BOUNDED ORBITS OF NONQUASIUNIPOTENT FLOWS ON HOMOGENEOUS SPACES

D. Y. Kleinbock and G. A. Margulis

Yale University

Abstract. Let $\{g_t\}$ be a nonquasiunipotent one-parameter subgroup of a connected semisimple Lie group G without compact factors; we prove that the set of points in a homogeneous space G/Γ (Γ an irreducible lattice in G) with bounded $\{g_t\}$-trajectories has full Hausdorff dimension. Using this we give necessary and sufficient conditions for this property to hold for any Lie group G and any lattice Γ in G.

Introduction

Let G be a Lie group, Γ a lattice in G, $F = \{g_t \mid t \in \mathbb{R}\}$ a one-parameter subgroup of G. Then the action of F on the homogeneous space $\Omega = G/\Gamma$ by left translations defines a flow. Assume that g_1 is not *quasiunipotent*, that is, $\mathrm{Ad}\, g_1$ has an eigenvalue with modulus different from 1. The results of S.G. Dani [Dan2, Dan3] suggested the following

Conjecture (A) [Mrg2]**.** *For any nonempty open subset W of Ω the set*

$$\{x \in W \mid \textit{the } F\textit{-orbit of } x \textit{ is bounded}\}$$

is of Hausdorff dimension equal to the dimension of G.

This conjecture was proved by Dani in the two following cases:

(i) (see [Dan2]) $G = SL_n(\mathbb{R})$, $\Gamma = SL_n(\mathbb{Z})$, and

$$g_t = \mathrm{diag}(e^{-t}, \dots, e^{-t}, e^{\lambda t}, \dots, e^{\lambda t}), \tag{1}$$

where λ is such that the determinant of g_t is 1;

(ii) (see [Dan3]) G is a connected semisimple Lie group of $\mathbb{R}$-rank 1.

In the present paper we prove Conjecture (A) in the case

(iii) G is a connected semisimple Lie group without compact factors and Γ is an irreducible lattice in G (Theorem 1.1).

In fact, the statement of this theorem is stronger: we consider orbits which are bounded and stay away from a given closed $\{g_t\}$-invariant subset Z of Ω of Haar measure 0.

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In the general case we give necessary and sufficient conditions for Conjecture (A) to hold (Theorem 5.2 and Corollary 5.5), based on the reduction to the case (iii). In particular, these conditions are satisfied when F consists of semisimple elements.

The main idea of the proof is similar to that of Dani [Dan2]: he considers the *horospherical* (relative to g_{-1}) subgroup H – the abelian subgroup of matrices of the form

$$h = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} . \tag{2}$$

Then he shows that the trajectory $\{g_t h \mathbb{Z}^n\}$ in the space $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ of lattices in $\mathbb{R}^n$, with g_t as in (1) and h as in (2), is bounded if and only if the set of linear forms, corresponding to the matrix L, is badly approximable in the sense of Schmidt [S2]. The result of the latter implies that the set of points with bounded trajectories has full Hausdorff dimension.

In fact, what is proved in [Dan2] (as well as in [Dan3]) is an apparently stronger result that the above set is winning in Schmidt's sense (cf. [S1]). In the general case the horospherical subgroup H is a connected simply connected nilpotent Lie group admitting a one-parametric semigroup of expanding automorphisms Φ_t – conjugations by g_t for positive t. The main reason why the method of [Dan2] cannot be used (at least directly) in this generality is that the restriction of Φ_t to H does not have to be conformal with respect to a Riemannian metric on H. Still, the fact that Φ_t, $t > 0$, is expanding on H, is crucial for our approach. In §1 we reduce the problem to studying one-sided trajectories F^+x of points $x \in \Omega$ (here $F^+ = \{g_t \mid t \geq 0\}$); this makes it possible to imbed a sequence of sets with easy to estimate increasing dimensions into the set of points with bounded orbits.

Indeed, fix a large compact set K in Ω and some neighborhood V of identity in H. Our first purpose is to prove that, roughly speaking, for large enough T and for any $x \in K$, there exists $t(x) \in [T, 2T]$ such that the most part of the set $g_{t(x)}Vx$ lies in a smaller set $K' \subset K$, the quantitative meaning of "the most" being uniform in $x \in K$ (see Proposition 2.5). After that we divide H (up to a set of measure 0) into pieces which are right translations of V by elements γ of H (this can be done for suitable V, see §3). For any $x \in K$ we choose all the translations γ such that $V\gamma \subset \Phi_{t(x)}(V)$ and $V\gamma g_{t(x)} x \subset K$, and define a compact subset $\mathbf{A}_1(x)$ of $\overline{V}$ to be the union of $\Phi_{-t(x)}(\overline{V}\gamma)$ over all γ as above. By iteration of this procedure, a sequence of compact sets $\overline{V}x \supset \mathbf{A}_1(x) \supset \mathbf{A}_2(x) \supset \dots \mathbf{A}_j(x) \supset \dots$ ($\mathbf{A}_{j+1}(x)$ being the union of $\Phi_{-t(x)}\big(\mathbf{A}_j(\gamma g_{t(x)} x)\gamma\big)$ over all γ as above) is constructed. The *limit set* $\mathbf{A}_\infty(x) = \bigcap_{i=1}^\infty \mathbf{A}_j(x)$ then consists of elements h such that the F^+-orbit of hx lies in a certain compact subset of Ω. The detailed description of this construction is given in §4; a result of C. McMullen and M. Urbanski allows one to derive a lower estimate on the Hausdorff dimension of $\mathbf{A}_\infty$ from the information about relative measure of the union of pieces $V\gamma g_{t(y)} y$ chosen at each stage for each $y \in K$.

To prove Proposition 2.5 mentioned above, we consider three cases.

Case 1. $\{g_t\}$ consists of semisimple elements (see Section 2.2; in this case we will say that the flow $(\Omega, \{g_t\})$ is *semisimple*).

Case 2. All the essential simple factors of G have $\mathbb{R}$-rank 1 (a factor G' of G is said to be *essential* if the projection of the semisimple part $\{a_t\}$ of $\{g_t\}$ onto G' is not relatively compact). This is a special case of *essentially semisimple* flows, see Section 2.3.

Case 3. The flow (Ω, F) *has property* (EM), by which we mean that the representation of the product G^{e} of all the essential factors of G on the space $L_2^0(\Omega) \stackrel{\mathrm{def}}{=} \{f \in L^2(\Omega) \mid \int_\Omega f = 0\}$ is isolated from the trivial representation (see Section 2.4). In view of Kazhdan's results on property (T) (cf. Lemma 2.4.1), it covers the case when G^{e} contains at least one factor of $\mathbb{R}$-rank greater than 1.

In Case 1 we use mixing properties of the action of F on Ω. In Case 2 we combine the results of Case 1 with facts about nondivergence of orbits of unipotent flows. In Case 3 instead of mixing we employ stronger results on the decay of matrix coefficients of smooth functions (Section 2.4; see also Appendix for similar treatment of Hölder functions). Note that in Cases 1 and 3 one can choose $t(x)$ to be equal to T for all $x \in K$. Let us also note that it is probably enough to consider Case 3 only, because the representation of any simple factor of G on $L_0^2(\Omega)$ seems to be always isolated from the trivial representation. However, we were unable to prove it or find a reference in the literature.

Several generalizations of the main result, as well as some open questions, are considered in §5. An estimate on exponential decay of spherical averages of Hölder continuous functions on Ω (Corollary A.8) is deduced in Appendix from mixing-related results obtained in §2.

In what follows, $\dim(A)$ will denote the Hausdorff dimension of a metric space A. If F is a one-parameter ($F = \{g_t \mid t \in \mathbb{R}\}$) or cyclic ($F = \{g^i \mid i \in \mathbb{Z}\}$) subgroup of a Lie group G, we will use the notation F^+ and F^- for the subsemigroups of F corresponding to nonnegative (resp. nonpositive) values of t (resp. i). A subgroup F of G will be called *nonquasiunipotent* if it contains an element which is not quasiunipotent.

§1. Reduction to the subgroup H

1.1. Our principal goal is to prove the following

Theorem. *Let G be a connected semisimple Lie group of dimension n without compact factors, Γ an irreducible lattice in G, F a (one-parameter or cyclic) nonquasiunipotent subgroup of G, and $Z \subset \Omega \stackrel{\mathrm{def}}{=} G/\Gamma$ a closed set of (Haar) measure 0, which is invariant under either F^+ or F^-. Then for any nonempty open subset W of Ω*

$$\dim\bigl(\{x \in W \mid Fx \text{ is bounded and } \overline{Fx} \cap Z = \varnothing\}\bigr) = n\,.$$

This theorem, with $Z = \varnothing$, clearly implies Conjecture (A) for the case (iii). The proof of this theorem will occupy §§1–4; unless otherwise specified, G, Γ and Ω will be as in Theorem 1.1.

1.2. Remark. Without loss of generality one can assume in the above theorem that the center $Z(G)$ of G is trivial. Indeed, let us denote the quotient group $G/Z(G)$ by G', the homomorphism $G \to G'$ by p, and the induced map $\Omega \to \Omega' \stackrel{\mathrm{def}}{=}$

$G'/p(\Gamma)$ by $\bar{p}$. Since $\Gamma Z(G)$ is discrete [Rag, Corollary 5.17], $p(\Gamma)$ is also discrete, hence $Z(G)/\big(\Gamma \cap Z(G)\big)$ is finite. This means that $(\Omega, \bar{p})$ is a finite covering of Ω', therefore for any closed null subset Z of Ω which is invariant under either F^+ or F^-, $\bar{p}(Z)$ is a closed null subset of Ω' invariant under either $p(F)^+$ or $p(F)^-$. Let $W \subset \Omega$ be an open set small enough for $\bar{p}$ to be injective on W, and denote

$$A = \{x \in \bar{p}(W) \mid p(F)x \text{ is bounded and } \overline{p(F)x} \cap \bar{p}(Z) = \varnothing\}.$$

Then for any $y \in \bar{p}^{-1}(A)$ the trajectory Fy is bounded and $\overline{Fy} \cap Z = \varnothing$; since $\bar{p}$ is a local isometry, the set $W \cap \bar{p}^{-1}(A)$ has dimension equal to the dimension of A.

In particular, the above statement shows that it is enough to prove Theorem 1.1 in the case when F is a one-parameter subgroup. Indeed, if G is centerfree, any its cyclic subgroup contains a subgroup of finite index which can be imbedded in a one-parameter subgroup of G. On the other hand, the statement of Theorem 1.1 is stable with respect to passing to a cocompact subgroup of F (see Lemma 5.1(a); part (b) of the same lemma shows stability with respect to passing to a finite covering of Ω, which was implicitly used in the first part of this remark).

1.3. Let $\mathfrak{g}$ be a Lie algebra of G, $\mathfrak{g}_{\mathbb{C}}$ its complexification, and for $\lambda \in \mathbb{C}$, let E_λ be a *generalized eigenspace* of $\operatorname{Ad} g_1$:

$$E_\lambda = \{X \in \mathfrak{g}_{\mathbb{C}} \mid (\operatorname{Ad} g_1 - \lambda I)^j X = 0 \text{ for some } j\}.$$

Let $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{h}^-$ be the subalgebras of $\mathfrak{g}$ with complexifications

$$\mathfrak{h}_{\mathbb{C}} = \operatorname{span}\big(E_\lambda \bigm| |\lambda| > 1\big),\ \mathfrak{h}^0_{\mathbb{C}} = \operatorname{span}\big(E_\lambda \bigm| |\lambda| = 1\big),\ \mathfrak{h}^-_{\mathbb{C}} = \operatorname{span}\big(E_\lambda \bigm| |\lambda| < 1\big).$$

The fact that g_1 is not quasiunipotent implies that $\mathfrak{h} \neq \{0\}$ and $\mathfrak{h}^- \neq \{0\}$; denote by k the common dimension of $\mathfrak{h}$ and $\mathfrak{h}^-$. Let H, H^0, H^- be the corresponding subgroups of G. Note that H^- is a horospherical subgroup with respect to g_1, while H is horospherical with respect to g_{-1}. (Recall that a subgroup

$$U(g) = \{h \in G \mid g^l h g^{-l} \to e \text{ as } l \to +\infty\}$$

of G is called *horospherical* with respect to $g \in G$.)

Clearly the subalgebras $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{h}^-$ are invariant under $\operatorname{Ad} g_t$, which implies that the subgroups H, H^0, H^- are normalized by F. Moreover, it is easy to show that the inner automorphism $\Phi_t : G \to G$, $g \to g_t g g_{-t}$, $t > 0$, defines an *expanding* automorphism of H:

$$\forall \text{ compact } K \subset H, \quad \forall \text{ open } V \subset H,\, e \in V, \quad \exists\, t_0 \text{ such that } \quad t > t_0 \Rightarrow K \subset \Phi_t(V).$$

Similarly, the automorphism Φ_t, $t > 0$, is *contracting* on the subgroup H^- (that is, $\Phi_t^{-1}|_{H^-}$ is expanding).

Since the group G can be assumed to be centerfree, it makes sense to consider a Jordan decomposition $g_t = a_t u_t = u_t a_t$, a_t and u_t being semisimple and unipotent parts of g_t respectively. Say that the flow (Ω, F) is *semisimple* if $\{u_t\}$ is trivial; say also that (Ω, F) is *essentially semisimple* if $\{u_t\}$ is trivial modulo the centralizer $Z(H)$ of H in G (that is, if $\{u_t\}$ commutes with H). Let us note that these are in fact properties of the subgroup F, not of its action on Ω.

In the flow (Ω, F) is semisimple, the restriction of Φ_t to the subgroup $\tilde{H} \stackrel{\text{def}}{=} H^- H^0$ is (locally) *nonexpanding* (cf. [Bow, EP]):

$$\exists\, c > 1 \text{ such that } \forall\, r < 1\ \forall\, t > 0 \quad \Phi_t\big(\tilde{B}(r)\big) \subset \tilde{B}(cr) \tag{1.1}$$

(here and hereafter $\tilde{B}(r)$ stands for the open ball of radius r in $\tilde{H}$ centered at e).

