 1983).

The quotient manifold $M := \mathbb{E}^3/\Gamma$ is *a complete flat Lorentz manifold.*

The corresponding hyperbolic surface

$$\Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

is *not compact* (Mess 1990) so $\pi_1(\Sigma) \cong \Gamma$ is free.

Closed geodesics

Most elements $\gamma \in \Gamma$ are *boosts,* affine deformations of hyperbolic elements of $\mathsf{O}(2,1)$. Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

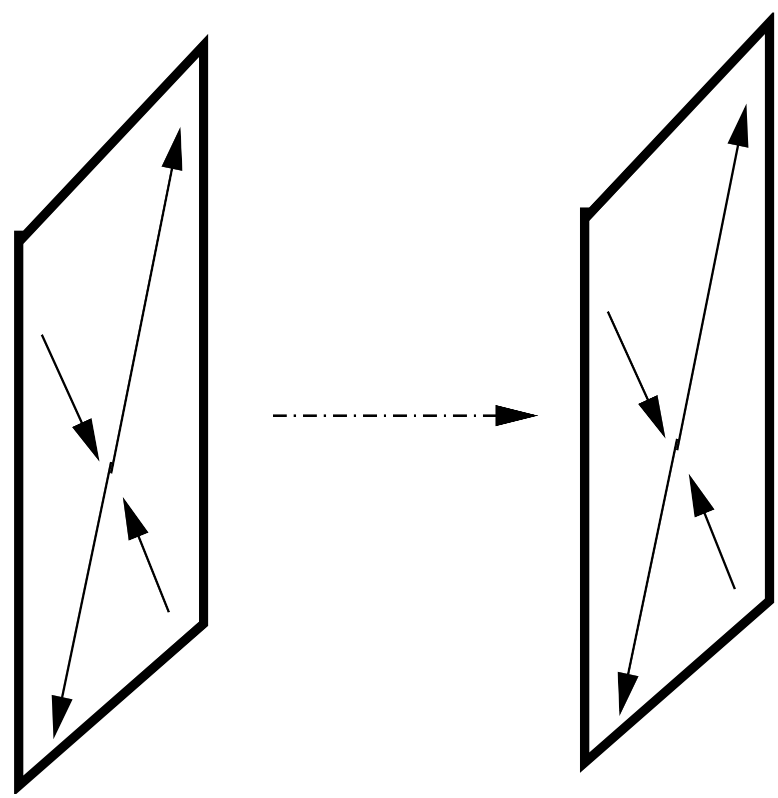

Figure 1. A boost identifying two parallel planes

Affine deformations

Equivalence classes of affine deformations comprise $H^1(\Gamma_0, \mathbb{R}^{2,1})$. Translational part

$$\Gamma_0 \xrightarrow{u} \mathbb{R}^{2,1}$$

defined by

$$\rho(\gamma)x = \mathbb{L}(\gamma)x + u(\gamma)$$

is a cocycle $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$.

Write $\rho = \rho_u$ and $\Gamma = \Gamma_u$.

Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.

Geometrically, $[u] \in H^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* of the hyperbolic surface Σ, that is a tangent vector to the Fricke-Teichmüller space $\mathfrak{F}_\Sigma$.

Classification

- Which discrete subgroups
$$\Gamma_0 := \mathbb{L}(\Gamma) \subset \mathsf{O}(2,1)$$
occur? Equivalently, which hyperbolic surfaces $\Sigma = \mathsf{H}^2/\Gamma_0$ arise? These comprise a subset of the Fricke-Teichmüller space
$$\mathfrak{F}_\Sigma \subset \mathsf{Hom}(\Gamma_0, \mathsf{O}(2,1))/\mathsf{O}(2,1).$$
- Given Fuchsian $\Gamma_0 \subset \mathsf{O}(2,1)$, which *affine deformations* $\Gamma_0 \xrightarrow{\rho_{[u]}} \mathsf{Aff}(\mathbb{E}^3)$ act *properly* on $\mathbb{E}^{2,1}$? These comprise a subset of the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

$$\begin{array}{ccc} & & \Gamma \\ & \overset{\rho_{[u]}}{\nearrow} \cong & \downarrow \mathbb{L} \\ \Gamma_0 & \hookrightarrow & \mathsf{O}(2,1). \end{array}$$

Such proper affine deformations $\Gamma = \Gamma_{[u]}$ of $\Gamma_0 \subset \mathsf{O}(2,1)$ correspond to *Margulis spacetimes* $\mathbb{E}^{2,1}/\Gamma_{[u]}$.

