...AND COUNTING

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G.A. Margulis: *On some applications of ergodic theory to the*
manifolds of negative curvature, Funct. Anal. Appl. **3**(1969),

G.A. Margulis: *Certain measures associated with U–flows on*
manifolds, Funct. Anal. Appl. **4** (1970), 133–144.

G.A. Margulis: *On some aspects of the theory of dynamical s*
Springer Monographs in Mathematics, (2004).

$\widetilde{M}$ simply connected manifold with pinched negative curva

Γ cocompact torsionfree group of isometries of $\tilde{M}$, $M = \Gamma$

$\{\Gamma y\}$, for some $y \in \tilde{M}$.

$$\pi_{x,y}(R) \ = \ \mathrm{Card}\{\gamma \in \Gamma : d(x, \gamma y) < R\}$$

Theorem 1 *(Margulis 70) There exist a function* $c : M \mapsto \mathbb{R}_+$ *positive number* H *such that, as* R *goes to infinity:*

$$\pi_{x,y}(R) \sim c(x)c(y)e^{HR}$$

H is the topological entropy of the geodesic flow,

c is a smooth function, constant for symmetric spaces,

c constant implies constant curvature in dimension 2.

Corollary 1 *(M 70)*

$$VolB(x,R) \sim c(x)\int_M c(y)dvol(y)e^{HR}$$

Proof.

Let D be a fundamental domain for the action of Γ on $\widetilde{M}$. The

$$\begin{aligned} \mathrm{VolB}(\mathrm{x},\mathrm{R}) &= \sum_{\gamma\in\Gamma} \mathrm{Vol}\left(\mathrm{B}(\mathrm{x},\mathrm{R})\cap\gamma\mathrm{D}\right) \\ &= \sum_{\gamma\in\Gamma}\int_D 1_{d(x,\gamma y)<R}\, dvol(y) \\ &= \int_D \pi_{x,y}(R)\, dvol(y). \end{aligned}$$

$\pi_{x,y}(R)$ is the number of geodesic arcs $\sigma(t), 0 \leq t \leq L$, such th
$\sigma(0) = x, \sigma(L) = y, L \leq R$.

For $A \subset T_x^1 M$, $B \subset T_y^1 M$, let $\pi_{A,B}(\delta, R)$ be the number of geod
$\sigma(t), 0 \leq t \leq L$, such that $\dot{\sigma}(0) \in A, \dot{\sigma}(L) \in B, R - \delta \leq L \leq R$.
Theorem 1 follows from the more precise:

Theorem 2 *(M70) There exists a family of measures μ_x on th spheres $T_x^1 M$ such that, for suitable A and B:*

$$\lim_{\delta \to 0} \lim_{R \to \infty} \frac{\pi_{A,B}(\delta, R)}{\delta e^{HR}} \sim \mu_x(A) \mu_y(B).$$

Theorem 2 implies that $\pi_{x,y}(R) = \int^R \mu_x(T_x^1 M) \mu_y(T_y^1 M) e^{Ht}$ d

Let now $A \subset W_x^{uu}$, $B \subset W_y^{ss}$, and let $\pi_{A,B}(\delta, R)$ be the number

arcs $\sigma(t), 0 \leq t \leq L$, such that $\dot{\sigma}(0) \in A, \dot{\sigma}(L) \in B, R - \delta \leq L$

Theorem 2 follows from:

Theorem 3 *(M70) There exist a family of measures μ_x^u on the unstable manifolds W^{uu} and a family of measures μ^s on the st manifolds W^{ss} such that, for suitable A and B:*

$$\lim_{\delta \to 0} \lim_{R \to \infty} \frac{\pi_{A,B}(\delta, R)}{\delta e^{HR}} = \mu_x^u(A) \mu_y^s(B).$$

One key point of the proof is that the measure defined locally

$$\mu^u \otimes \mu^s \otimes dt$$

is a geodesic flow invariant probability measure which is MI

Corollary 2 *(M70) The number $\pi(R)$ of closed geodesics of*
length smaller that R satisfies, as $R \to \infty$,

$$\pi(R) \sim \frac{e^{HR}}{HR}.$$

Proof.

Compare the number of closed geodesics of length close to R
neighbourhood of x with

$\pi_{W^u_\delta(x), W^s_\delta(x)}(2\delta, R)$

and cover T^1M by almost disjoint $\left(W^u_\delta(x) \times W^s_\delta(x) \times [-\delta, \delta]\right)$

Corollary 3 *(Bowen 72) Equidistribution of closed geodesic*
measure is the above Margulis measure, the measure of maxi
H.