If $\{u_t\}$ is not trivial, (1.1) need not be true. However, Φ_t is still contracting on H^-, while the restriction of $\operatorname{Ad} g_t$ to $\mathfrak{h}^0$ lies in the product of a compact and a unipotent subgroup of $GL(\mathfrak{h}^0)$. This means that Φ_t defines (locally) *at most polynomially expanding* automorphism of $\tilde{H}$:

$$\exists\, c > 1,\ \varkappa \in \mathbb{N} \text{ such that } \forall\, t > 1\ \forall\, r < t^{-\varkappa} \quad \Phi_t\big(\tilde{B}(r)\big) \subset \tilde{B}(ct^{\varkappa} r) \tag{1.2}$$

1.4. An important property of Hausdorff dimension that we will use is the following

Lemma (Marstrand Slicing Theorem, [Mrs or F, Theorem 5.8]**).** *Let A and B be metric spaces, and let C be a subset of the direct product $A \times B$. Assume that either*

(a) *A has positive α-dimensional Hausdorff measure $\mathcal{H}_\alpha$, and*

$$\dim\big(C \cap (x \times B)\big) \ge \beta \tag{1.3}$$

for $\mathcal{H}_\alpha$-almost every $x \in A$, or

(b) *A is a Borel set in a manifold,* $\dim(A) \ge \alpha$, *and the condition (1.3) is satisfied for all $x \in A$.*

Then

$$\dim(C) \ge \alpha + \beta\,. \tag{1.4}$$

Proof. The standard version, as in [Mrs or F], assumes (a); to derive (1.4) from (b), note that by [Dav] (see also [F, Theorem 5.6]), A contains a subset of positive α'-dimensional Hausdorff measure for any $\alpha' < \dim(A)$. $\square$

1.5. We will repeatedly use the fact that locally (in the neighborhood of identity) G is bi-Lipschitz equivalent to the direct product of the subgroups H, H^0 and H^-. In particular, it makes it possible to reduce Theorem 1.1 to the following statement:

Theorem. *For any $x \in \Omega \smallsetminus Z$ and for any neighborhood V of identity in H*

$$\dim\big(\{h \in V \mid F^+hx \text{ is bounded and } \overline{F^+hx} \cap Z = \varnothing\}\big) = k\,.$$

Reduction of Theorem 1.1 to Theorem 1.5. Given a nonempty open $W \subset \Omega$, choose a point $x \in W$ not contained in Z. After that take $U \subset G$ of the form $V^- V V^0$, where V^-, V and V^0 are neighborhoods of identity in H^-, H and H^0 respectively, such that the multiplication map $V^- \times V \times V^0 \to U$, $(h^-, h, h_0) \to h^- h h^0$, is bi-Lipschitz, the quotient map $\pi_x : G \to \Omega$, $g \to gx$, is injective on U, and Ux is inside $W \smallsetminus Z$. Then it is enough to show that

$$\dim\big(\{g \in U \mid Fgx \text{ is bounded and } \overline{Fgx} \cap Z = \varnothing\}\big) = n\,. \tag{1.5}$$

Let C be the set $\{h \in U \mid F^+hx \text{ is bounded and } \overline{F^+hx} \cap Z = \varnothing\}$. From Theorem 1.5 it follows that for any $h^0 \in V^0$, $\dim(C \cap Vh^0) = k$. In view of Lemma 1.4, this implies that the set $C \cap VV^0$ has dimension equal to $n - k$.

Claim. *For all $h \in C$ there exists a neighborhood $V^-(h)$ of identity in H^- such that $V^-(h)h \subset C$.*

Proof. For $h \in C$, let $\varepsilon(h)$ be the distance between disjoint closed sets Z and $\overline{F^+hx}$. Since the map Φ_t is contracting on H^- for $t > 0$, one can find an open neighborhood $V^-(h)$ of identity in H^- such that $V^-(h)h \subset U$ and $\operatorname{diam}\big(\Phi_t(V^-(h))\big) \le \varepsilon(h)/2$ for any $t \ge 0$. Then $g_t V^-(h)hx = \Phi_t\big(V^-(h)\big)g_t hx$ is disjoint from Z for any $t \ge 0$, and clearly $\operatorname{diam}\big(F^+V^-(h)hx\big) < \varepsilon(h) + \operatorname{diam}(F^+hx)$. Therefore the trajectory of any point from $V^-(h)hx$ is bounded and stays away from Z. $\square$

Applying Theorem 1.5 (with F^- in place of F^+ and H^- in place of H) to each of the sets $V^-(h)$, $h \in C \cap VV^0$, one gets that

$$\forall\, h \in C \cap VV^0 \quad \dim\big(\{h^- \in V^- \mid Fh^-hx \text{ is bounded and } \overline{Fh^-hx} \cap Z = \varnothing\}\big) = k\,,$$

and another application of Lemma 1.4 yields the equality (1.5). $\square$

1.6. Since Hausdorff dimension of the union of a countable family of sets is equal to the supremum of dimensions of these sets, Theorem 1.5 is equivalent to the following statement, which is in fact what we will be proving in the course of the paper:

Theorem. *For any $x \in \Omega$, $x \notin Z$, there exist a sequence of neighborhoods V_s of identity in H and a sequence of compact subsets C_s of $\Omega \smallsetminus Z$, $s \in \mathbb{N}$, such that*

$$\operatorname{diam}(V_s) \to 0 \text{ as } s \to \infty\,, \tag{1.6}$$

and

$$\dim\big(\{h \in \overline{V_s} \mid F^+hx \subset C_s\}\big) \to k \text{ as } s \to \infty\,. \tag{1.7}$$

§2. Mixing and its consequences

2.1. Preliminaries

2.1.1. Let μ, $\bar\mu$, m, m^0, m^- be Haar measures on G, Ω, H, H^0, H^- respectively, normalized so that $\bar\mu(\Omega) = 1 = \mu(\textit{any fundamental domain of } \Gamma\textit{-action on } G)$, and μ is locally almost the product of m^-, m^0 and m. The latter, in view of [Bou, Ch. VII, §9, Proposition 13], means that μ can be expressed via m, m^0 and m^- in the following way: for any $\varphi \in L^1(G)$

(2.1.1)

$$\int_{H^-H^0H} \varphi(g)\, d\mu(g) = \int_{H^-\times H^0 \times H} \varphi(h^-h^0h)\Delta_{\tilde H}(h^0)\, dm^-(h^-)\, dm^0(h^0)\, dm(h)\,,$$

where $\Delta_{\tilde H}$ is the modular function of the group $\tilde H = H^-H^0$.

Observe that the measures μ and m^0 are preserved by the automorphism Φ_t, while for m and m^- one has $\Phi_t(m^-) = e^{-\chi t}m^-$, and

$$\Phi_t(m) = e^{\chi t} m\,, \tag{2.1.2}$$

where χ is a positive number (equal to $\operatorname{Tr}\operatorname{ad} Y|_{\mathfrak h}$, if $Y \in \mathfrak g$ is such that $g_1 = \exp(Y)$).

2.1.2. It is well-known (cf. [Mo1]) that the action of F on Ω is *mixing*. We will use the following generalized version:

Theorem. *Let Φ, $\Psi \subset L^2(\Omega)$ be two compact (in the topology of $L^2(\Omega)$) families of functions. Then for any $\varepsilon > 0$ there exists $T > 0$ such that*

$$t \ge T \Rightarrow \forall\, \varphi \in \Phi\ \forall\, \psi \in \Psi \qquad \left|(g_t\varphi, \psi) - \int_\Omega \varphi\, d\bar\mu \int_\Omega \psi\, d\bar\mu\right| \le \varepsilon\,.$$

(here $(\cdot,\cdot)$ means the inner product in $L^2(\Omega)$).

Proof. Since F is not quasiunipotent, $\operatorname{Ad} F$ is not relatively compact, so by [Mo1] the statement is true for one-element sets $\Phi = \{\varphi\}$ and $\Psi = \{\psi\}$, φ, $\psi \in L^2(\Omega)$. The general case then follows from the unitarity of the regular representation of G on $L^2(\Omega)$. $\square$

2.1.3. Our purpose now is to derive an analogue of the mixing property for functions ϕ supported on certain proper submanifolds of Ω. Denote by v the volume form on H corresponding to the Haar measure m. Since $\mathfrak g = \operatorname{Lie}(G)$ is a direct sum of $\mathfrak h = \operatorname{Lie}(H)$ and $\tilde{\mathfrak h} = \operatorname{Lie}(\tilde H)$, the projection $\mathfrak g \to \mathfrak h$ is well-defined and induces the map $p_x : T_x\Omega \to T_x(Hx)$ for any $x \in \Omega$. Thus the form v induces a k-form ω_H on Ω given by $(\omega_H)_x = v \circ d\pi_x^{-1} \circ p_x$ (here π_x as before stands for the quotient map $G \to \Omega$, $g \to gx$).

Let M be a smooth k-dimensional manifold, and let π be a C^∞ immersion $M \to \Omega$ such that $\pi(M)$ is transversal to the orbit $\tilde H\pi(y)$ for all $y \in M$. Then one can pull the form ω_H back to get a k-form ω_π on M. Further, let f be a function on M and ψ a function on Ω. Then one can integrate the product $f(y)\psi\big(\pi(y)\big)$ with respect to ω_π and, furthermore, look at the asymptotics of $\int_M f(y)\psi\big(g_t\pi(y)\big)\,\omega_\pi$ (which is by (2.1.2) equal to $e^{-\chi t}\int_M f(y)\psi\big(g_t\pi(y)\big)\,\omega_{g_t\circ\pi}$) as $t \to \infty$. We will return to this level of generality in Appendix (Section A.7), while for the proof of Theorem 1.6 it suffices to consider a special case $M = H$ and $\pi = \pi_x$, where $x \in \Omega$.

If $f \in L^2(H)$ and $x \in \Omega$, the integral $\int_H f(h)\psi\big(g_t\pi_x(h)\big)\,\omega_{\pi_x}$ can be written as $\int_H f(h)\psi(g_thx)\, dm(h)$. Using (2.1.2), one gets

$$\int_H f(h)\psi(g_thx)\, dm(h) = e^{-\chi t}\int_H f(g_thg_{-t})\psi(hg_tx)\, dm(h)\,, \tag{2.1.3}$$

the last integral being equal to $\int_H f(h)\psi\big(g_t\pi_x(h)\big)\,\omega_{g_t\circ\pi_x}$.

We will describe asymptotics of integrals of type (2.1.3) for large enough classes of functions f and ψ; moreover, our estimates will be uniform in x lying in a fixed compact subset of Ω. However, additional assumptions on the subgroup F will be necessary.

2.2. Semisimple flows

2.2.1. The proof of the next proposition is based on Theorem 2.1.2 and the non-expanding property (1.1) of $\Phi_t|_{\tilde H}$. Similar methods were developed in [Bow, EM, EP, Mrc, Mrg1].

Proposition. *Assume that F consists of semisimple elements. Let $f \in L^2(H)$ with compact support and a uniformly continuous $\psi \in L^2(\Omega)$ be given. Then for any compact subset L of Ω and any $\varepsilon > 0$ there exists $T > 0$ such that*

$$\left|\int_H f(h)\psi(g_t hx)\,dm(h) - \int_H f\,dm \int_\Omega \psi\,d\bar\mu\right| \le \varepsilon \tag{2.2.1}$$

for all $x \in L$ and $t \ge T$.

Proof. Since $\operatorname{supp}(f)$ and L are compact, and Γ is discrete, f can be written as a sum of functions f_j, $1 \le j \le N$, with π_x injective on $\operatorname{supp}(f_j)$ for all $x \in L$ and for each j. Hence one can without loss of generality assume that the maps π_x are injective on $\operatorname{supp}(f)$ for all $x \in L$.

If $f \equiv 0$ a.e., there is nothing to prove. Otherwise, denote by a the positive number $\varepsilon\big(2\int_H |f|\,dm\big)^{-1}$. Since ψ is uniformly continuous, there exists r, $0 < r < 1$, such that

$$\forall\, \tilde h \in \tilde B(r)\ \forall\, x \in \Omega \quad \big|\psi(\tilde h x) - \psi(x)\big| \le a\,. \tag{2.2.2a}$$

We will choose this r small enough to ensure that

$$\forall\, x \in L \quad \pi_x \text{ is injective on } \tilde B(r)\cdot \operatorname{supp}(f)\,. \tag{2.2.2b}$$

Pick two nonnegative functions $f^- \in L^2(H^-)$, $f^0 \in L^2(H^0)$ such that

$$\int_{H^-} f^-\,dm^- = 1 = \int_{H^0} f^0\,dm^0\,, \tag{2.2.3}$$

and

$$\operatorname{supp}(f^-)\cdot\operatorname{supp}(f^0) \subset \tilde B(r/c) \quad (\text{with } c > 1 \text{ from } (1.1))\,. \tag{2.2.4}$$

Then for any $t \in \mathbb{R}$ by (2.1.1)

$$\begin{aligned}\int_H f(h)\psi(g_t hx)\,dm(h) &= \int_{H^-} f^-\,dm^- \int_{H^0} f^0\,dm^0 \int_H f(h)\psi(g_t hx)\,dm(h)\\ &= \int_G f^-(h^-)f^0(h^0)f(h)\psi(g_t hx)\Delta_{\tilde H}(h^0)^{-1}\,d\mu(h^-h^0h)\,.\end{aligned} \tag{2.2.5}$$

Now observe that

$$\tag{2.2.6} \begin{gathered} \left|\psi(g_t hx) - \psi(g_t h^- h^0 hx)\right| = \left|\psi(g_t hx) - \psi\left(\left(\Phi_t(h^- h^0)\right) g_t hx\right)\right| \le a \\ \text{whenever } h^- \in \operatorname{supp}(f^-) \text{ and } h^0 \in \operatorname{supp}(f^0) \end{gathered}$$

(this follows from (2.2.4), nonexpanding property (1.1) and (2.2.2a)). Hence
(2.2.7)

$$\begin{aligned} &\left|\int_H f(h)\psi(g_t hx)\, dm(h) - \int_G f^-(h^-) f^0(h^0) f(h)\psi(g_t h^- h^0 hx)\Delta_{\tilde H}(h^0)^{-1}\, d\mu(h^- h^0 h)\right| \\ &= \left|\int_G f^-(h^-) f^0(h^0) f(h)\big(\psi(g_t hx) - \psi(g_t h^- h^0 hx)\big)\Delta_{\tilde H}(h^0)^{-1}\, d\mu(h^- h^0 h)\right| \\ &\le a\int_G \left|f^-(h^-) f^0(h^0) f(h)\Delta_{\tilde H}(h^0)^{-1}\right|\, d\mu(h^- h^0 h) = a\int_H |f|\, dm = \frac{\varepsilon}{2}\,. \end{aligned}$$

Let φ be a function on $H^- H^0 H$ defined by $\varphi(h^- h^0 h) = f^-(h^-) f^0(h^0) f(h)\Delta_{\tilde H}(h^0)^{-1}$, and denote by φ_x the function $\varphi \circ \pi_x^{-1}$. Then, by (2.2.2b) and (2.2.4), φ_x is well defined for all $x \in L$, and

$$\tag{2.2.8} \int_\Omega \varphi_x\, d\bar\mu = \int_G \varphi\, d\mu = \int_H f\, dm$$

(the last equality follows from (2.1.1) and (2.2.3)). Moreover,
(2.2.9)

$$\int_G f^-(h^-) f^0(h^0) f(h)\psi(g_t h^- h^0 hx)\Delta_{\tilde H}(h^0)^{-1}\, d\mu(h^- h^0 h) = (\varphi_x, g_{-t}\psi) = (g_t\varphi_x, \psi)\,.$$

Since the family $\{\varphi_x \mid x \in L\}$ is compact in $L^2(\Omega)$, by Theorem 2.1.2 there exists $T > 0$ such that for $t \ge T$ and for all $x \in L$

$$\left|(g_t\varphi_x, \psi) - \int_H f\, dm \int_\Omega \psi\, d\bar\mu\right| \le \frac{\varepsilon}{2}\,,$$

and the claim follows from (2.2.7), (2.2.9) and the above inequality. □

2.2.2. The assumption of the uniform continuity of the function ψ was essential in the above proof. However, it is not hard to see that Proposition 2.2.1 also holds for functions ψ which can be approximated by uniformly continuous functions in a suitable way. To better describe it, we make the following definition: for any property $\mathcal{P}$ of measurable functions on a measure space X, say that a function φ *almost has* property $\mathcal{P}$ if there exist two sequences of functions $\{\varphi_j\}$ and $\{\varphi'_j\}$ having property $\mathcal{P}$, such that one of them is nondecreasing ($\varphi_{j+1} \ge \varphi_j$ a.e.), the other one nonincreasing ($\varphi'_{j+1} \le \varphi'_j$ a.e.), and both converge to φ almost everywhere. The main example: characteristic functions of sets with null boundary in a metric space with Borel measure almost have many nice properties, such as continuity or infinite differentiability.