Affine Schottky Groups

Theorem (Drumm)**.** *Every free discrete subgroup* $\Gamma_0 \subset \mathsf{O}(2,1)$ *admits a proper affine deformation* Γ *with quotient* $\mathbb{E}^{2,1}/\Gamma$ *a solid handlebody.*

Drumm's construction derived a fundamental polyhedron for Γ acting on $\mathbb{E}^{2,1}$ from a fundamental polygon for Γ_0 acting on H^2. He constructed fundamental polyhedra out of *crooked planes*.

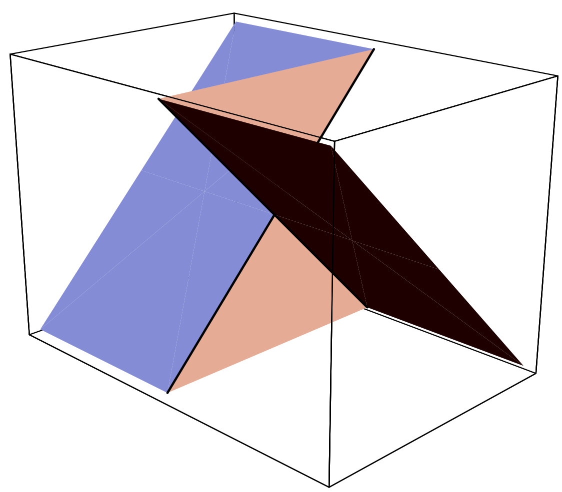

Figure 2. A crooked plane

Drumm's Fundamental Polyhedra

Disjoint crooked planes bound *crooked slabs.* Boosts identify the faces of a crooked slab.

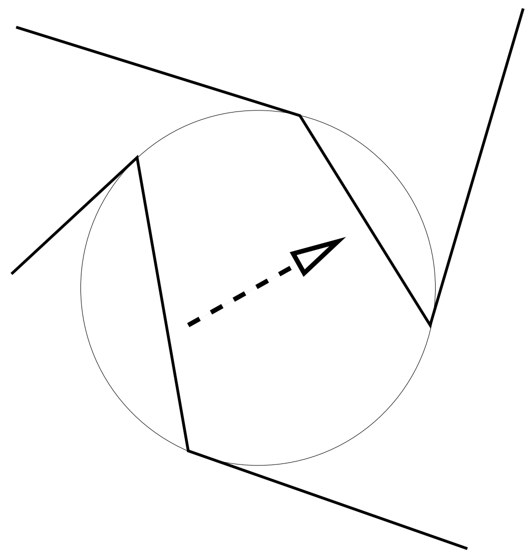

Figure 3. Two disjoint crooked planes

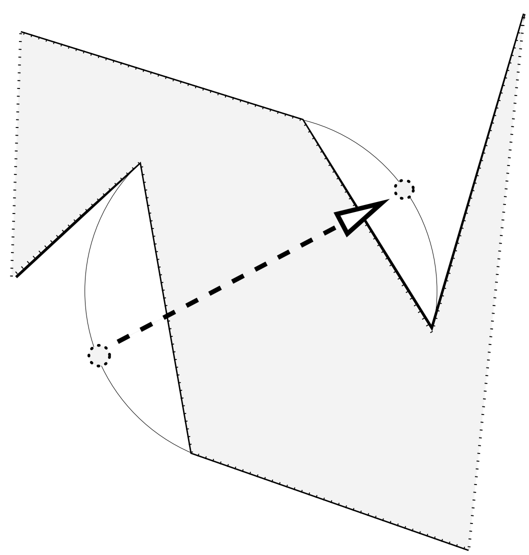

Figure 4. A crooked slab is a fundamental domain for a boost

Cyclic groups

Orbits of the crooked slab fill an open dense subset of $\mathbb{E}^{2,1}$.

