Rigidity Theory of Discrete Groups

Study countable groups Γ via methods which are analytic, measure theoretic, probabilistic, geometric etc., by relating them to "continuous" objects (groups, spaces) with rich structure.

One Method: Embed Γ as a discrete and co-compact subgroup of a locally compact (l.c.) group G.

What does Γ "inherit" from G ?

Example: For $\Gamma < G$ as above, if G is compactly generated then Γ is finitely generated.

Definition. A group Γ is called just infinite if every proper quotient of it is finite.

* "Just infinite" sometimes upgraded to "simple".
* In many situations (e.g. $\Gamma < GL_n(F)$) there are always "many" finite quotients – *cannot be simple*.
* Every infinite f.g. group Γ has an infinite just infinite quotient (which is not true for "simple").

Definition. A l.c. group G is called just non-compact if every proper quotient of it is compact.

Phenomenon: "Many" automorphism (l.c.) groups of "symmetric" structures are just non-compact.

NB: G can be simple, $\Gamma < G$ not even just infinite e.g. $G = PSL_2(\mathbb{R}), Aut$(tree) $\quad \Gamma = \pi_1(\Sigma_g)$, free

Main Theorem (with Uri Bader, to appear Invent.)

The following holds, excluding one counter-example:

Let G_1, G_2 be comp. gen. non-discrete l.c. groups, and $\Gamma < G = G_1 \times G_2$ be a discrete co-compact subgroup, with dense projections to both G_i's.

If each G_i is just non-compact, Γ is just infinite.

The unique counter-example: $G_1 = G_2 = \mathbb{R}$, $\Gamma = \mathbb{Z}^2 < \mathbb{R}^2 = G$ "irrationally embedded".

Margulis Normal Subgroup Theorem (75): For G algebraic of split rank ≥ 2 over local fields, e.g., $G = SL_{n\geq 3}(\mathbb{R})$. For $\Gamma < G$ only known proof !

Burger-Mozes (IHES 2000): Thm for $G_i <$ Aut(tree). Upgrade to get simple, torsion free, f.p. ... grps.

Main Thm holds more generally when $\Gamma < G$ is a <u>lattice</u> (G/Γ admits a finite G-invariant measure), provided a certain technical condition holds.

Example: $SL_2(\mathbb{Z}[\sqrt{2}]) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

Kac-Moody Groups: "Infinite dim. algebraic grp" $\mathbf{\Lambda}$, associated to a "generalized Cartan matrix".

If F is a finite field, $\mathbf{\Lambda}(F)$ is an infinite, f.g. group.

Theorem (Remy): Assume $\mathbf{\Lambda}$ has an irreducible Weyl group generated by s Coxeter generators. Then for $q \geq s$ $\Gamma = \mathbf{\Lambda}(F_q)$ fits the setting of the Main Theorem (above extension to lattices).

Caprace–Remy(2006): If $\mathbf{\Lambda}$ is of *non-affine* type, $\mathbf{\Lambda}(F_q)$ has **no** (non-trivial) finite quotients.

...+ Remy + Main Thm $\Rightarrow \mathbf{\Lambda}(F_q)$ is <u>*simple*</u> ($q \geq s$).

Definition: Γ has Kazhdan's Property (T) if every isometric Γ-action on a Hilbert space fixes a point.

Examples: Finite (compact) groups, $SL_{n\geq 3}(\mathbb{Z})$.

Definition: Γ is amenable if $\forall$ continuous action on a compact space X, $\exists$ Γ-invariant probability measure m: $m(\gamma A) = m(A) \quad \forall \gamma \in \Gamma, A \subseteq X$.

Basic Fact: $(T) \cap$ (Amenable) $=$ (Finite)

Proof of Main Theorem follows the original remarkable strategy of Margulis, implemented differently: For $N \triangleleft \Gamma$ show Γ / N is both amenable and (T) .

Two completely independent "halves" of the proof.