From Proposition 2.2.1 we now deduce the following

Corollary. *Let F and f be as in Proposition 2.2.1, and let $\psi \in L^2(\Omega)$ be almost uniformly continuous. Then for any compact $L \subset \Omega$ and any $\varepsilon > 0$, there exists $T = T(L, \varepsilon) > 0$ such that* (2.2.1) *holds for all $x \in L$ and $t \geq T$.*

Proof. Without loss of generality one can assume that the function f is a.e. nonnegative. Indeed, if $f = f_1 - f_2$ is a difference of a.e. nonnegative functions, take T such that

$$t \geq T \Rightarrow \forall\, x \in L \qquad \left| \int_H f_i(h)\psi(g_t hx)\, dm(h) - \int_H f_i\, dm \int_\Omega \psi\, d\bar{\mu} \right| \leq \frac{\varepsilon}{2}$$

for $i = 1, 2$.

Let $\{\psi_j\}$ be a nondecreasing sequence of uniformly continuous functions converging to ψ. For $\varepsilon > 0$, find j such that

$$\int_H f\, dm \left| \int_\Omega \psi_j\, d\bar{\mu} - \int_\Omega \psi\, d\bar{\mu} \right| \leq \frac{\varepsilon}{2}\,.$$

Then, using Proposition 2.2.1, find $T' > 0$ such that

$$\left| \int_H f(h)\psi_j(g_t hx)\, dm(h) - \int_H f\, dm \int_\Omega \psi_j\, d\bar{\mu} \right| \leq \frac{\varepsilon}{2}$$

for any $t \geq T'$ and for all $x \in L$.

Since $f \geq 0$ and $\psi \geq \psi_j$ by construction, $\int_H f(h)\psi(g_t hx)\, dm(h)$ is not less than $\int_H f(h)\psi_j(g_t hx)\, dm(h)$. Thus from the above inequalities it follows that

$$\int_H f(h)\psi(g_t hx)\, dm(h) \geq \int_H f\, dm \int_\Omega \psi\, d\bar{\mu} - \varepsilon$$

for any $t \geq T'$ and for all $x \in L$. Repeating all the above argument with nonincreasing sequence of uniformly continuous functions approximating ψ, one gets $T'' > 0$ such that

$$\int_H f(h)\psi(g_t hx)\, dm(h) \leq \int_H f\, dm \int_\Omega \psi\, d\bar{\mu} + \varepsilon$$

for any $t \geq T''$ and for all $x \in L$, and it suffices to take $T = \max(T', T'')$. $\square$

2.3. Essentially semisimple flows

2.3.1. Recall that if the unipotent part $\{u_t\}$ of $\{g_t\}$ is nontrivial, the restriction $\Phi_t|_{\tilde{H}}$ does not have to be locally nonexpanding, so the above proof of Proposition 2.2.1 cannot be carried out. However, it turns out that if the subgroup $\{u_t\}$ is small enough, one can prove a weaker (but still good enough for our purposes) version of the estimates from the preceding section. Here "small enough" means "lies in the centralizer $Z(H)$ of H in G", that is, the flow (Ω, F) is essentially semisimple as defined in Section 1.3.

The following lemma gives a large class of nontrivial examples of essentially semisimple flows:

Lemma. *Let G be a direct product of G' and G'' such that*
(a) *the projection of $\{a_t\}$ onto G' is relatively compact, and*
(b) *the projection of $\{u_t\}$ onto G'' is trivial.*
Then the flow (Ω, F) is essentially semisimple.

Proof. Indeed, (a) implies that $H \subset G''$, while $\{u_t\} \subset G'$ by (b). □

2.3.2. To apply the above lemma, decompose the group G (assumed to be centerfree) into a direct product of simple groups. Then say that a factor G_i of G is *essential* with respect to F if the projection of the semisimple part $\{a_t\}$ of F onto G_i is not relatively compact. Denote by G^{e} the product of all the essential factors, and by G^{i} the product of all other factors.

Corollary. *Assume that all the essential factors of G are of $\mathbb{R}$-rank 1. Then the flow is essentially semisimple.*

Proof. Indeed, the projection of u_t onto G^{e} is a product of unipotent elements of groups of rank 1, and each of these elements centralizes a non-compact torus in the corresponding factor. Since the centralizer of a maximal torus for groups of rank 1 coincides with the torus itself, the projection of u_t on each factor of G^{e} must be semisimple, hence trivial. On the other hand, the projection of $\{a_t\}$ onto G^{i} is by definition relatively compact, and the claim follows from Lemma 2.3.1. □

2.3.3. Observe now that if $\{u_t\}$ commutes with H, the integral (2.1.3) can be written in the form

$$\int_H f(h)\psi\big(g_t\pi_x(h)\big)\,\omega_{\pi_x} = \int_H f(h)\psi(a_t u_t h x)\,dm(h)$$
$$= \int_H f(h)\psi(a_t h u_t x)\,dm(h) = \int_H f(h)\psi\big(a_t\pi_{u_t x}(h)\big)\,\omega_{\pi_{u_t x}}\,.$$

This suggests that in order to estimate the asymptotics of the above integral, one may wish to apply Proposition 2.2.1 to the semisimple part $\{a_t\}$ of $\{g_t\}$. However, to get a uniform estimate, one has to be sure that the points $u_t x$ lie in a fixed compact set. This is made possible by the following theorem, describing nondivergence of orbits of unipotent flows $(\Omega, \{u_t\})$:

Theorem [Dan4, DM]**.** *Let L be a compact subset of Ω and let $\sigma > 0$ be given. Then there exists a compact subset $Q = Q(L, \sigma)$ of Ω such that for any unipotent one-parameter subgroup $\{u_t\}$ of G, any $x \in L$ and any $T \geq 0$*

$$l\big(\{t \in [0, T] \mid u_t x \in Q\}\big) \geq (1 - \sigma)T\,,$$

where l denotes the Lebesgue measure on $\mathbb{R}$.

2.3.4. Proposition. *Assume that the flow (Ω, F) is essentially semisimple. Then for any compactly supported $f \in L^2(H)$, almost uniformly continuous $\psi \in L^2(\Omega)$, a compact set $L \subset \Omega$ and positive ε and σ, there exists $T_0 > 0$ such that for all $x \in L$ and $T \geq T_0$*

$$l\big(\{t \in [T, 2T] \mid (2.2.1)\ \textit{holds}\}\big) \geq (1 - \sigma)T\,.$$

Proof. Given $\sigma > 0$ and a compact set $L \subset \Omega$, take $Q = Q(L, \sigma/2)$ from Theorem 2.3.3. After that for any $\varepsilon > 0$ put T_0 to be equal to $T(Q, \varepsilon)$ from Corollary 2.2.2. Then it follows from the latter corollary that (2.2.1) is satisfied whenever $u_t x \in Q$. Therefore for any $x \in L$ and $T \geq T_0$, one has

$$\begin{aligned} l\big(\{t \in [T, 2T] \bigm| (2.2.1) \text{ holds}\}\big) &\geq l\big(\{t \in [T, 2T] \bigm| u_t x \in Q\}\big) \\ &\geq l\big(\{t \in [0, 2T] \bigm| u_t x \in Q\}\big) - T \geq 2T(1 - \sigma/2) - T\,, \end{aligned}$$

which finishes the proof. □

2.4. Flows with property (EM)

2.4.1. We will show in this section that certain properties of representations of semisimple groups allow one to derive an analogue of Proposition 2.2.1, valid, in particular, for flows which are not essentially semisimple.

Denote by ρ_0 the regular representation of G on the subspace $L^2_0(\Omega)$ of $L^2(\Omega)$ orthogonal to constant functions, and say that the flow (Ω, F) *has property* (EM) (an abbreviation for exponential mixing, see Theorem 2.4.5 below for justification) if

$$\text{(EM)} \qquad \begin{array}{c} \text{the restriction of } \rho_0 \text{ to } G^{\mathrm{e}} \text{ is isolated} \\ \text{(in the Fell topology) from the trivial representation.} \end{array}$$

Note that this condition, in particular, means that G^{e} is not trivial (there exists at least one essential factor), hence F is not quasiunipotent .

Lemma. *Assume that there exists an essential factor G' of G not locally isomorphic to $SO(m, 1)$ or $SU(m, 1)$, $m \in \mathbb{N}$. Then the flow (Ω, F) has property* (EM).

Proof. Write $G^{\mathrm{e}} = G' \times G''$ (G'' being the product of all the essential factors of G except G'), and consider a direct integral decomposition

$$\rho_0|_{G^{\mathrm{e}}} = \int_X \rho'_x \otimes \rho''_x \, d\mu_X\,, \tag{2.4.1}$$

where ρ'_x and ρ''_x are irreducible representations of G' and G'' respectively. From the irreducibility of the lattice Γ it follows that μ_X-almost all the representations ρ'_x are nontrivial. Since G' has Kazhdan's property (T) (see e.g. [Cow, §2.3]), all those representations are separated from the trivial representation of G', which, together with the decomposition (2.4.1), implies (EM). □

Note that, in view of Corollary 2.3.2, this lemma implies that the flow (Ω, F) has property (EM) whenever it is not essentially semisimple.

2.4.2. Remark. It is well known (although we failed to identify the exact reference) that ρ_0 itself is isolated from the trivial representation. This means that (EM) is satisfied whenever $G = G^{\mathrm{e}}$ (all the simple factors of G are essential with respect to F). In fact, it seems likely that the condition (EM) is always satisfied, though we are not aware of the proof, if it exists.

2.4.3. It turns out that for the flows with property (EM) one can formulate a refinement of the mixing property. Indeed, the condition (EM) allows one to make use of the results on the decay of matrix coefficients of semisimple Lie groups. The following exponential estimate is a modification of Theorem 3.1 from [KS].

Fix a maximal compact subgroup $\mathcal{K}$ of a connected semisimple Lie group G, and denote by $\mathfrak{k}$ its Lie algebra. Take an orthonormal basis $\{Y_j\}$ of $\mathfrak{k}$, and set $\Upsilon = 1 - \sum Y_j^2$. Then Υ belongs to the center of the universal enveloping algebra of $\mathfrak{k}$ and acts on smooth vectors of any representation space of G.

Let $\mathfrak{a}$ be a maximal split Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. Fix an order on the roots, and let $\mathfrak{c}$ be a positive Weyl chamber; denote by ϑ half the sum of the positive roots on $\mathfrak{c}$.

Theorem. *Let G be a connected semisimple Lie group with finite center, and let Π be a family of unitary representations of G such that the restriction of Π to any simple factor of G is isolated from the trivial representation. Then there exist a universal constant $B > 0$, a positive integer l (dependent only on G) and a positive integer p such that for any $\rho \in \Pi$, any C^∞-vectors v, w in a representation space of ρ, all $Y \in \bar{\mathfrak{c}}$ and $t \geq 0$*

$$\big|\big(\rho\big(\exp(tY)\big)v, w\big)\big| \leq Be^{-\frac{t}{2p}\vartheta(Y)}\|\Upsilon^l(v)\|\|\Upsilon^l(w)\|\,. \tag{2.4.2}$$

Proof. Decomposing any $\rho \in \Pi$ into a direct integral of irreducible representations and using Cauchy-Schwartz inequality, one can without loss of generality assume that Π consists of irreducible representations. A. Katok and R. Spatzier, using R. Howe's estimates [H] for matrix coefficients of $\mathcal{K}$-finite vectors, proved in [KS] that (2.4.2) holds whenever an irreducible representation ρ of G with discrete kernel is strongly L^p. Then from M. Cowling's results [Cow] they deduced that p depends only on G if the latter contains no factors locally isomorphic to $SO(m,1)$ or $SU(m,1)$, $m \in \mathbb{N}$. However, using the argument from [Cow, §3.1], one can show that all irreducible representations of G outside a fixed neighborhood of the trivial representation are strongly L^p for some p. □

2.4.4. Let us fix a Riemannian metric $\mathrm{dist}(\cdot,\cdot)$ on G bi-invariant with respect to $\mathcal{K}$.

Corollary. *Let G, Π, ρ, v and w be as in Theorem 2.4.3. Then there exist constants $E > 0$, $l \in \mathbb{N}$ (dependent only on G) and $\alpha > 0$ (dependent only on G and Π) such that for any $g \in G$*

$$\big|\big(\rho(g)v, w\big)\big| \leq Ee^{-\alpha \mathrm{dist}(e,g)}\|\Upsilon^l(v)\|\|\Upsilon^l(w)\|\,.$$

Proof. If $k_1 \exp(tY) k_2$, with $\vartheta(Y) = 1$, is the Cartan decomposition of g, one can apply Theorem 2.4.3 to the subgroup $\exp(tY)$. □

2.4.5. We now return to the original problem and apply the above corollary to simple factors of G^{e}.

Theorem. *Let the condition* (EM) *be satisfied. Then there exist constants* $\gamma > 0$, $E > 0$, $l \in \mathbb{N}$ *such that for any two functions* φ, $\psi \in C^\infty_{comp}(\Omega) \cap L^2_0(\Omega)$ *and for any* $t \geq 0$

$$|(g_t\varphi, \psi)| \leq Ee^{-\gamma t}\|\varphi\|_{_l}\|\psi\|_{_l}\,, \tag{2.4.3}$$

where $\|\cdot\|_{_l}$ *means the norm in the Sobolev space* $W^2_l(\Omega)$.

In other words, if the flow has property (EM), the rate of mixing for smooth functions is at least exponential.

Proof. Let G_j be the simple factors of G, and let $\{g_{t,j}\}$ and $\{g'_{t,j}\}$ denote the one-parameter subgroups of G which are natural projections of $\{g_t\}$ onto G_j and $\prod_{i\neq j} G_j$ respectively. From the definition of G^{e} it follows that $\mathrm{dist}(e, g_{t,j})$ is for any $t \geq 0$ bounded from below by βt for some $\beta > 0$ whenever $G_j \subset G^{\mathrm{e}}$. Hence by Corollary 2.4.4 and property (EM), for any irreducible component ρ of ρ_0 one can choose j such that $G_j \subset G^{\mathrm{e}}$ and for any two C^∞-vectors v and w of $\rho|_{G_j}$

$$\big|\big(\rho(g_{t,j})v, w\big)\big| \leq Ee^{-\alpha\beta t}\|\Upsilon^l(v)\|\|\Upsilon^l(w)\|\,, \tag{2.4.4}$$

where $E > 0$, $l \in \mathbb{N}$ and $\alpha > 0$ do not depend on ρ.

Now restrict ρ to the group $G_j \times \{g'_{t,j}\}$ and then decompose this restriction into a direct integral of irreducible representations. Each of them will be a tensor product of $\rho|_{G_j}$ and a one-dimensional representation of $\{g'_{t,j}\}$. Therefore the estimate (2.4.3) for matrix coefficients $(g_t\varphi, \psi)$ can be obtained by integration of inequalities of type (2.4.4). □

Note that for Theorem 2.4.3 (resp. Theorem 2.4.5) to hold, it clearly suffices for v and w (resp. φ and ψ) to be C^l for large enough l. Moreover, similar estimates can be proved for Hölder vectors – see Appendix.

