MEASURE RIGIDITY BEYOND UNIFORM HYPERBOLICITY: RESULTS AND OPEN PROBLEMS

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Measure rigidity

Algebraic actions

Goal: every ergodic Borel probability invariant measure for an action of a higher rank abelian group is either essentially of algebraic nature or comes from a rank one factor on an algebraic invariant set.

This problem was discussed in the talk by Elon Lindenstrauss primarily for homogeneous actions.

All existing results (at least for smooth actions) use some form of positive entropy assumption. Its weakest form:

($\mathfrak{P}$) *Some element of the action has positive metric entropy*

The geometric approach to measure rigidity is based on the study of conditional measures on various invariant foliations for the action which expanded by some elements and contracted by others.

The entropy assumption is necessary for that approach since for zero entropy invariant measures all those invariant measure are simply δ–measures. The crucial distinction between the rank one and higher rank cases is that in higher rank

Some elements of the action act isometrically along some of those foliations

In order to use this one needs a form of ergodicity assumption for those elements.

There are three essential tools or methods within this approach in order of their chronological appearance:

- Geometry of Lyapunov exponents and derivative objects, in particular Weyl chambers (A.K., R.Spatzier, B.Kalinin)

  We will use a far–reaching generalization of this method.

- The non-commutativity and specific commutation relations between various invariant foliations (M. Einsiedler, A.K., E. Lindenstrauss)

  This includes the high entropy method and the low entropy method discussed by Linderstrauss. Those are crucial for number–theoretic applications.

- Diophantine properties of global recurrence (M. Einsiedler, E. Lindenstrauss)

  This method gives best results for actions by automorphisms of a torus

Anosov actions

There is rigidity program for *Anosov actions* of higher rank abelian groups separate from the measure rigidity program.

All standard Anosov algebraic (homogeneous or affine) actions of higher rank abelian groups are locally rigid (M. Guysinsky, A.K., R. Spatzier) There are some remarkable (but not complete) global rigidity results (F. Rodriguez Hertz; B. Kalinin, R. Spatzier).

Measure rigidity sometimes can be shown even without proving differentiable conjugacy to an algebraic model.

EXAMPLE. Let α_0 be a linear $\mathbb{Z}^k$ action on a torus which contains a $\mathbb{Z}^2$ subaction all of whose elements other than identity are ergodic. Any Anosov action α homotopic to α_0 preserves a smooth measure. This follows from two facts:

1. The action α is Hölder conjugate to α_0
2. Hölder cocycles over α_0 and hence over α are rigid and hence the logarithm of the Jacobian for α is cohomologous to a constant which must be equal to zero by preservation of the total volume

If α_0 is irreducible, i.e. contains an element with irreducible over $\mathbb{Q}$ characteristic polynomial Lebesgue measure is the only positive entropy ergodic invariant measure for α_0 (Einsiedler–Linderstrauss). Hence the smooth invariant measure is the only positive entropy invariant measure for α.

Invariant Measures for Cartan actions on Tori

joint work with Boris Kalinin

http://www.math.psu.edu/katok_a/papers.html

An action of $\mathbb{Z}^k$ on $\mathbb{T}^{k+1}$, $k \geq 2$, by automorphisms which are ergodic with respect to the Lebesgue measure is called a (linear) *Cartan action*.

Every element of a Cartan action other than identity is hyperbolic and has distinct real eigenvalues, and the centralizer of a Cartan action in the groups of automorphisms of the torus is a finite extension of the action itself. (Dirichlet Unit Theorem about number fields).

Lyapunov characteristic exponents of the linear action α_0 are independent of an invariant measure and are equal to the logarithms of the absolute values of the eigenvalues. They all have multiplicity one and no two of them are proportional.

MAIN THEOREM. Any action α homotopic to a linear cartan action α_0 preserves an absolutely continuous ergodic measure μ. Furthermore,

- The Lyapunov exponents (and hence the entropy function) for α wrt μ are equal to those for α_0.
- There exists a partition of a set A of full measure μ into finitely many sets $A_1, \ldots, A_m$ of equal measure such that every element of α permutes these sets. Furthermore, there is a subgroup of finite index $\Gamma \subset \mathbb{Z}^k$ such that for any $\gamma \in \Gamma$ other than identity $\alpha(\gamma)A_i = A_i,\ i = 1, \ldots, m$, and the restriction of $\alpha(\gamma)$ to each set A_i is Bernoulli.

In particular, if generators of α are ergodic then they are Bernoulli.

This is a first known result concerning abelian groups of diffeomorphisms where existence of an invariant geometric structure is obtained from homotopy data.

Ingredients of the proof

- Topological semi-conjugacy between α and α_0. (J. Franks).
- Pesin theory for non-uniformly hyperbolic systems: Lyapunov characteristic exponents, contracting, expanding and Lyapunov "foliations".
- Invariant affine parameters on Lyapunov manifolds.
- Entropy calculations (F. Ledrappier, L-S Young, extending Y. Pesin, going back to Margulis; higher rank argumetns based on H. Hu)
- THE KEY STEP: rigidity expansion coefficients based on distortion estimates along singular directions.
- π–partition trick to verify ergodicity along critical directions.

The semi–conjugacy

There is a unique surjective continuous map $h : \mathbb{T}^{k+1} \to \mathbb{T}^{k+1}$ homotopic to identity such that

$$h \circ \alpha = \alpha_0 \circ h.$$

Consider the set of all Borel probability measures ν on $\mathbb{T}^{k+1}$ such that $(h)_*\nu = \lambda$, where λ is Lebesgue measure on $\mathbb{T}^{k+1}$. This set is convex, weak* compact, and α invariant. Hence by Tychonoff theorem it contains a nonempty subset $\mathcal{M}$ of measures invariant under α. Since α_0 is ergodic with respect to λ, almost every ergodic component of an α-invariant measure $\nu \in \mathcal{M}$ also belongs to $\mathcal{M}$. Let μ be such an ergodic measure.

THEOREM. Any ergodic α-invariant measure μ such that $(h)_*\mu = \lambda$, is absolutely continuous.

Since any α-invariant measure whose ergodic components are absolutely continuous is itself absolutely continuous we obtain the following.

COROLLARY. Every measure $\nu \in \mathcal{M}$ is absolutely continuous and has no more than countably many ergodic components. Hence $\mathcal{M}$ contains at most countably many ergodic measures.

THEOREM. For any ergodic measure $\mu \in \mathcal{M}$ the semiconjugacy h is finite-to-one in the following sense. There is an α-invariant set A of full measure μ such that for λ almost every $x \in \mathbb{T}^{k+1}$, $A \cap h^{-1}(\{x\})$ consists of equal number s of points and the conditional measure induced by μ assigns every point in $A \cap h^{-1}(\{x\})$ equal measure $1/s$.

Open problems

- *More about the semi-conjugacy in the homotopically Cartan case.*

  CONJECTURE. The set $\mathcal{M}$ consists of a single measure. The semiconjugacy h is a measurable isomorphism between actions α and α_0.

- *Structure of the action.* Periodic orbits blowup. What else?

- *Actions in other homotopy classes.*

- *Measure rigidity without semi-conjugacy.*

  MODEL CONJECTURE. Let α be a $\mathbb{Z}^2$ action on a three-dimensional manifold with an ergodic hyperbolic measure μ whose Lyapunov exponents are not pairwise proportional (Lyapunov lines are different) and at least one elemnt has positive entropy. Then μ is absolutely continuous.