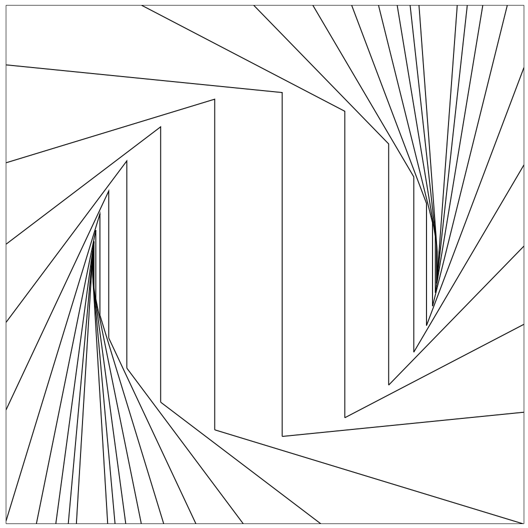

Figure 5. Images of crooked planes under a linear hyperbolic cyclic group

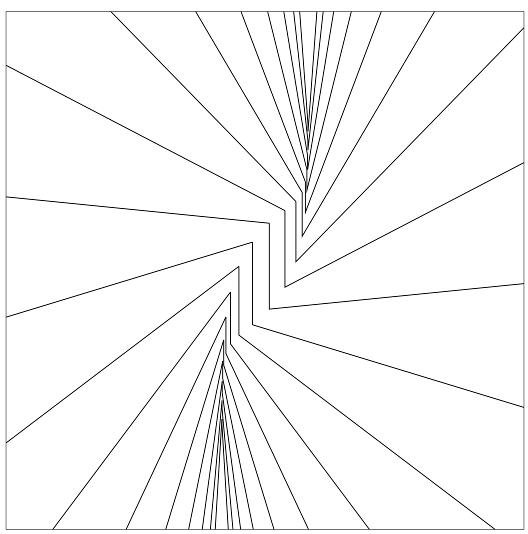

Figure 6. Images of crooked planes under an affine hyperbolic cyclic group

Free groups

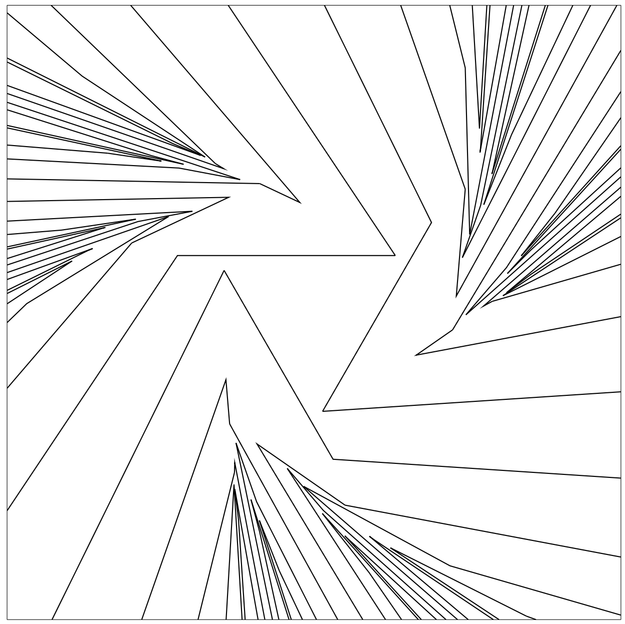

Figure 7. Affine action of Schottky group

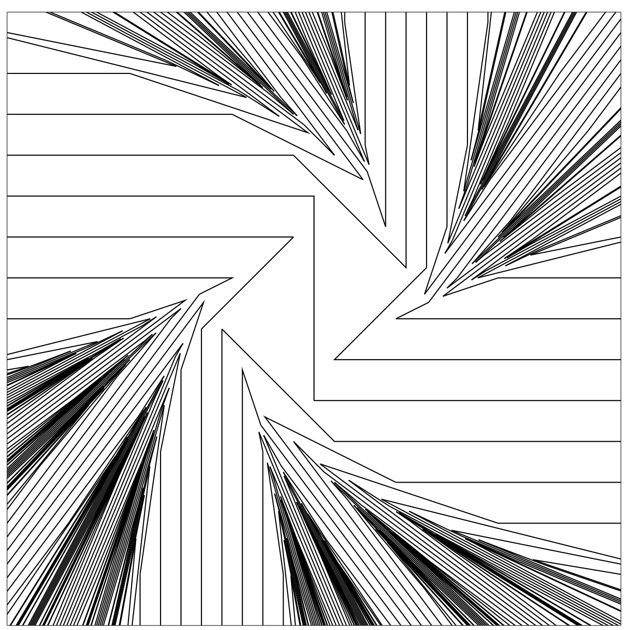

Figure 8. Affine action of modular group

Tameness Questions

Question. *Does every proper affine deformation of a free group admit a fundamental polyhedron bounded by crooked planes?*

Question. *Is every Margulis space-time homeomorphic to a solid handlebody?*

Classification

Theorem. *For a given finitely generated free discrete subgroup* $\Gamma_0 \subset \mathsf{O}(2,1)$*, the subset of*

$$[u] \in H^1(\Gamma_0, \mathbb{R}^{2,1})$$

corresponding to **proper** *affine deformations is an open convex cone.*

- Γ_0 Schottky (Goldman-Labourie-Margulis)
- Extended to when Γ_0 has parabolics (Charette-Goldman).

Margulis's invariant

$$\Gamma_0^H := \{\text{hyperbolic elements in} \Gamma_0\}.$$

For an affine deformation $\Gamma = \Gamma_{[u]}$, Margulis defined

$$\Gamma_0^H \xrightarrow{\alpha_{[u]}} \mathbb{R}.$$

- (Class function) $\alpha(\gamma) = \alpha(\eta^{-1}\gamma\eta)$;
- $\alpha(\gamma) = 0 \iff \gamma$ fixes a point in $\mathbb{E}^{2,1}$.
- (Absolute linearity) $\alpha(\gamma^n) = |n|\alpha(\gamma)$;