2.4.6. Corollary. *With the assumptions of Theorem 2.4.5, for any two functions* φ, $\psi \in C^\infty_{comp}(\Omega)$ *and for any* $t \geq 0$

$$\left|(g_t\varphi, \psi) - \int_\Omega \varphi d\bar\mu \int_\Omega \psi d\bar\mu\right| \leq Ee^{-\gamma t}\|\varphi\|_{_l}\|\psi\|_{_l}\,.$$

Proof. Apply (2.4.3) to the functions $\varphi - \int_\Omega \varphi d\bar\mu$ and $\psi - \int_\Omega \psi d\bar\mu$. □

2.4.7. To make use of the above facts, let us list the simple properties of the Sobolev norm $\|\cdot\|_{_l}$ we will need.

Lemma. (a) *Let* X, Y *be Riemannian manifolds,* $\varphi \in C^\infty_{comp}(X)$, $\psi \in C^\infty_{comp}(Y)$*; consider* $\varphi\cdot\psi$ *as a function on* $X \times Y$*. Then* $\|\varphi\cdot\psi\|_{_l} \leq c(X, Y)\|\varphi\|_{_l}\|\psi\|_{_l}$, *where* $c(X, Y)$ *is a constant independent on the functions* φ *and* ψ.

(b) *Let* X *be a Riemannian manifold of dimension* N, x *a point in* X*. Then for any* r, $0 < r < 1$, *there exists a nonnegative function* $f \in C^\infty_{comp}(X)$ *such that* $\mathrm{supp}(f)$ *is inside the ball of radius* r *centered at* x, $\int_X f = 1$, *and* $\|f\|_{_l} \leq c(X, x)r^{-(l+N/2)}$, *where* $c(X, x)$ *is a constant independent on* r.

2.4.8. Proposition. *Let the condition* (EM) *be satisfied. Then for any* $f \in C^\infty_{comp}(H)$*, for any* $\psi \in C^\infty_{comp}(\Omega)$ *and for any compact subset* L *of* Ω *there exists a constant* $C = C(f, \psi, L)$ *such that for all* $x \in L$ *and for any* $t \geq 0$

$$\left|\int_H f(h)\psi(g_t hx)\, dm(h) - \int_H f\, dm \int_\Omega \psi\, d\bar{\mu}\right| \leq Ct^q e^{-\lambda t}\,, \tag{2.4.5}$$

where $\lambda = \gamma/(2l+1+\frac{n-k}{2})$ *(*γ *and* l *from Theorem 2.4.5), and* $q = \varkappa/(2l+1+\frac{n-k}{2})$ *(*$\varkappa$ *from* (1.2)*).*

Proof. The proof will basically go along the same lines as that of Proposition 2.2.1. It clearly suffices to prove (2.4.5) for large enough t and for f such that the maps π_x are injective on $\operatorname{supp}(f)$ for all $x \in L$; then it can be extended to hold for arbitrary $f \in C^\infty_{comp}(H)$ and $t \geq 0$ by appropriate variation of the constant $C(f, \psi, L)$.

If either $f \equiv 0$ or $\psi \equiv 0$, there is nothing to prove. Otherwise, pick $t > 1$, denote by a the positive number $t^q e^{-\lambda t} \cdot \big(\int_H |f|\, dm\big)^{-1}$, and by r the positive number $a \cdot d(\psi)^{-1}$, where $d(\psi) \overset{\text{def}}{=} \max_{x\in\Omega} |\nabla\psi(x)|$. Then (2.2.2a) is satisfied, and making t large enough one can satisfy (2.2.2b) as well.

Using Lemma 2.4.7, choose nonnegative functions $f^- \in C^\infty_{comp}(H^-)$, $f^0 \in C^\infty_{comp}(H^0)$ satisfying (2.2.3) such that

$$\operatorname{supp}(f^-) \cdot \operatorname{supp}(f^0) \subset \tilde{B}(r/ct^\varkappa) \quad (c \text{ and } \varkappa \text{ from } (1.2))\,, \tag{2.4.6}$$

and at the same time[1]

$$\|\tilde{f}\|_l \leq \text{const} \cdot (r/ct^\varkappa)^{(2l+(n-k)/2)}\,, \tag{2.4.7}$$

where $\tilde{f} \in C^\infty_{comp}(\tilde{H})$ is defined by $\tilde{f}(h^-h^0) = f^-(h^-)f^0(h^0)\Delta_{\tilde{H}}(h^0)^{-1}$. Then the formula (2.1.1) implies (2.2.5), while (2.2.6) follows from (2.2.2a), (2.4.6) and at most polynomial expanding property (1.2). Hence

(2.4.8)

$$\begin{aligned}
&\left|\int_H f(h)\psi(g_t hx)\, dm(h) - \int_G f^-(h^-)f^0(h^0)f(h)\psi(g_t h^-h^0hx)\Delta_{\tilde{H}}(h^0)^{-1}\, d\mu(h^-h^0h)\right| \\
&= \left|\int_G f^-(h^-)f^0(h^0)f(h)\big(\psi(g_t hx) - \psi(g_t h^-h^0hx)\big)\Delta_{\tilde{H}}(h^0)^{-1}\, d\mu(h^-h^0h)\right| \\
&\leq a\int_G \left|f^-(h^-)f^0(h^0)f(h)\Delta_{\tilde{H}}(h^0)^{-1}\right|\, d\mu(h^-h^0h) = a\int_H |f|\, dm = t^q e^{-\lambda t}\,.
\end{aligned}$$

Define the functions φ and φ_x as in the proof of Proposition 2.2.1; then (2.2.8) and (2.2.9) are satisfied, and by Corollary 2.4.6

$$\left|(g_t\varphi_x, \psi) - \int_H f\, dm \int_\Omega \psi\, d\bar{\mu}\right| \leq Ee^{-\gamma t}\|\varphi_x\|_l\|\psi\|_l\,.$$

[1] The values of constants in the proof are different and independent on t, f and ψ.

Observe now that

$$
\begin{aligned}
\|\varphi_x\|_l &= \|\tilde{f}\cdot f\|_l \\
\text{(by Lemma 2.4.7)} \quad &\le \text{const}\cdot \|\tilde{f}\|_l \|f\|_l \\
\text{(by (2.4.7))} \quad &\le \text{const}\cdot \big(t^{\varkappa}/r\big)^{2l+\frac{n-k}{2}} \|f\|_l \\
\text{(by definition of } r) \quad &\le \text{const}\cdot \left(t^{\varkappa-q}e^{\gamma t}d(\psi)\int_H |f|\,dm\right)^{2l+\frac{n-k}{2}} \|f\|_l \\
\text{(by definition of } q \text{ and } \lambda) \quad &\le \text{const}\cdot t^q e^{(\gamma-\lambda)t}\left(d(\psi)\int_H |f|\,dm\right)^{2l+\frac{n-k}{2}} \|f\|_l\,.
\end{aligned}
$$

Thus

$$
\left|(g_t\varphi_x,\psi) - \int_H f\,dm\int_\Omega \psi\,d\bar{\mu}\right| \le \text{const}\cdot t^q e^{-\lambda t}\left(d(\psi)\int_H |f|\,dm\right)^{2l+\frac{n-k}{2}} \|f\|_l\|\psi\|_l\,,
$$

which, together with (2.4.8) and (2.2.9), finishes the proof. □

2.4.9. Corollary. *Let the condition* (EM) *be satisfied. Let f be almost in $C^\infty_{comp}(H)$, ψ almost in $C^\infty_{comp}(\Omega)$, and let L be a compact subset of Ω. Then given any $\varepsilon > 0$ there exists $T > 0$ such that* (2.2.1) *holds for any $x \in L$ and $t \ge T$.*

Proof. If $f \in C^\infty_{comp}(H)$ and $\psi \in C^\infty_{comp}(\Omega)$, for any $\varepsilon > 0$ the estimate (2.4.5) gives the value of T such that (2.2.1) is satisfied for all $x \in L$ and $t \ge T$. After that one can argue as in the proof of Corollary 2.2.2. □

2.5. An application

We now unify the results of three preceding sections in the form which will be used in the proof of Theorem 1.6. Let V be a subset[2] of H, K a subset of Ω, $x \in \Omega$ and $t \in \mathbb{R}$. Define the set

$$
V(x,K,t) = \{h \in g_tVg_{-t} \mid hg_tx \in K\} = \Phi_t\big(V \cap \pi_x^{-1}(g_{-t}K)\big)\,.
$$

In other words, $h \in V(x,K,t)$ if and only if $h = \Phi_t(h')$, where $h' \in V$ and $g_th'x \in K$. If f is the characteristic function of V and ψ the characteristic function of K, the measure of this set is equal to

$$
\text{(2.5)}\quad m\big(V(x,K,t)\big) = \int_H f(g_thg_{-t})\psi(hg_tx)\,dm(h) = e^{\chi t}\int_H f(h)\psi(g_thx)\,dm(h)\,.
$$

[2] All the sets considered are Borel subsets of the corresponding measure spaces.

Proposition. *Let V be a bounded subset of H with $m(\partial V) = 0$, K a bounded subset of Ω with $\bar{\mu}(\partial K) = 0$, L a compact subset of Ω. Then for any $\varepsilon > 0$ and $\sigma > 0$ there exists $T_1 = T_1(V, K, L, \varepsilon, \sigma) > 0$ such that*

$$T \geq T_1 \Rightarrow \forall\, x \in L \quad l\left(\left\{t \in [T, 2T] \;\middle|\; |e^{-\chi t} m\big(V(x, K, t)\big) - m(V)\bar{\mu}(K)| \leq \varepsilon\right\}\right) \geq (1-\sigma)T\,.$$

Proof. Let f be the characteristic function of V, ψ the characteristic function of K. Clearly f is almost in $C^\infty_{comp}(H)$, and ψ is almost in $C^\infty_{comp}(\Omega)$, in particular, almost uniformly continuous. Thus the proposition follows from Proposition 2.3.4 in the essentially semisimple case or from Corollary 2.4.9 in the (EM) case, once the substitution (2.5) is made. □

Note that if the flow is either semisimple or with property (EM), Corollaries 2.2.2 and 2.4.9 imply the validity of the above proposition for $\sigma = 0$ as well.

§3. Tesselations in nilpotent Lie groups

3.1. Let X be a (locally compact separable) topological space, m a Borel measure on X, G a group of measure-preserving homeomorphisms of X. Say that an open subset V of X is a *tesselation domain* (cf. [Mag]) for G-action on X relative to a (countable) subset Λ of G if

(a) $m(\partial V) = 0$,
(b) $\gamma_1(V) \cap \gamma_2(V) = \varnothing$ for different $\gamma_1, \gamma_2 \in \Lambda$, and
(c) $X = \bigcup_{\gamma \in \Lambda} \gamma(\overline{V})$.

For brevity, we will refer to the pair (V, Λ) as to the *tesselation* of X. The following fact follows easily from the definition:

Lemma. *If (V, Λ_V) is a tesselation of X, $U \subset V$, and (U, Λ_U) is a tesselation of V (with the topology and the measure coming from X), then $(U, \Lambda_V \cdot \Lambda_U)$ is a tesselation of X.*

3.2. Example. The interior of a fundamental domain for a discrete group Λ acting on X properly discontinuously is a tesselation domain relative to Λ. In particular, the set $I_k = \left\{(x_1, \dots, x_k) \;\middle|\; |x_j| < 1/2\right\} \subset \mathbb{R}^k$ is a tesselation domain relative to $\mathbb{Z}^k \subset G$, where G is any subgroup of isometries of $\mathbb{R}^k$ containing translations. Or, more generally, for any $R > 0$ the set $\frac{1}{R} I_k \subset \mathbb{R}^k$ is a tesselation domain relative to $\frac{1}{R}\mathbb{Z}^k$.

3.3. Let now H denote a connected simply connected nilpotent Lie group of dimension k, acting on itself by right translations. The purpose of the next proposition is to give an explicit construction of tesselation domains for this action. Note that any nilpotent Lie group considered below will be understood to be equipped with (left = right) Haar measure and right invariant Riemannian metric "dist".

Let $\mathfrak{h}$ be the Lie algebra of H. Pick a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{h}$ such that $\mathbb{R}$-spans $\mathfrak{h}_j$ of $\{X_1, \dots, X_j\}$, $j = 1, \dots, k$, are ideals in $\mathfrak{h}$ satisfying $[\mathfrak{h}, \mathfrak{h}_j] \subseteq \mathfrak{h}_{j-1}$, that is, a *strong Malcev basis* of $\mathfrak{h}$ (cf. [CG]). Here $\mathfrak{h}_0 = \{0\}$, and (at least) $\mathfrak{h}_1$ lies in the center of $\mathfrak{h}$. Note also that for any j, $0 \leq j \leq k$, $\{p(X_{j+1}), \dots, p(X_k)\}$ is a strong Malcev basis for $\mathfrak{h}/\mathfrak{h}_j$, where p is the quotient map $\mathfrak{h} \to \mathfrak{h}/\mathfrak{h}_j$.

Let $(e) = H_0 \subset H_1 \subset \cdots \subset H_{k-1} \subset H_k = H$ be the central series of H corresponding to the ideals $\mathfrak{h}_j$, so that $H_j = \exp(\mathfrak{h}_j)$, $j = 0, \dots, k$.

Proposition. *For any $r > 0$ there exists a neighborhood V of identity in H, $\operatorname{diam}(V) < r$, which is a tesselation domain for the right action of H on itself.*

Proof. As in the above example, let $I_k = \{ \sum_{j=1}^k x_j X_j \mid |x_j| < 1/2 \}$ be the unit cube in the Lie algebra $\mathfrak{h}$ of H. We will put $V = \exp(\frac{1}{R} I_k)$, where R is large enough to ensure that $\operatorname{diam}(V) < r$, and then use induction on the dimension k of H to prove that V is a tesselation domain for the right action of H on itself.

The basis of the induction is treated in Example 3.2. Suppose that the statement is true for groups of dimension less than k. Let $\hat{H}$ be the quotient group H/H_1, $\hat{\mathfrak{h}}$ its Lie algebra (isomorphic to the span of $X_2, \ldots, X_k$ as a linear space), p the quotient map $\mathfrak{h} \to \hat{\mathfrak{h}}$, π the quotient map $H \to \hat{H}$. Then the set $\hat{I} = p(\frac{1}{R} I_k)$ is the cube of sidelength $\frac{1}{R}$ in $\hat{\mathfrak{h}}$. Clearly $p \circ \exp = \exp \circ \pi$, hence by the induction assumption $\pi(V)$ is a tesselation domain for the right action of $\hat{H}$ on itself; denote by $\hat{\Lambda} \subset \hat{H}$ the corresponding set of translations.