Kazhdan's Property (T)

Any isometric Γ-action on a Hilbert space V is of the form

$\gamma v = \pi(\gamma)v + b(\gamma)$ where

π: $\Gamma \to U(V)$ (the linear part) is a unitary Γ-representation

b: $\Gamma \to V$ (the affine part) is a 1-cocycle $(\in Z^1(\Gamma, \pi))$

Action has a fixed point $(v_0 \in V) \Leftrightarrow b(\gamma) = v_0 - \pi(\gamma)v_0$ is a co-boundary $(\in B^1(\Gamma, \pi))$

$\exists$ Γ-action without fixed points	$\Leftrightarrow$	$\exists$ unitary Γ-representation π with $H^1(\Gamma, \pi) = Z^1/B^1 \neq 0$

Reduced 1-cohomology : $\bar{H}^1(\Gamma, \pi) = Z^1(\Gamma, \pi)/\overline{B^1(\Gamma, \pi)}$

Existence Theorem (Sh. Invent. 00:) If Γ is a f.g. group then:
No $(T) \Rightarrow \exists$ irreducible Γ-rep. π with $\bar{H}^1(\Gamma, \pi) \neq 0$.

* H^1 instead of $\bar{H}^1$ conjectured by Vershik-Karpushev ('83).
* For many Γ, such π is *unique* (e.g. Γ abelian).
* Proof implies: $\Gamma \in (T) \Rightarrow \Gamma$ is a quotient of a finitely presented group with (T) (question of Grigorchuk and of Zuk).

"Property (T) half" of proof of Main Thm (Sh. Invent. '00)

Theorem. Let $\Gamma < G = G_1 \times G_2$ be as in Main Theorem and assume $\mathrm{Hom}_{cont}(G_i, \mathbb{R}) = 0$ for both i. If $N \triangleleft \Gamma$ then:

Γ/N has $(T) \Leftrightarrow G_i/\overline{pr_i(N)}$ has (T) for both i.

Strategy for $\Leftarrow$ (non-trivial part): By previous existence Theorem need: $\forall$ unitary rep. π of Γ/N: $\bar{H}^1(\Gamma/N, \pi) = 0$.

Theorem. Let $\Gamma < G = G_1 \times G_2$ be as above. Let (π, V_π) be **any** unitary Γ-representation. Then $\exists$ Γ- subrepresentation $\sigma \subseteq \pi$ on a Γ-subspace $V_\sigma \subseteq V_\pi$, such that:

* $\bar{H}^1(\Gamma, \sigma) \hookrightarrow \bar{H}^1(\Gamma, \pi)$ is also onto.

* σ extends from Γ to a unitary G-representation on V_σ, and:

$$\bar{H}^1(\Gamma, V_\sigma) \cong \bar{H}^1(G, V_\sigma) \cong \bar{H}^1(G_1, V_\sigma^{G_2}) \oplus \bar{H}^1(G_2, V_\sigma^{G_1})$$

"All the reduced Γ cohomology comes from the factors G_i"

This + previous existence Thm $\Rightarrow$ "property (T) half".

So where do we stand now in this talk ??

Want: Normal Subgroup Theorem:

The following holds, excluding one counter-example:

Let G_1, G_2 be comp. gen. non-discrete l.c. groups, and $\Gamma < G = G_1 \times G_2$ be a discrete co-compact subgroup, with dense projections to both G_i's.

If each G_i is just non-compact, Γ is just infinite.

The unique counter-example: $G_1 = G_2 = \mathbb{R}$, $\Gamma = \mathbb{Z}^2 < \mathbb{R}^2 = G$ "irrationally embedded".

How: Show $\Gamma/N \in (T) \cap$ (Amenable) = (finite).

"Proved": $\Gamma/N \in (T)$.

Remains: Γ/N is amenable.

Amenability

Let μ be a prob. measure on Γ . $f : \Gamma \to \mathbb{R}$ is μ-harmonic if

$$\forall \gamma_0 \in \Gamma \qquad \sum_{\gamma \in \Gamma} f(\gamma_0 \gamma) d\mu(\gamma) = f(\gamma_0)$$

Assume Γ acts continuously on a compact metric space K.

$P(K)$ = convex set of probability measures on K.

If $m = \sum_{\gamma \in \Gamma} \mu(\gamma) \underbrace{\gamma m}$ "translation" of m by γ

call m μ-stationary (such m always exists). In that case:

$\forall \varphi : X \to \mathbb{R} \qquad \gamma \to \int_X \varphi(\gamma x) dm(x)$ is μ-harmonic.