- If Γ acts properly, $|\alpha(\gamma)| =$ Lorentzian length of the unique closed geodesic in $\mathbb{E}^{2,1}/\Gamma$ corresponding to γ.
- (Isospectrality) Affine deformations with same $\alpha_{[u]}$ are conjugate (Charette-Drumm-Goldman).
- If $[u]$ represents a deformation Σ_t of hyperbolic surfaces, then

  $$\alpha(\gamma) = \left.\frac{d}{dt}\right|_{t=0} \ell_{\Sigma_t}(\gamma)$$

  where $\ell_{\Sigma_t}(\gamma)$ is the length of the closed geodesic γ in the surface Σ_t.

Dynamics and Margulis's invariant

Margulis (1983) deduced properness from:

Theorem. *Suppose* $\exists g_0 \in \mathsf{O}(2,1))$ *such that:*

- *(ϵ-Hyperbolicity) The invariant axes of* $g_0 g$ *stay bounded,* $\forall g \in \Gamma_0$*;*
- $|\alpha(g_0\gamma)|$ *grows linearly with respect to the word-length of* $\gamma \in \Gamma$*.*

Then Γ *acts properly.*

and non-properness from:

Theorem (Opposite Sign Lemma)**.**

$$\alpha(\gamma_1)\alpha(\gamma_2) < 0$$

implies $\langle \gamma_1, \gamma_2 \rangle$ *does* not *act properly.*

Margulis (1983): If $\alpha(\gamma_1), \alpha(\gamma_2)$ are either both positive or both negative, then $\exists m, n$ such that the *subgroup*

$$\langle \gamma_1^m, \gamma_2^n \rangle \subset \langle \gamma_1, \gamma_2 \rangle$$

acts properly.

Extension of Margulis's invariant: Parabolics

Charette and Drumm extended α to all of Γ_0 — however only the *sign* of $\alpha(\gamma)$ is defined when γ is parabolic — and the Opposite Sign Lemma now holds for all Γ_0.