Let $\Lambda \subset H$ be any section for $\hat{\Lambda}$. Then it is clear that $\hat{V} = \pi^{-1}(\pi(V))$ is a tesselation domain of H relative to Λ. But the group $H_1 = \exp(\mathfrak{h}_1)$ acts on $\hat{V}$ by left (= right, since it is in the center of G) translations, and one can easily show that V is a tesselation domain for this action relative to, say, $\Lambda_1 = \exp(\frac{1}{R}\mathbb{Z}X_1)$. By Lemma 3.1, $(V, \Lambda_1 \cdot \Lambda)$ is a tesselation of H. □

3.4. Let now (V, Λ) be any tesselation of H, Φ an expanding automorphism of H, and let c be the *contraction bound* of Φ^{-1} on V, that is,

$$c = \sup_{g,h \in V,\, g \neq h} \frac{\operatorname{dist}(\Phi^{-1}(g), \Phi^{-1}(h))}{\operatorname{dist}(g, h)} \,.$$

Define a function

$$f_V(r) = m(\{h \in H \mid \operatorname{dist}(h, \partial V) \leq r\})\,;$$

since $m(\partial V) = 0$, $f_V(r) \to 0$ as $r \to 0$. The next inequality helps one to estimate the number of translates $V\gamma, \gamma \in \Lambda$, which lie entirely inside the expansion of V by the map Φ.

Proposition.

$$\#\{\gamma \in \Lambda \mid V\gamma \subset \Phi(V)\} \geq \frac{m(\Phi(V))}{m(V)} \left(1 - \frac{f_V(c \cdot \operatorname{diam}(V))}{m(V)}\right) . \tag{3.1}$$

Proof. One has

$$\begin{aligned} &\#\{\gamma \in \Lambda \mid V\gamma \subset \Phi(V)\} = \#\{\gamma \in \Lambda \mid \Phi^{-1}(V\gamma) \subset V\} \\ &= \#\{\gamma \in \Lambda \mid \Phi^{-1}(V\gamma) \cap V \neq \varnothing\} - \#\{\gamma \in \Lambda \mid \Phi^{-1}(V\gamma) \cap \partial V \neq \varnothing\}\,. \end{aligned}$$

Since (V, Λ) is a tesselation of H, the minuend is not less than $m(\Phi(V))/m(V)$, while the substrahend is not greater than

$$\frac{m\left(\{h \in H \mid \operatorname{dist}(h, \partial V) \leq \operatorname{diam}(\Phi^{-1}(V))\}\right)}{m(\Phi^{-1}(V))} \leq \frac{f_V(c \cdot \operatorname{diam}(V)) \cdot m(\Phi(V))}{m(V)^2}\,,$$

and (3.1) follows. □

3.5. Corollary. *Let $\{\Phi_t,\ t \ge 0\}$ be a one-parameter semigroup of expanding automorphisms of H. Then for any tesselation (V,Λ) of H and any $\varepsilon > 0$, there exists $T_2 = T_2(V,\Lambda,\varepsilon) > 0$ such that*

$$t \ge T_2 \quad \Rightarrow \quad \#\{\gamma \in \Lambda \mid V\gamma \subset \Phi_t(V)\} \ge \frac{m\big(\Phi_t(V)\big)}{m(V)}(1-\varepsilon)\,.$$

Proof. Indeed, since the contraction bounds c_t of Φ_t tend to 0 as $t \to \infty$, one has $f_V\big(c_t \cdot \operatorname{diam}(V)\big) \to 0$ as $t \to \infty$. Thus one can find T_2 such that $f_V\big(c_t \cdot \operatorname{diam}(V)\big) \le \varepsilon m(V)$ for $t \ge T_2$. $\square$

3.6. We return now to the setting of §1, i.e. take H to be the horospherical subgroup of $\{g_t\}$, and Φ_t to be the restriction of the inner automorphism $g \to g_t g g_{-t}$ to H. Let (V,Λ) be any tesselation of H, K a subset of Ω, $x \in \Omega$, $t \in \mathbb{R}$. Say that a translation $\gamma \in \Lambda$ is (x,K,t)-*marked* (or just *marked*, if the prefix is clear from the context) if $V\gamma$ lies entirely inside $V(x,K,t)$, in other words, $V\gamma \subset \Phi_t(V)$ and $V\gamma g_t x = g_t\Phi_t^{-1}(V\gamma)x \subset K$. We will also call *marked* the translate $V\gamma$ or $\overline{V}\gamma$ for a marked γ, and denote the nimber of (x,K,t)-marked translates by $N(x,K,t)$.

The above results now allow one to deduce the following fact about tesselations from the measure-theoretical estimate of Proposition 2.5:

Proposition. *Let K be a subset of Ω with $\bar\mu(\partial K) = 0$, L a compact subset of Ω. Then for any $r > 0$ there exists a tesselation (V,Λ) of H such that*

(a) $\operatorname{diam}(V) < r$,

(b) *for any $\varepsilon > 0$ and $\sigma > 0$ there exists $T_0 > 0$ such that*

(3.2)
$$T \ge T_0 \Rightarrow \forall\, x \in L \quad l\left(\left\{t \in [T,2T] \;\middle|\; N(x,K,t) \ge e^{\chi t}\big(\bar\mu(K)-\varepsilon\big)\right\}\right) \ge (1-\sigma)T\,,$$

and

(c) *assuming $\sigma < 1$ and $T \ge T_0$,*

(3.3) $$\forall\, x \in L \quad \exists\, t(x) \in [T,(1+\sigma)T] \quad \textit{such that} \quad N\big(x,K,t(x)\big) \ge e^{\chi t}\big(\bar\mu(K)-\varepsilon\big)\,.$$

Proof. If $\bar\mu(K) = 0$, there is nothing to prove. Otherwise, pick a compact subset K' of K with $\bar\mu(\partial K') = 0$, which satisfies $\bar\mu(K') \ge \bar\mu(K) - \varepsilon/3$ and lies at a positive distance from the complement of K. Using Proposition 3.3, find a tesselation domain $V \subset H$ such that $VV^{-1}K' \subset K$ and $\operatorname{diam}(V) < r$. Then (a) is satisfied; moreover, for any $t \in \mathbb{R}$ and $x \in \Omega$

$$\begin{aligned} N(x,K,t) &\ge N(x,VV^{-1}K',t) \\ &= \#\{\gamma \in \Lambda \mid V\gamma \subset \Phi_t(V)\ \&\ V\gamma g_t x \subset VV^{-1}K'\} \\ &\ge \#\{\gamma \in \Lambda \mid V\gamma \subset \Phi_t(V)\ \&\ \gamma g_t x \subset V^{-1}K'\} \\ &\ge \#\{\gamma \in \Lambda \mid V\gamma \subset \Phi_t(V)\ \&\ V\gamma g_t x \cap K' \ne \varnothing\} \\ &= \#\{\gamma \in \Lambda \mid V\gamma \subset \Phi_t(V)\} - \#\{\gamma \in \Lambda \mid V\gamma g_t x \cap K' = \varnothing\}\,. \end{aligned}$$

Now take $T_0 = \max\big(T_1(V, K', L, \varepsilon m(V)/3, \sigma)$ *from Proposition* 2.5, $T_2(V, \Lambda, \varepsilon/3)$ *from Corollary* 3.5$\big)$. Then the minuend is for all $x \in L$ and $t \geq T_0$ not less than $e^{\chi t}(1 - \varepsilon/3)$ by Corollary 3.5. On the other hand, for any $x \in L$ and $T \geq T_0$, the substrahend is by Proposition 2.5 not greater than

$$\frac{m\big(\Phi_t(V) \smallsetminus V(x, K', t)\big)}{m(V)} \leq e^{\chi t}\left(1 - \frac{1}{m(V)} e^{-\chi t} m\big(V(x, K', t)\big)\right)$$
$$\leq e^{\chi t}\left(1 - \frac{1}{m(V)}\big(m(V)\bar{\mu}(K') - \varepsilon m(V)/3\big)\right) \leq e^{\chi t}\big(1 - \bar{\mu}(K) + 2\varepsilon/3\big)$$

for a set of values of $t \in [T, 2T]$ with measure not less than $(1 - \sigma)T$, and part (b) follows; the statement (3.3) in (c) is clearly a direct consequence of (b). $\square$

As before, one can remark that the above statements are true for $\sigma = 0$ if the flow is either semisimple or with property (EM); in particular, one can then take $t(x) = T$ for all $x \in L$.

3.7. In order to deduce a dimension estimate from the above proposition, it will be convenient to use the *density* (relative measure) $\delta(x, K, t)$ of all the marked translates instead of the number $N(x, K, t)$:

$$(3.4)\quad \delta(x, K, t) = \frac{m\big(\bigcup_{\gamma \text{ is } (x,K,t)\text{-marked}} V\gamma\big)}{m\big(\Phi_t(V)\big)} = \frac{m\big(\bigcup_{\gamma \text{ is } (x,K,t)\text{-marked}} \Phi_t^{-1}(V\gamma)\big)}{m(V)}\,;$$

this notation allows one to rewrite (3.3) as follows:

(3.5)
$$t \geq T \quad \Rightarrow \quad \forall\, x \in L \quad \exists\, t(x) \in [T, (1+\sigma)T] \quad \text{such that} \quad \delta\big(x, K, t(x)\big) \geq \bar{\mu}(K) - \varepsilon\,.$$

§4. The main construction

4.1. We start with the formal description of a construction (cf. [F, Mc, U, PW]) which has served as a basic source for producing fractal sets since the times of Cantor and Hausdorff. In what follows, $\mathbb{N}_0$ will stand for the set of nonnegative integers. Let X be a Riemannian manifold, m a Borel measure on X, A_0 a compact subset of X. Say that a countable collection $\mathcal{A}$ of compact subsets of A_0 is *tree-like* relative to m if $\mathcal{A}$ is the union of finite nonempty subcollections $\mathcal{A}_j$, $j \in \mathbb{N}_0$, such that $\mathcal{A}_0 = \{A_0\}$ and the following two conditions are satisfied:

(TL1) $\qquad \forall\, j \in \mathbb{N} \quad \forall\, A, B \in \mathcal{A}_j \quad \text{either } A = B \quad \text{or} \quad m(A \cap B) = 0\,;$

(TL2) $\qquad \forall\, j \in \mathbb{N} \quad \forall\, B \in \mathcal{A}_j \quad \exists\, A \in \mathcal{A}_{j-1} \quad \text{such that} \quad B \subset A\,.$

Say also that $\mathcal{A}$ is *strongly tree-like* if it is tree-like and in addition

(STL) $\qquad d_j(\mathcal{A}) \overset{\text{def}}{=} \sup_{A \in \mathcal{A}_j} \operatorname{diam}(A) \to 0 \quad \text{as} \quad j \to \infty\,.$

Let $\mathcal{A}$ be a tree-like collection of sets. For each $j \in \mathbb{N}_0$, let $\mathbf{A}_j = \bigcup_{A \in \mathcal{A}_j} A$. These are nonempty compact sets, and from (TL2) it follows that $\mathbf{A}_j \subset \mathbf{A}_{j-1}$ for any $j \in \mathbb{N}$. Therefore one can define the (nonempty) *limit set* of $\mathcal{A}$ to be

$$\mathbf{A}_\infty = \bigcap_{j \in \mathbb{N}_0} \mathbf{A}_l \,.$$

Further, for any subset B of A_0 and $j \in \mathbb{N}$, define the *jth stage density* $\delta_j(B, \mathcal{A})$ of B in $\mathcal{A}$ by

$$\delta_j(B, \mathcal{A}) = \begin{cases} 0\,, \text{ if } m(B) = 0 \\ \dfrac{m(\mathbf{A}_j \cap B)}{m(B)}\,, \text{ if } m(B) > 0\,; \end{cases}$$

the condition (TL1) implies that $\delta_j(B, \mathcal{A}) \leq 1$ for any $B \subset A_0$ and $j \in \mathbb{N}$. Then for any $j \in \mathbb{N}_0$ define the *jth stage density* $\Delta_j(\mathcal{A})$ of $\mathcal{A}$ by

$$\Delta_j(\mathcal{A}) = \inf_{B \in \mathcal{A}_j} \delta_{j+1}(B, \mathcal{A})\,.$$

The following estimate, based on an application of Frostman's Lemma, is essentially proved in [Mc] and [U]:

Lemma. *Assume that there exist constants $D > 0$ and $k > 0$ such that*

$$m\big(B(x, r)\big) \leq D r^k \tag{4.1}$$

for any $x \in A_0$ ($B(x, r)$ being a ball of radius r centered at x). Then for any strongly tree-like (relative to m) collection $\mathcal{A}$ of subsets of A_0

$$\dim(\mathbf{A}_\infty) \geq k - \limsup_{j \to \infty} \frac{\sum_{i=0}^{j-1} \log(\frac{1}{\Delta_i(\mathcal{A})})}{\log(\frac{1}{d_j(\mathcal{A})})}\,.$$

4.2. We are now ready to give a

Proof of Theorem 1.6. Recall that a point $x \in \Omega \smallsetminus Z$ is given. Pick a compact set $K \subset \Omega \smallsetminus Z$ with $\bar{\mu}(\partial K) = 0$ containing some neighborhood of x, and choose a sequence $r_s \to 0$, $r_s < \operatorname{dist}(x, \Omega \smallsetminus K)$. Then, using Proposition 3.6 with $r = r_s$ and $L = K$, find a corresponding tesselation domain V, which will play the role of V_s in Theorem 1.6. By the choice of the sequence $\{r_s\}$, the set Vx is contained in K and (1.6) holds.

Choose ε and σ such that $0 < \varepsilon < \bar{\mu}(K)$ and $0 < \sigma < 1$. Using part (b) of Proposition 3.6, find $T_0 > 0$ such that (3.2) holds. Choose $T = T_s \geq T_0$ such that $T_s \to \infty$ as $s \to \infty$, and denote $(1 + \sigma)T$ by T'. Then by (3.5), for any $y \in K$ there exists $t(y) \in [T, T']$ such that

$$\delta\big(y, K, t(y)\big) \geq \bar{\mu}(K) - \varepsilon\,. \tag{4.2}$$

Now for all $y \in K$ define strongly tree-like (relative to the Haar measure m on H) collections $\mathcal{A}(y)$ inductively as follows. We first let $\mathcal{A}_0(y) = \{\overline{V}\}$ for all $y \in K$, then define

$$\mathcal{A}_1(y) = \{\Phi_{-t(y)}(\overline{V}\gamma) \mid \gamma \text{ is } \big(y, K, t(y)\big)\text{-marked}\}\,. \tag{4.3}$$

More generally, if $\mathcal{A}_i(y)$ is defined for all $y \in K$ and $i \le j$, we let

$$\mathcal{A}_{j+1}(y) = \{\Phi_{-t(y)}(A\gamma) \mid \gamma \text{ is } \big(y, K, t(y)\big)\text{-marked},\ A \in \mathcal{A}_j(\gamma g_{t(y)} y)\} \tag{4.4}$$

(the sets $\mathcal{A}_j(\gamma g_{t(y)} y)$ are defined, since $\gamma g_{t(y)} y \in K$ by virtue of γ being a marked translation).