Corollary 4 *(Bowen-Marcus 77) There is only one (ray of) f*
measures on transversals to W^{uu} *which is invariant under* W^{u}
holonomies.

Proof would use an averaging

convergence theorem along

sets $\phi_T^{-1} B_\varepsilon^u(\phi_T x)$

Same is true for Anosov diffeomorphisms or flows (M70),

Axiom A diffeomorphisms and flows (Parry-Pollicott 83).

Analog of volume result with another mixing invariant measu

Same $\widetilde{M}$ and Γ.

Set $G(x,y)$ for the Green function of the Laplacian on $\widetilde{M}$, $S_R($
at distance R from x. Then,

Theorem 4 *There is a positive number L such that, as $R \to$*

$$\int_{S_R(x)} G(x,y)dy \to \frac{1}{L}.$$

Remark: $L \leq H$, with equality only for locally symmetric sp

Case when Γ is NOT cocompact, nonelementary,

divergent: the Poincaré series $\sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x)}$ diverges at the crit

and nonarithmetic: the lengths of the periodic orbits generate subgroup of $\mathbb{R}$.

Then the Patterson-Sullivan invariant measure m_x on $X = \Gamma \backslash$ conservative, ergodic and mixing, and

Theorem 5 *(Roblin 03)*

$$If\ m_x(X) < +\infty,\ then\ Card(\Gamma x \cap B(x,R)) \ \sim \ \frac{e^{\delta_\Gamma R}}{\delta_\Gamma m_x(X)}\ a$$

$$If\ m_x(X) = +\infty,\ then\ Card(\Gamma x \cap B(x,R)) \ = \ o(e^{\delta_\Gamma R})\ as$$

Analog of counting and distribution of periodic orbits for rank manifolds.

A geodesic in a manifold of non-positive curvature is called *r* doesn't admit any nontrivial parallel Jacobi field.

A compact manifold of non-positive curvature is called *rank*- contains at least one regular closed geodesic.

Theorem 6 *(Gunesch) Consider a rank -one compact manifo number* $\pi(R)$ *of regular closed geodesics of primitive length s* R *satisfies, as* $R \to \infty$*,*

$$\pi(R) \sim \frac{e^{HR}}{HR},$$

where H *is the topological entropy of the geodesic flow. Thos geodesics are equidistributed according to the measure of ma entropy.*

Higher rank. G a connected semisimple Lie group with finite
closed subgroup, Γ a lattice in G such that $\Gamma \backslash G/H$ is a finite
affine locally symmetric space. Then,

Theorem 7 *(Eskin-McMullen (93)) The translates* Yg *of the*
$Y = (\Gamma \cap H) \backslash H$ *become equidistributed as g goes to infinity.*
Furthermore, for any wellrounded sequence of subsets $B_n \subset$
$n \to \infty$,

$$Card\,(\Gamma eH \cap B_n) \sim Vol B_n.$$

Wellrounded means more and more invariant by the action of
neighborhood of the identity.

Many further applications...

Gorodnik and Oh: Higher rank analog of Theorems 2 and 3, v

$\widetilde{M}$ is a symmetric space of noncompact type,

and Γ a lattice.

Gorodnik, Maucourant and Oh: Counting K rational points of absolutely almost simple group G over a field K.

Here mixing (with speed) comes from the mixing in $G(K) \setminus G$

Quint: Higher rank and Γ a Schottky group.

Here mixing comes from symbolic dynamics and spectral the transfer operator.

L + Babillot, Sarig. $\mathbb{Z}^d$ covers of compact surfaces.

From counting periodic orbits to estimates for integrals along

Theorem 8 *Let M be a $\mathbb{Z}^d$–cover of a compact hyperbolic su is a number σ such that for every $f \in L^1(m_0)$, for m_0 a.e. ω:*

$$\lim_{N\to\infty} \frac{1}{\ln\ln N}\int_3^N \frac{1}{T\ln T}\left(\frac{1}{a(T)}\int_0^T f(h^s\omega)ds\right)dT = \int$$

where $a(T) := \dfrac{1}{(4\pi\sigma)^{d/2}}\dfrac{T}{(\ln T)^{d/2}}$.

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