Properness of affine deformations

Question. *Is Margulis's necessary condition, that $\alpha(\gamma)$ are all positive (or all negative), sufficient for properness of $\langle\gamma_1, \gamma_2\rangle$ itself?*

If Σ is a 3-holed sphere with boundary $\partial_1, \partial_2, \partial_3$, then

$$\alpha(\partial_1), \alpha(\partial_2), \alpha(\partial_3) > 0 \quad \text{or}$$
$$\alpha(\partial_1), \alpha(\partial_2), \alpha(\partial_3) < 0$$

implies:

- Γ acts properly,
- Γ has a crooked fundamental domain,
- $\mathbb{E}^{2,1}/\Gamma$ is homeomorphic to a solid handlebody of genus 2.

(Brenner-Charette-Drumm–Jones).

Extension of Margulis's invariant: Geodesic Currents

Margulis's invariant *extends* from a class function on Γ_0 to a function defined on the convex set $\mathcal{C}(\Sigma)$ of *geodesic currents* on Σ.

Theorem (Goldman-Labourie-Margulis)**.** *Let* $\Gamma_0 \subset \mathsf{O}(2,1)$ *be a Schottky group.* $\exists$ *continuous biaffine map*

$$\mathcal{C}(\Sigma) \times H^1(\Gamma_0, \mathbb{R}^{2,1}) \xrightarrow{\Psi} \mathbb{R}.$$

- *If* $\gamma \in \Gamma_0$*, and* μ *is the corresponding geodesic current, then*
$$\Psi(\mu, [u]) = \frac{\alpha_{[u]}(\gamma)}{\ell_\Sigma(\gamma)}$$
where $\ell_\Sigma(\gamma)$ *is the length of the closed geodesic on* Σ *corresponding to* γ*.*
- $\Gamma_{[u]}$ *acts properly on* $\mathbb{E}^{2,1} \iff \Psi(\mu, [u]) \neq 0$ *for all* $\mu \in \mathcal{C}(\Sigma)$*.*

Corollary. *The set of proper affine deformations of* Γ_0 *is the open convex cone in* $H^1(\Gamma_0, \mathbb{R}^{2,1})$ *defined by the functionals*

$$[u] \xmapsto{\alpha^\mu} \Psi(\mu, [u])$$

for $\mu \in \mathcal{C}(\Sigma)$*.*

Deformation space of Margulis space-times

The set of proper affine deformation identifies with a bundle of open convex cones in $H^1(\Gamma_0, \mathbb{R}^{2,1})$ over $\mathfrak{F}_\Sigma$.

Sufficient to test *measured geodesic laminations* μ. Thus, proper affine deformations correspond to infinitesimal deformations of Σ which *lengthen* all measured geodesic laminations.

Figures 9 and 10 depict the subspaces of the parameter space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ defined by the linear functionals

$$\begin{aligned} H^1(\Gamma_0, \mathbb{R}^{2,1}) &\xrightarrow{\alpha^\gamma} \mathbb{R} \\ [u] &\longmapsto \alpha_{[u]}(\gamma) \end{aligned}$$

when γ corresponds to a free generator of Γ. Functionals α^μ are limits of functionals α^γ, where γ are closed geodesics on Σ.