The properties (TL1) and (TL2) follow readily from the construction and V being a tesselation domain. Also, since $t(y)$ is for all $y \in K$ not less than T, from (4.4) it follows that for all $j \in \mathbb{N}_0$ and $y \in K$, the constant $d_j\big(\mathcal{A}(y)\big)$ is not greater than $\operatorname{diam}(V) \cdot (c_T)^j$, where $c_T < 1$ is the contraction bound of Φ_T^{-1} on V, and therefore (STL) is satisfied. Moreover, (4.3) and the definition (3.4) of $\delta(x, K, t)$ imply that the density $\delta_1\big(\overline{V}, \mathcal{A}(y)\big)$ is for all $y \in K$ exactly equal to $\delta\big(y, k, t(y)\big)$. Hence using (4.4), (4.2) and the relative Φ_t-invariance (2.1.2) of m, one can show by induction that the jth density $\Delta_j\big(\mathcal{A}(y)\big)$ is for all $y \in K$ and $j \in \mathbb{N}_0$ bounded from below by $\bar{\mu}(K) - \varepsilon$. Finally, the measure m clearly satisfies (4.1) with some positive D and $k = \dim(H)$, and an application of Lemma 4.1 yields that for all $y \in K$

$$\dim\big(\mathbf{A}_\infty(y)\big) \ge k - \limsup_{j\to\infty} \frac{j \log\big(\frac{1}{\bar{\mu}(K)-\varepsilon}\big)}{\log\big(\frac{1}{\operatorname{diam}(V)}\big) + j \log(\frac{1}{c_T})} = k - \frac{\log\big(\frac{1}{\bar{\mu}(K)-\varepsilon}\big)}{\log(\frac{1}{c_T})}\,. \tag{4.5}$$

We now claim that for any h in the limit set $\mathbf{A}_\infty(x)$ of $\mathcal{A}(x)$, the trajectory F^+hx is contained in some compact subset of Ω. Indeed, for all $y \in K$ define a sequence $t_j(y)$, $j \in \mathbb{N}_0$, as follows: let $t_0(y) = 0$ for all $y \in K$, and then, if $t_i(y)$ is defined for all $y \in K$ and $i \le j$, let

$$t_{j+1}(y) = t(y) + t_j(g_{t(y)} y)\,. \tag{4.6}$$

From (4.4) and (4.6) it immediately follows that $g_{t_j(y)}\mathbf{A}_j(y)y \subset K$ for all $y \in K$ and $j \in \mathbb{N}$. Recall now that V was chosen so that $\mathbf{A}_0(x)x = \overline{V}x \subset K$, therefore $g_{t_j(x)}\mathbf{A}_j(x)x \subset K$ for all $j \in \mathbb{N}_0$, which means that

$$\forall\, j \in \mathbb{N}_0\ \forall\, h \in \mathbf{A}_\infty(x) \quad g_{t_j(x)} hx \subset K\,. \tag{4.7}$$

Now define the set

$$C = C_s = \begin{cases} \displaystyle\bigcup_{t=-T'}^{0} g_t K, \text{ if } Z \text{ is } F^+\text{-invariant} \\ \displaystyle\bigcup_{t=0}^{T'} g_t K, \text{ if } Z \text{ is } F^-\text{-invariant.} \end{cases}$$

Clearly C is compact and has empty intersection with Z. From (4.6) it easily follows that the difference $t_j - t_{j-1}$ is for any $j \in \mathbb{N}$ bounded from above by T', therefore

$$\forall\, h \in \mathbf{A}_\infty(x) \quad F^+hx \subset C\,. \tag{4.8}$$

As s goes to ∞, $T = T_s$ also tends to infinity, thus the contraction bound c_T decreases to 0. Hence the right hand side of (4.5) tends to k as $s \to \infty$, which, together with (4.8), finishes the proof of (1.7). □

4.3. One can notice that the invariance of the set Z to be avoided was used only once – when deducing a statement about continuous orbits F^+hx (cf. (4.8)) from the corresponding statement about the sequence $\{g_{t_j(x)}hx\}$ (cf. (4.7); see also the proof of Lemma 5.1(a) below, which is a generalization of the same argument). In other words, the construction presented above leads to the proof of the following

Theorem. *Let G, Γ, Ω and $F = \{g_t\}$ be as in Theorem 1.1, and let K be a subset of Ω with $\bar\mu(K) > 0$ and $\bar\mu(\partial K) = 0$. Then given any nonempty open subset W of Ω, $\varepsilon > 0$ and $\sigma > 0$, there exists $T_0 > 0$ such that for any $T \geq T_0$*

$$\dim\left(\left\{x \in W \,\middle|\, \begin{array}{l} \text{there exists a sequence } 0 = t_0 < t_1 < \dots < t_j < \dots \text{ such that} \\ \forall\, j \in \mathbb{N}_0 \qquad T \leq t_j - t_{j-1} \leq (1+\sigma)T \qquad \text{and} \qquad g_{t_j}x \in K \end{array}\right\}\right) > n - \varepsilon\,.$$

Moreover, if the flow is either semisimple or with property (EM), *one can take $\sigma = 0$ and $t_j = jT$. In other words, for any nonempty open $W \subset \Omega$ and any $\varepsilon > 0$, there exists $T > 0$ such that for any $t \geq T$*

$$\dim\left(\{x \in W \mid \forall\, j \in \mathbb{N}_0 \quad g_{jt}x \in K\}\right) > n - \varepsilon\,.$$

§5. Generalizations and concluding remarks

5.1. To make the results below easier to state, let us introduce the following notation. For a locally compact metric space X, let X^* denote the topological space $X \cup \{\infty\}$, with the topology defined so that the complements to all the compact sets constitute the basis of the neighborhoods of $\{\infty\}$. In other words, for $A \subset X$, the closure (in X^*) of A contains ∞ iff A is not bounded. Thus if X is compact, ∞ is an isolated point of X^*; otherwise X^* is a one-point compactification of X.

Let now F be any set of maps $X \to X$. Say that a subset Z of X^* is *F-escapable* if for any nonempty open subset W of X

$$\dim\big(\{x \in W \mid \overline{Fx} \cap Z = \varnothing\}\big) = \dim(W)\,,$$

with the closure taken in the topology of X^*.

The main result of the paper, Theorem 1.1, can now be stated as follows:

Let G be a connected semisimple Lie group without compact factors, Γ an irreducible lattice in G, F a (one-parameter or cyclic) nonquasiunipotent subgroup of G. Then for any closed null subset Z of G/Γ which is invariant under either F^+ or F^-, $Z \cup \{\infty\}$ is F-escapable.

Clearly any subset of an F-escapable set is F-escapable, and for any subset F' of F, any F-escapable set is F'-escapable. The partial converse for the latter statement, as well as another easy but useful (see e.g. Remark 1.2) stability property of escapable sets, are stated in the following

Lemma. (a) *Let F and F' be sets of continuous transformations of X, and let F'' be a compact (in the compact-open topology) set of homeomorphisms $X \to X$; assume that $F \subset F''F'$. Then any F'-escapable subset of X^* which is invariant under $(F'')^{-1}$ (we set $f(\infty) = \infty$ for any map $X \xrightarrow{f} X$) is F-escapable.*

(b) *Let $\tilde{X} \xrightarrow{\varphi} X$ be a locally bi-Lipschitz covering of metric spaces, and F a set of maps $\tilde{X} \to \tilde{X}$ which factor through φ (so that elements of ${}_\varphi F \stackrel{\text{def}}{=} \varphi \circ F \circ \varphi^{-1}$ are transformations of X). Then $Z \subset X^*$ is ${}_\varphi F$-escapable if and only if $\varphi^{-1}(Z)$ (here $\varphi(\infty) = \infty$) is F-escapable.*

Proof. For part (a), let $Z \subset X^*$ be $(F'')^{-1}$-invariant; we will show that the set $\{x \in \Omega \mid \overline{Fx} \cap Z = \varnothing\}$ contains $\{x \in \Omega \mid \overline{F'x} \cap Z = \varnothing\}$. Take x from the latter set, then for any $f \in F''$, $f(\overline{F'x}) \cap Z \subset f(\overline{F'x}) \cap f(Z) = f(\overline{F'x} \cap Z) = \varnothing$, and by compactness of F'', the closure of $F''F'x$ for x as above has empty intersection with Z.

For (b), let $Z \in X^*$ be such that $\varphi^{-1}(Z)$ is F-escapable. Take a nonempty open set $W \subset X$ small enough for $\varphi^{-1}(W)$ to consist of finitely many open sets bi-Lipschitz homeomorphic to W; let $\tilde{W}$ be one of these sets. Then φ gives a one-to-one correspondence between $\{x \in \tilde{W} \mid \overline{Fx} \cap \varphi^{-1}(Z) = \varnothing\}$ and $\{x \in W \mid \overline{{}_\varphi Fx} \cap Z = \varnothing\}$, while the dimension of the former set is equal to $\dim(\tilde{W}) = \dim(W)$ by the assumption; hence Z is ${}_\varphi F$-escapable. The proof of the converse statement goes along the same lines; see also Remark 1.2. □

5.2. Our goal now is to extend the class of groups G for which the statement similar to that of Theorem 1.1 is true. First let us consider the special case $Z = \varnothing$, the subject of Conjecture (A) from [Mrg2].

Let G be a connected Lie group, Γ a lattice in G, F a subgroup of G. Say that the flow $(G/\Gamma, F)$ *has property* (Q) if for any connected normal subgroup $N \subset G$ with the quotient map $p : G \to G' \stackrel{\text{def}}{=} G/N$ such that G' is semisimple without compact factors and $p(\Gamma)$ is an irreducible lattice in G', at least one of the following three conditions is satisfied:

(Q1) $p(\Gamma)$ is cocompact in G';

(Q2) $\operatorname{Ad} p(F)$ is relatively compact;

(Q3) $p(F)$ is not quasiunipotent.

Note that if the flow is semisimple, then for any such p the subgroup $p(F)$ satisfies either (Q2) or (Q3); therefore any semisimple flow has property (Q).

Theorem. *Let G be a connected Lie group, Γ a lattice in G, $\Omega = G/\Gamma$, F a (one-parameter or cyclic) subgroup of G. The following are equivalent:*

(a) $\{\infty\}$ *is F-escapable;*

(b) (Ω, F) *has property* (Q).

Proof. First let us show that (a) implies (b). Let p be such that neither of the conditions (Q1), (Q2) and (Q3) holds, in other words, $\Omega' \overset{\text{def}}{=} G'/p(\Gamma)$ is not compact, $F' \overset{\text{def}}{=} p(F)$ is quasiunipotent, and $\operatorname{Ad} F'$ is not relatively compact. Then F' can be expressed as $F_c F_u$, where F_c and F_u are commuting (one-parameter or cyclic) subgroups such that $\operatorname{Ad} F_c$ is relatively compact and F_u is a nontrivial unipotent subgroup. Dividing by the center of G' and applying the argument similar to that of Remark 1.2 and the proof of Lemma 5.1(b), one can show that for any $x \in \Omega'$, the orbit $F'x$ is bounded if and only if $F_u x$ is bounded. Hence one can assume that F' is unipotent. The results on closures of unipotent orbits [Rat2, see also DM, Theorem 3] imply that the set $A' \overset{\text{def}}{=} \{x \in \Omega' \mid F'x \text{ is bounded}\}$ lies in a countable union of proper submanifolds of Ω'. Denote by $\bar{p}$ the induced map of homogeneous spaces $\Omega \to \Omega'$. Then the set $\{x \in \Omega \mid Fx \text{ is bounded}\}$ is contained in $\bar{p}^{-1}(A')$, and by Wegmann's Product Theorem [Weg], the latter set has dimension at most $\dim(G) - 1$.

Now assume that (b) is satisfied. First consider the case when G is connected semisimple without compact factors and (see Remark 1.2) with trivial center. Let $G_1, \dots, G_l$ be connected normal subgroups of G such that $G = \prod_{i=1}^{l} G_i$, $G_i \cap G_j = \{e\}$ if $i \neq j$, $\Gamma_i = G_i \cap \Gamma$ is an irreducible lattice in G_i for each i, and $\prod_{i=1}^{l} \Gamma_i$ has finite index in Γ.

Take any $i \in \{1, \dots, l\}$. Denote by p_i the projection $G \to G_i$, and by Ω_i the homogeneous space $G_i/p_i(\Gamma)$. By assumption (b), at least one of the conditions (Q1), (Q2) or (Q3) (with p_i in place of p and G_i in place of G') is satisfied. Denote by A_i the set $\{x \in \Omega_i \mid p_i(F)x \text{ is bounded}\}$. If either (Q1) or (Q2) holds, then every orbit is bounded, so clearly for any nonempty open $W_i \subset \Omega_i$, the dimension of $W_i \cap A_i$ is equal to $\dim(G_i)$. By virtue of Theorem 1.1, this equality still holds if (Q3) is satisfied. This implies, in view of Wegmann's Theorem or Lemma 1.4, that for any nonempty open $W \subset \prod_{i=1}^{l} \Omega_i$, the dimension of the set $\{x \in W \mid Fx$ is bounded$\}$ is equal to the dimension of G. Then (a) follows from Lemma 5.1(b) and the fact that Ω is a finite covering of $\prod_{i=1}^{l} \Omega_i = G/\prod_{i=1}^{l} p_i(\Gamma)$.

Consider now the general case. Denote by $R(G)$ the radical of G. Then $G/R(G) = G_0 \times \hat{G}$, where G_0 is compact and $\hat{G}$ is connected semisimple without compact factors. Let $\pi : G \to \hat{G}$ be the canonical projection, then (cf. [Rag, Chapter 9] or [Dan1, Lemma 5.1]) $\hat{\Gamma} \overset{\text{def}}{=} \pi(\Gamma)$ is a lattice in $\hat{G}$. Denote by $\hat{\Omega}$ the homogeneous space $\hat{G}/\hat{\Gamma}$, and let $\hat{A}$ be the set $\{x \in \hat{\Omega} \mid \pi(F)x \text{ is bounded}\}$. Since $\hat{G}$ is a quotient group of G, property (Q) for the flow (Ω, F) implies the same property for $\big(\hat{\Omega}, \pi(F)\big)$. Therefore, by the first part of the proof, for any nonempty open $\hat{W} \subset \hat{\Omega}$ the dimension of $\hat{A} \cap \hat{W}$ is equal to $\dim(\hat{G})$.

Denote by $\bar{\pi}$ the induced map of homogeneous spaces $\Omega \to \hat{\Omega}$. Since (cf. the second part of the same lemma from [Dan1]) $\operatorname{Ker} \pi \cap \Gamma$ is a uniform lattice in

$\operatorname{Ker}\pi$, the preimage $\bar{\pi}^{-1}(x)$ of any point $x \in \hat{\Omega}$ is compact. Therefore the F-orbit of any point from $\bar{\pi}^{-1}(\hat{A})$ is bounded, while Wegmann's Theorem implies that the intersection of $\bar{\pi}^{-1}(\hat{A})$ with any nonempty open subset of Ω has full dimension. $\square$

5.3. The same argument as in the proof of (b)⇒(a) leads to the following

Theorem. *Let G, Γ, Ω and F be as in Theorem 5.2, and let $Z \subset \Omega$ be invariant under either F^+ or F^-. Assume that for any connected normal subgroup $N \subset G$ with the quotient map $p : G \to G' \stackrel{\text{def}}{=} G/N$ such that G' is semisimple without compact factors and $p(\Gamma)$ is an irreducible lattice in G',*

(a) *$p(F)$ is not quasiunipotent, and*

(b) *the closure of $\bar{p}(Z)$ (here $\bar{p}$ is the induced map of homogeneous spaces $\Omega \to G'/p(\Gamma)$) has Haar measure 0.*

Then $Z \cup \{\infty\}$ is F-escapable.

We conjecture that this theorem is still true if the cumbersome condition (b) is replaced by "the closure of Z has measure 0" as in Theorem 1.1.