Furstenberg ('63): <u>All</u> bounded μ-harmonic functions on Γ can be accounted for by some (X, m). There is even a space which is <u>minimal</u>, and hence <u>unique</u>. This is the:

$B(\Gamma, \mu) =$ (Furstenberg-)Poisson boundary of (Γ, μ).

"Compactification" of Γ where the μ-random walk converges.

Kaimanovich-Vershik, Rosenblatt('80) – Existence Thm (ii):

Γ amenable $\Rightarrow \exists \mu$ s.t. $\forall$ bdd μ-harmonic functions constant.

“Amenability Half” of pf (Bader-Sh. Invent. to appear)

Thm. Let $\Gamma < G = G_1 \times G_2$ be as in Main Thm. If $N \triangleleft \Gamma$:

Γ/N is amenable $\Leftrightarrow G_i/\overline{pr_i(N)}$ is amenable for both i.

$\Leftarrow$: Let Γ/N act on a cpt space K. Want a fixed point (=invariant measure) for action on $P(K)$. Take $Y = P(K)$ in:

Factor Theorem. Let μ_1, μ_2 be “admissible” probability measures on the l.c. groups G_1, G_2. Let $B_1 = B(G_1, \mu_1)$ $B_2 = B(G_2, \mu_2),\ B = B_1 \times B_2 = B(G, \mu_1 \times \mu_2)$.
Let $\Gamma < G = G_1 \times G_2$ irreducible lattice, (Y, ν) a Γ-measure space, with $\varphi : B \to Y$ a Γ-equivariant (“factor”) map. Then:

* The Γ-action on Y extends (ν-a.e) to a G-action.
* (Y, ν) splits as $(Y_1, \nu_1) \times (Y_2, \nu_2)$ each Y_i is a G_i-space.
* φ is a G-equivariant (ν a.e.) map.

Factor Thm $\Rightarrow$ Thm, with μ_i chosen via Existence Thm (ii) for amenable $G_i/\overline{pr_i(N)}$ (existence of $\varphi : B \to P(K)$ - “soft”).

When $G_1, G_2 \in (T)$, a *purely measure theoretic* generalization of the normal subgroup Thm holds:

Theorem. Assume $G_i \in (T)$ and are just non-compact. If $G = G_1 \times G_2$ acts measure preservingly on a prob. space, with each G_i ergodic, then: *either the G-action is transitive or it is free (a.e.)*.

* Generally not true for an action of one simple G.

* For G_i non-discrete, any G satisfying the conclusion has all its irreducible lattices Γ just infinite.

* Stuck-Zimmer (Annals'94): G algebraic rank≥ 2. Our proof modeled on theirs, based on the IFT.

* Gromov: $\exists$ continuum of *simple* (T)*-groups* G_i.

Work in (good ...) progress, with Tim Steger

$\tilde{A}_2$ buildings: Apartments are tessellations of the plane by equilateral triangles (= chambers).

Mostly known examples: Bruhat-Tits buildings of $GL_3(F)$, F a non-archimedian local field $(Q_p \ldots)$.

Cartwright, Mantero, Steger, Zappa: $\exists$ *non linear* $\tilde{A}_2$ buildings $\mathbb{B}$ with Γ =Iso$(\mathbb{B})$ *discrete* co-compact.

$\Gamma \ncong$ linear lattice. Aim: show they are just infinite.

Known to have (T). Again, “amenability half “ in Margulis strategy reduces to appropriate measurable Factor Theorem (on the “boundary” of $\mathbb{B}$).

Run the “$\Gamma \leftrightarrow G$ rigidity theory” without G !

(Next: are they linear, simple, superrigidity ? ...)

Some Thoughts and Speculations . . .

Definition: A Burnside group is a f.g. group Γ satisfying for some m: $\gamma^m = 1_\Gamma \quad \forall \gamma \in \Gamma$.

Adyan,Novikov,Olshanski: They can be infinite, and in all such known cases they are non-amenable.

Conjecture: Any Burnside group has property (T).
$\Rightarrow$ Any infinite Burnside group is non-amenable.

Restricted Burnside Problem-Zelmanov's Thm('92):
Any residually finite Burnside group is finite.

Can $(T)\cap$(amenable) approach be used here ?
Strategy: Show that any infinite sequence of finite quotients will be both an *expander* family (T), and a *non-expander* family (amenability).