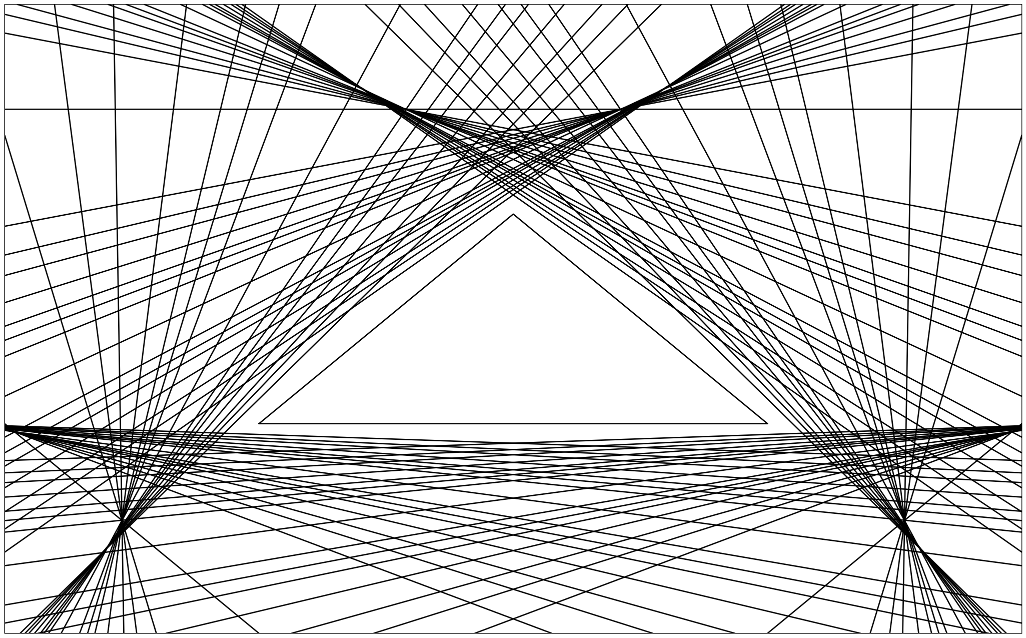

FIGURE 9. Linear functionals α^γ when Σ is a three-holed sphere

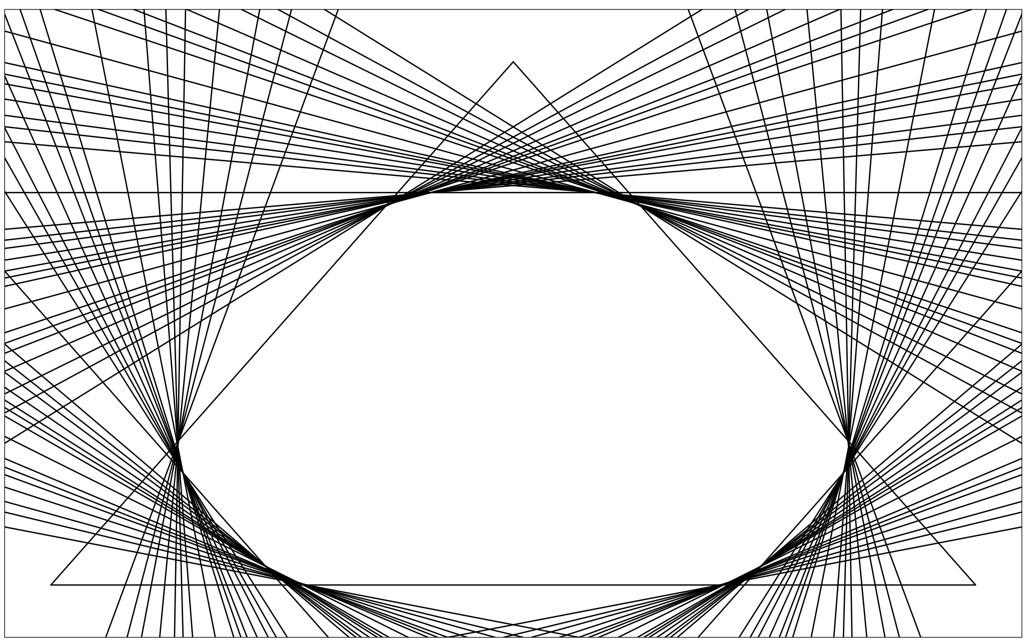

FIGURE 10. Linear functionals α^γ when Σ is a one-holed torus

Geometry of the deformation space

Figure 9 depicts the case when Σ is a three-holed sphere; the central triangle is the region where all $\alpha^{\partial_i} > 0$ for $i = 1, 2, 3$, and all other functionals α^γ are positive there. The only measured laminations are the ∂_i, and the deformation is a finite polyhedron.

Figure 10 depicts the case when Σ is a one-holed torus; here infinitely many functionals define the deformation space. Points on the boundary either lie on intervals (corresponding to simple closed curves) or are points of strict convexity (corresponding to irrational laminations) (Goldman-Margulis-Minsky).

In general, a third type of single boundary point corresponds to parabolic elements of Γ_0, that is, the *cusps* of Σ.