5.4. Our next objective is to treat the case of disconnected groups. Let G^0 be the connected component of identity in a Lie group G. If $\Gamma \subset G$ is a lattice, the subgroup $G^0\Gamma$ is of finite index in G, thus $\Omega = G/\Gamma$ consists of finitely many connected components $\Omega_1, \dots, \Omega_l$, the component Ω_1 of $[e]$ being naturally identified with $\Omega^0 \stackrel{\text{def}}{=} G^0/(G^0 \cap \Gamma) \cong G^0\Gamma/\Gamma$. In this situation, the case $F = \{g^i \mid i \in \mathbb{Z}\}$, with g not contained in G^0, requires separate treatment. First let us consider subgroups F which are contained in G^0 (in particular, one-parameter subgroups of G).

Theorem. *Let $\mathcal{F}$ be a class of subsets of G^0 which is invariant under all inner automorphisms of G, and for any $F \in \mathcal{F}$, let $\mathcal{G}(F)$ be another class of subsets of G^0 such that the correspondence $\mathcal{G}$ is also invariant under inner automorphisms of G. Then the following are equivalent:*

(a) *for any $F \in \mathcal{F}$ and for any closed null $Z \subset \Omega$ which is invariant under at least one of the sets in $\mathcal{G}(F)$, $Z \cup \{\infty\}$ is F-escapable;*

(b) *for any $F \in \mathcal{F}$ and for any closed null $Z^0 \subset \Omega^0$ which is invariant under at least one of the sets in $\mathcal{G}(F)$, $Z^0 \cup \{\infty\}$ is F-escapable.*

Proof. Since Ω^0 is one of the connected components of Ω, (a) trivially implies (b). For the converse, choose an element g_j in each coset of $G/G^0\Gamma$ ($g_1 = e$), then left multiplication by g_j gives a bijective map $\Omega^0 \to \Omega_j$. Given an open nonempty set $W \subset \Omega$ and a closed null $Z \subset \Omega$, for any j, $1 \le j \le l$, define $W_j = W \cap \Omega_j$ and $Z_j = Z \cap \Omega_j$.

Take any $F \in \mathcal{F}$; since $F \subset G^0$, it leaves each of the components Ω_j invariant. Therefore, the set $\{x \in W \mid \overline{Fx} \cap (Z \cup \{\infty\}) = \varnothing\}$ is the union of l sets

(5.1)
$$\{x \in W_j \mid \overline{Fx} \cap (Z_j \cup \{\infty\}) = \varnothing\} = g_j\{x \in g_j^{-1}W_j \mid \overline{g_j^{-1}Fg_jx} \cap (g_j^{-1}Z_j \cup \{\infty\}) = \varnothing\}.$$

Let $F' \in \mathcal{G}(F)$ be such that Z is F'-invariant. Take j, $1 \le j \le l$, such that W_j is nonempty. Clearly the closed null set Z_j is F'-invariant, hence the closed null subset $g_j^{-1}Z_j$ of Ω^0 is invariant under $g_j^{-1}F'g_j$. Since $\mathcal{F}$ and $\mathcal{G}$ are invariant under inner automorphisms of G, $g_j^{-1}Fg_j \in \mathcal{F}$ and $g_j^{-1}F'g_j \in \mathcal{G}(F)$. Thus, by assumption (b), at least one (with j as above) of the sets (5.1) has full Hausdorff dimension. $\square$

5.5. Corollary. *Let G, Ω, G^0 and Ω^0 be as in Section 5.4, and let F be a (one-parameter or cyclic) subgroup of G^0. Then the following are equivalent:*

(a) $\{\infty\} \subset \Omega^*$ *is F-escapable;*

(b) (Ω^0, F) *has property* (Q).

Proof. Consider the class $\mathcal{F}$ of subgroups $F \subset G^0$ such that the flow (Ω^0, F) has property (Q), with the correspondence $\mathcal{G}(F) = \{G^0\}$ for any $F \in \mathcal{F}$. Clearly both $\mathcal{F}$ and $\mathcal{G}$ are invariant under inner automorphisms of G, therefore one can combine Theorems 5.2 and 5.4 to get the desired statement. $\square$

5.6. Before considering subgroups of the form $\{g^i \mid i \in \mathbb{Z}\}$ with g not contained in G^0, we will prove an auxillary result which seems to be of independent interest, indicating yet another direction for the generalization of Theorem 1.1.

Theorem. *Let G, Γ and Ω be as in §1, and let g and γ be two elements of G such that g is not quasiunipotent and γ normalizes Γ. Define the map $f : \Omega \to \Omega$ by $f(h\Gamma) = gh\gamma\Gamma$, $h \in G$, and denote by F the cyclic subgroup generated by f. Then for any closed null subset Z of Ω which is invariant under either f or f^{-1}, $Z \cup \{\infty\}$ is F-escapable.*

Proof. Denote by Γ' the normalizer of Γ in G. Then it is known [Rag, Corollary 5.17] that Γ' is a lattice in G containing Γ, therefore Ω is a finite covering of the homogeneous space $\Omega' \stackrel{\text{def}}{=} G/\Gamma'$, and the covering map φ sends the transformation f of Ω to the left multiplication by g in Ω'. In other words, ${}_{\varphi}F$ can be identified with the cyclic subgroup generated by g. For any closed null subset Z of Ω which is invariant under either f or f^{-1}, $\varphi(Z)$ will be closed, null and invariant under either g or g^{-1}, hence ${}_{\varphi}F$-escapable by Theorem 1.1. Lemma 5.1(b) now implies that the set $\varphi^{-1}\big(\varphi(Z)\big)$ (which contains Z) is F-escapable. $\square$

5.7. Theorem. *Let G be a semisimple Lie group such that G^0 is without compact factors, Γ a lattice in G such that $\Gamma^0 \stackrel{\text{def}}{=} G^0 \cap \Gamma$ is irreducible, and let F be a (one-parameter or cyclic) nonquasiunipotent subgroup of G. Then for any closed null subset Z of Ω which is invariant under either F^+ or F^-, $Z \cup \{\infty\}$ is F-escapable.*

Proof. The case $F \subset G^0$ follows from Theorems 1.1 and 5.4 (in the latter, $\mathcal{F}$ stands for the class of (one-parameter or cyclic) nonquasiunipotent subgroups of G^0, with $\mathcal{G}(F) = \{F^+, F^-\}$ for $F \in \mathcal{F}$). Now pick any (nonquasiunipotent) $g \in G$; since $G^0\Gamma$ is of finite index in G, one can find $r \in \mathbb{N}$ such that $g^r \in \bigcap_{h\in G} h^{-1}G^0\Gamma h$, then g^r leaves invariant every connected component Ω_j of Ω. Arguing as in the proof of Theorem 5.4 and using Lemma 5.1(a), one can reduce the problem to studying the action of g^r on Ω^0.

Write g^r in the form $h\gamma$, where $h \in G^0$ and $\gamma \in \Gamma$. For any $s \in \mathbb{N}$ and any $g^0 \in G^0$, multiplication by g^{rs} sends $g^0\Gamma^0$ to $g^{rs}g^0\gamma^{-s}\Gamma^0 = \tilde{h}\gamma^s g^0\gamma^{-s}\Gamma^0$, where $\tilde{h}$ is an element of G^0 dependent on s. Since G^0 is a connected semisimple group assumed to be centerfree (see Remark 1.2), the group of its inner automorphisms has finite index in $\operatorname{Aut} G^0$. Thus for some $s \in \mathbb{N}$, $\gamma^s = az$, where $a \in G^0$ and z

lies in the centralizer of G^0 in G. For this s, multiplication by g^{rs} sends $g^0\Gamma^0$ to $\tilde{h}ag^0a^{-1}\Gamma^0$.

Observe now that $a^{-1} = \gamma^{-s}z$ lies in the normalizer of Γ^0 in G^0, and $\tilde{h}a = g^{rs}z^{-1}$ is a nonquasiunipotent element of G^0. Therefore Theorem 5.6 implies that for any closed null subset Z of Ω^0 which is invariant under the action of either g or g^{-1}, $Z \cup \{\infty\}$ is $\{g^{rsi} \mid i \in \mathbb{Z}\}$-escapable, and another application of Lemma 5.1(a) finishes the proof. □

The above proof shows that the statement of the main result (the analogue of Theorem 5.2 and Corollary 5.5) for the general case of a Lie group G and any nonquasiunipotent cyclic subgroup F of G would be more complicated. However, one can modify the proof of Theorem 5.2 to show that $\{\infty\}$ is F-escapable for any (one-parameter or cyclic) F which consists of semisimple elements.

5.8. Let G be semisimple without compact factors, and Γ irreducible and nonuniform (that is, the homogeneous space Ω noncompact). Then from Theorem 1.1 and the ergodicity of F-action on Ω it follows that for a dense subset Ω_F of Ω the closure of the orbit Fx has Haar measure 0 for any $x \in \Omega_F$. Applying Theorem 1.1 with $Z = \bigcup_{j=1}^{l} \overline{Fx_j}$, where $x_1, \dots, x_l$ are from Ω_F, one can estimate the dimension of the set of points whose orbits are bounded and stay away from any given finite subset of this dense set.

Theorem. *Let G, Ω, F and Z be as in Theorem 1.1; assume that Ω is not compact. Then there exist a set Ω_F satisfying $\dim(W \cap \Omega_F) = n$ for any nonempty open subset W of Ω, such that for any l points $x_1, \dots, x_l \in \Omega_F$, $l \in \mathbb{N}$, the set $Z \cup \{x_1, \dots, x_l, \infty\}$ is F-escapable.*

Conjecture (B) in [Mrg2] states that the above (with $Z = \varnothing$) should be true without any restrictions on G, Ω and the choice of the points x_j; this is going to be the topic of a forthcoming paper. Let us note that related problems were considered in [Dan5] for endomorphisms of tori and in [U] for Anosov flows on compact Riemannian manifolds.

5.9. A special case of the flows studied above is the geodesic flow on the unit tangent bundle SM of a Riemannian manifold M of constant negative curvature and finite Riemannian volume. Since the ambient group in this case has $\mathbb{R}$-rank 1, the fact that the set of points in SM with bounded geodesics has full Hausdorff dimension follows from the result of Dani [Dan3] (see also [AL] for a strengthening of this result). In [Dan3] it was asked whether an analogous statement would be true for manifolds of variable negative curvature. A certain progress in this direction has been recently obtained by D. Dolgopiat [Do].

It is worthwhile to note that in [Do] the case of infinite volume is also considered, the goal being to prove that the set of points in SM with bounded geodesics has the same Hausdorff dimension as the set of points with geodesics returning to some compact subset of SM infinitely often. In the case when M is a rank-1 locally symmetric space this is done in [BJ] and [Do]. It would be certainly interesting to know whether a similar assertion holds for flows on homogeneous spaces of higher rank semisimple Lie groups.

Appendix

A.1. The first estimates on the decay of matrix coefficients of $\mathcal{K}$-finite vectors ($\mathcal{K}$ a maximal compact subgroup of a semisimple Lie group G) appeared in the work of Harish-Chandra, and then were refined by several people. Katok and Spatzier [KS] used Howe's estimates [H] for the $\mathcal{K}$-finite case, as well as Cowling's results [Cow] on strongly L^p representations, to prove their exponential estimate (cf. Theorem 2.4.3 above) for C^∞-vectors. Here we will use their treatment of C^∞-case to show exponential decay of matrix coefficients of Hölder vectors. This was done earlier for certain special cases by C.C. Moore [Mo2] and M. Ratner [Rat1].

Let G be a Lie group with left Haar measure μ, ρ a unitary representation of G on a Hilbert space $\mathcal{H}$. Say that a vector $v \in \mathcal{H}$ is *Hölder* with exponent $\alpha > 0$ if

$$C = \sup_{g \in G - \{e\}} \frac{\|\rho(g)v - v\|}{\operatorname{dist}(e,g)^\alpha} < \infty\,;$$

we will refer to the number C as to the *α-Hölder coefficient* of v, and say that the vector v is (C,α)*-Hölder*, or *α-Hölder*, if the coefficient C is irrelevant.

A.2. Remark. For $r > 0$, denote by $B(r)$ the open ball in G of radius r centered in e. The representation ρ can be extended to the Banach algebra $L^1(G)$: for $\varphi \in L^1(G)$ and $v \in \mathcal{H}$, $\rho(\varphi)v \stackrel{\text{def}}{=} \int_G \varphi(g)\rho(g)v\,d\mu$. Clearly the norm of $\rho(\varphi)$ is not greater than the L^1-norm of φ: for any $v \in \mathcal{H}$, $\|\rho(\varphi)v\| \le \|\varphi\|_{L^1}\|v\|$. It is also clear that if v is (C,α)-Hölder and φ is a.e. nonnegative with $\|\varphi\|_{L^1} = 1$ and $\operatorname{supp}(\varphi) \subset B(r)$, then $\|\rho(\varphi)v - v\| \le Cr^\alpha$.

A.3. Let $\mathfrak{k}$ be the Lie algebra of a maximal compact subgroup $\mathcal{K}$ of G, and, as in Section 2.4, denote by Υ the element $1 - \sum Y_j^2$, where $\{Y_j\}$ is an orthonormal basis of $\mathfrak{k}$. We will need the following simple

Lemma. (cf. Lemma 2.4.7(b)) *There exists a constant A_1 dependent only on G such that if $0 < r < 1$ and $l \in \mathbb{N}$, one can find a nonnegative function $\varphi \in C^\infty_{comp}(G)$ with* $\operatorname{supp}(\varphi) \subset B(r)$, $\int_G \varphi\,d\mu = 1$, *and* $\|\Upsilon^l(\varphi)\|_{L^1} \le A_1 r^{-2l}$.

A.4. Theorem. *Let G, $\mathfrak{a}$, $\mathfrak{c}$, ϑ and Π be as in Theorem 2.4.3, and let v_i, $i = 1,2$, be (C_i,α_i)-Hölder vectors in the representation space of $\rho \in \Pi$. Then for any $Y \in \bar{\mathfrak{c}}$ and for any $t \ge 0$*

$$|\big(\rho\big(\exp(tY)\big)v_1, v_2\big)| \le \big(A_1^2 B\|v_1\|\|v_2\| + C_1\|v_2\| + C_2\|v_1\| + C_1C_2\big)e^{-\frac{t\xi}{2p}\vartheta(Y)}\,,$$

where $\xi = \left(1 + \frac{2l}{\alpha_1} + \frac{2l}{\alpha_2}\right)^{-1}$, B, p and l are from Theorem 2.4.3, and A_1 is from Lemma A.3.

Proof. Take $a_1 > 0$, $a_2 > 0$, $r_1 = e^{-\frac{ta_1}{2p}\vartheta(Y)}$, $r_2 = e^{-\frac{ta_2}{2p}\vartheta(Y)}$. Using Lemma A.3, find nonnegative C^∞-functions φ_1, φ_2 on G with $\int_G \varphi_i\,d\mu = 1$, $\operatorname{supp}(\varphi_i) \subset B(r_i)$ and

$$\|\Upsilon^l(\varphi_i)\|_{L^1} \le A_1 r_i^{-2l} \tag{A.1}$$

$(i = 1, 2)$. Clearly $\rho(\varphi_1)v_1$ and $\rho(\varphi_2)v_2$ are C^∞-vectors, and (see Remark A.2)

$$\|\rho(\varphi_i)v_i - v_i\| \le C_i r_i^{\alpha_i} \tag{A.2}$$

$(i = 1, 2)$.

We now apply Theorem 2.4.3 to the vectors $\rho(\varphi_1)v_1$ and $\rho(\varphi_2)v_2$:

$$\begin{aligned}
|\big(\rho\big(\exp(tY)\big)\rho(\varphi_1)v_1, \rho(\varphi_2)v_2\big)| &\le Be^{-\frac{t}{2p}\vartheta(Y)}\|\Upsilon^l\big(\rho(\varphi_1)v_1\big)\|\|\Upsilon^l\big(\rho(\varphi_2)v_2\big)\| \\
&= Be^{-\frac{t}{2p}\vartheta(Y)}\|\rho\big(\Upsilon^l(\varphi_1)\big)v_1\|\|\rho\big(\Upsilon^l(\varphi_2)\big)v_2\| \\
\text{(see Remark A.2)} \quad &\le Be^{-\frac{t}{2p}\vartheta(Y)}\|\Upsilon^l(\varphi_1)\|_{L^1}\|\Upsilon^l(\varphi_2)\|_{L^1}\|v_1\|\|v_2\| \\
\text{(by (A.1))} \quad &\le A_1^2 Be^{-\frac{t}{2p}\vartheta(Y)} r_1^{-2l} r_2^{-2l}\|v_1\|\|v_2\| \\
\text{(by definition of } r_1 \text{ and } r_2) \quad &\le A_1^2 Be^{-\frac{t}{2p}\vartheta(Y)(1-2la_1-2la_2)}\|v_1\|\|v_2\|\,.
\end{aligned}$$

From (A.2) and the unitarity of ρ it follows that

$$\begin{aligned}
|\big(\rho\big(\exp(tY)\big)\big(\rho(\varphi_1)v_1 - v_1\big), \rho(\varphi_2)v_2\big)| &\le C_1 r_1^{\alpha_1}\|v_2\| = C_1\|v_2\|e^{-\frac{t}{2p}\vartheta(Y)a_1\alpha_1}\,, \\
|\big(\rho\big(\exp(tY)\big)v_1, \rho(\varphi_2)v_2 - v_2\big)| &\le C_2 r_2^{\alpha_2}\|v_1\| = C_2\|v_1\|e^{-\frac{t}{2p}\vartheta(Y)a_2\alpha_2}\,, \\
|\big(\rho\big(\exp(tY)\big)\big(\rho(\varphi_1)v_1 - v_1\big), \rho(\varphi_2)v_2 - v_2\big)| &\le C_1C_2 r_1^{\alpha_1} r_2^{\alpha_2} = C_1C_2 e^{-\frac{t}{2p}\vartheta(Y)(a_1\alpha_1 + a_2\alpha_2)}\,.
\end{aligned}$$

Therefore

(A.3)

$$\begin{aligned}
&|\big(\rho\big(\exp(tY)\big)v_1, v_2\big)| \le |\big(\rho\big(\exp(tY)\big)\rho(\varphi_1)v_1, \rho(\varphi_2)v_2\big)| + |\big(\rho\big(\exp(tY)\big)v_1, \rho(\varphi_2)v_2 - v_2\big)| \\
&+|\big(\rho\big(\exp(tY)\big)\big(\rho(\varphi_1)v_1 - v_1\big), \rho(\varphi_2)v_2\big)| + |\big(\rho\big(\exp(tY)\big)\big(\rho(\varphi_1)v_1 - v_1\big), \rho(\varphi_2)v_2 - v_2\big)| \\
&\qquad \le A_1^2 Be^{-\frac{t}{2p}\vartheta(Y)(1-2la_1-2la_2)}\|v_1\|\|v_2\| + C_2\|v_1\|e^{-\frac{t}{2p}\vartheta(Y)a_2\alpha_2} \\
&\qquad + C_1\|v_2\|e^{-\frac{t}{2p}\vartheta(Y)a_1\alpha_1} + C_1C_2e^{-\frac{t}{2p}\vartheta(Y)(a_1\alpha_1+a_2\alpha_2)}\,.
\end{aligned}$$

Now choose a_1 and a_2 such that

$$\alpha_1 a_1 = \alpha_2 a_2 = 1 - 2la_1 - 2la_2 = \xi = \left(1 + \frac{2l}{\alpha_1} + \frac{2l}{\alpha_2}\right)^{-1}\,.$$

Then from (A.3) one gets

$$\begin{aligned}
|\big(\rho\big(\exp(tY)\big)v_1, v_2\big)| &\le \big(A_1^2 B\|v_1\|\|v_2\| + C_1\|v_2\| + C_2\|v_1\|\big)e^{-\frac{t\xi}{2p}\vartheta(Y)} + C_1C_2e^{-\frac{2t\xi}{2p}\vartheta(Y)} \\
&\le \big(A_1^2 B\|v_1\|\|v_2\| + C_1\|v_2\| + C_2\|v_1\| + C_1C_2\big)e^{-\frac{t\xi}{2p}\vartheta(Y)}\,,
\end{aligned}$$

which is exactly the desired statement. □

A.5. Let now G, Γ and Ω be as in Theorem 1.1, and let $\{g_t \mid t \in \mathbb{R}\}$ be a one-parameter subgroup of G such that the condition (EM) is satisfied.

Theorem. *There exists a constant $A_2 > 0$, dependent only on G, such that if $\psi_i \in L^2_0(\Omega)$, $i = 1, 2$, are (C_i, α_i)-Hölder, then for any $t \geq 0$*

$$\left|(g_t\psi_1, \psi_2)\right| \leq \left(A_2 E\|\psi_1\|\|\psi_2\| + C_1\|\psi_2\| + C_2\|\psi_1\| + C_1C_2\right)e^{-\gamma\eta t},$$

where γ, E and l are from Theorem 2.4.5, and $\eta = \left(1 + \frac{l}{\alpha_1} + \frac{l}{\alpha_2}\right)^{-1}$.

The proof is based on the same kind of argument that was used in the proofs of Theorems 2.4.5 and A.4. (Instead of functions φ_1, φ_2 on G one should consider C^∞ functions supported in suitable neighborhoods of identity in the group G^{e}.)

A.6. Say that a function on a metric space X with distance "dist" is *α-Hölder*, if

$$\sup_{x,y\in X,\, x\neq y} \frac{|f(x) - f(y)|}{\mathrm{dist}(x,y)^\alpha} < \infty\,.$$

Clearly if Γ is a closed subgroup of a Lie group G, then any Hölder square-integrable function on G/Γ is a Hölder vector of a regular representation of G on $L^2(G/\Gamma)$.

The following Hölder analogue of Proposition 2.4.8 can be proved:

Proposition. *Let the condition* (EM) *be satisfied. Then for any α_1-Hölder compactly supported function $f \in L^2(H)$, for any α_2-Hölder function $\psi \in L^2(\Omega)$ and for any compact subset L of Ω, there exists a constant $C = C(f, \psi, L)$ such that for all $x \in L$ and for any $t \geq 0$*

$$\left|e^{-\chi t}\int_H f(g_t h g_{-t})\psi(hg_t x)\, dm(h) - \int_H f\, dm \int_\Omega \psi\, d\bar{\mu}\right| \leq Ce^{-\zeta t},$$

where

$$\zeta = \gamma\left(1 + \frac{l}{\alpha_1} + \frac{n-k+3l}{\alpha_2} + \frac{3(n-k)l}{\alpha_1\alpha_2} + \frac{(n-k)l}{\alpha_2^2}\right)^{-1},$$

with γ and l from Theorem 2.4.5.

An indication for the proof is given in the next section, where a more general theorem is considered. Note that one of the reasons why the Hölder setting for this situation seems to be natural is that the constant $d(\psi)$ which appears in the proof of Proposition 2.4.8 is nothing but the Hölder coefficient of ψ corresponding to the exponent 1.

A.7. Finally, we note that similar methods can be applied to the situation of Section 2.1.3, where we considered integrals of type $\int_M f(y)\psi\big(g_t\pi(y)\big)\,\omega_\pi(y)$ (M being a k-dimensional manifold and π an immersion $M \to \Omega$ with image transversal to the $\tilde{H}$-orbit foliation at any point). More generally, one can prove the following generalization of the above proposition:

Theorem. *Let M be a smooth k-dimensional Riemannian manifold, and let $\{\pi_q \mid q \in Q\}$ be a compact family of C^∞ immersions $M \to \Omega$ such that $\pi_q(M)$ is transversal to the orbit $\tilde{H}\pi(y)$ for every $q \in Q$ and $y \in M$. Denote by ω_q the pullback of the form ω_H to M via π_q. Further, let $\{f_q \mid q \in Q\}$ be a family of functions on M with common compact support M_0 such that f_q are uniformly α_1-Hölder for some $\alpha_1 > 0$, that is*

$$\sup_{q\in Q,\, x,y\in M,\, x\neq y} \frac{|f_q(x) - f_q(y)|}{\operatorname{dist}(x,y)^{\alpha_1}} < \infty\,.$$

Assume that the condition (EM) *is satisfied. Then for any α_2-Hölder function $\psi \in L^2(\Omega)$ there exists a constant $C = C(\{\pi_q\},\{f_q\},\psi)$ such that for all $q \in Q$ and $t \geq 0$*

$$\Big|\int_M f_q(y)\psi\big(g_t\pi_q(y)\big)\,\omega_q(y) - \int_M f_q\,\omega_q \int_\Omega \psi\, d\bar{\mu}\Big| \leq Ce^{-\zeta t}\,, \tag{A.4}$$

where ζ is as in Proposition A.6.

Sketch of proof. Using a smooth partition of unity on M_0, one can assume that all the immersions π_q are injective when restricted to M_0. Similarly to what was done in the proof of Proposition 2.4.8, for large enough t choose $r = e^{-at}$ and α_1-Hölder nonnegative functions f^0 on H^0 and f^- on H^- satisfying

$$\int_{H^-} f^-\,dm^- = 1 = \int_{H^0} f^0\,dm^0 \quad \text{and} \quad \operatorname{supp}(f^-)\cdot\operatorname{supp}(f^0) \subset \tilde{B}(r^2)\,,$$

and such that the Hölder coefficient of $f^-\cdot f^0$ is bounded from above by $\mathrm{const}\cdot r^{-2(n-k+\alpha_1)}$. Then, using f^-, f^0 and f_q, build an α_1-Hölder function on Ω, and apply the averaging trick (with radii $r_1 = e^{-a_1\gamma t}$ and $r_2 = e^{-a_2\gamma t}$) from the proof of Theorem A.4 to the latter function and ψ to get an upper estimate for the left hand side of (A.4), which can be then minimized by choosing appropriate values of the parameters a, a_1 and a_2. The critical values satisfy the following system of equations:

$$\zeta = \alpha_2 a = \gamma(1 - la_1 - la_2) - (n-k)a = \gamma\alpha_2 a_2 - (n-k)a = \gamma\alpha_1 a_1 - 2(n-k+\alpha_1)a\,,$$

which yields the value of ζ as in the Proposition A.6. □

A.8. Corollary. *Let, as before, $\mathcal{K}$ stand for a maximal compact subgroup of G, and let ν be a normalized Haar measure on $\mathcal{K}$. Let $\{g_t\}$ be a one-parameter subgroup of a maximal split torus of G such that the flow $(\Omega,\{g_t\})$ has property* (EM). *Then for any α-Hölder function ψ in $L^2(\Omega)$ the spherical averages of ψ converge exponentially to $\int_\Omega \psi\, d\bar{\mu}$. More precisely, for any compact subset L of Ω there is a constant $C = C(\psi, L)$ such that for all $t \geq 0$*

$$\Big|\int_{\mathcal{K}} \psi(g_t h x)\,d\nu(h) - \int_\Omega \psi\,d\bar{\mu}\Big| \leq Ce^{-\zeta_1 t}\,, \tag{A.5}$$

where

$$\zeta_1 = \gamma\left(1 + l + \frac{n-k+3l+3(n-k)l}{\alpha} + \frac{(n-k)l}{\alpha^2}\right)^{-1}$$

is the value of ζ from Proposition A.6 corresponding to $\alpha_1 = 1$ and $\alpha_2 = \alpha$.

Sketch of proof. Consider a smooth finite partition of unity $\{f_j\}$ on $(\mathcal{K} \cap H^0)\backslash\mathcal{K}$ subordinate to a covering $\{U_j\}$ such that the bundle $\mathcal{K} \to (\mathcal{K} \cap H^0)\backslash\mathcal{K}$ is trivial over each of U_j. For each j choose a smooth section $\varphi_j : U_j \to \mathcal{K}$, then for any $x \in \Omega$, the set $\varphi(U_j)x$ will be transversal to the orbit $\tilde{H}x$. Thus one can apply Theorem A.7 to f_j, the immersions $\{\pi_x \circ \varphi_j \mid x \in L\}$ and to the function obtained from ψ by averaging over the translations by elements of $\mathcal{K} \cap H^0$. □

Note that under the assumption that the flow (Ω, F) is semisimple, one can prove in the above corollary (resp. Theorem A.7) that the left hand side of (A.5) (resp. (A.4)) tends to zero as $t \to \infty$.

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References

[AL] C. S. Aravinda and F. E. Leuzinger, *Bounded geodesics in rank-1 locally symmetric spaces*, Ergodic Theory Dynamical Systems (to appear).

[Bou] N. Bourbaki, *Éléments de mathematique*, Livre VI: Integration, Chapitre 7: Mesure de Haar, Chapitre 8: Convolution et representations, Hermann, Paris, 1963.

[Bow] R. Bowen, *Weak mixing and unique ergodicity on homogeneous spaces*, Israel J. Math. **23** (1976), 267–273.

[BJ] C. Bishop and P. W. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math (to appear).

[Cow] M. Cowling, *Sur le coefficients des représentations unitaries des groupes de Lie simple*, Lecture Notes in Math., vol. 739, Springer-Verlag, Berlin and New York, 1979, pp. 132–178.

[CG] L. J. Corwin and F. P. Greenleaf, *Representations of nilpotent Lie groups and their applications.* Part I, Cambridge Stud. Adv. Math., vol. 18, Cambridge Univ. Press, Cambridge, 1990.

[Dan1] S. G. Dani, *Invariant measures of horospherical flows on noncompact homogeneous spaces*, Invent. Math. **47** (1978), 101–138.

[Dan2] ———, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. Reine Angew. Math. **359** (1985), 55–89.

[Dan3] ———, *Bounded orbits of flows on homogeneous spaces*, Comment. Math. Helv. **61** (1986), 636–660.

[Dan4] ———, *On orbits of unipotent flows on homogeneous spaces,* II, Ergodic Theory Dynamical Systems **6** (1986), 167–182.

[Dan5] ———, *On orbits of endomorphisms of tori and the Schmidt game*, Ergodic Theory Dynamical Systems **8** (1988), 523–529.

[Dav] R. O. Davies, *Subsets of finite measure in analytic sets*, Indag. Math. **14** (1952), 488–489.

[DM] S. G. Dani and G. A. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, Adv. in Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 91–137.

[Do] D. Dolgopiat, *Bounded orbits of Anosov flows*, submitted to Duke Math. J.

[EM] A. Eskin and C. McMullen, *Mixing, counting and equidistribution in Lie groups*, Duke Math. J. **71** (1993), 181–209.

[EP] R. Ellis and W. Perrizo, *Unique ergodicity of flows on homogeneous spaces*, Israel J. Math. **29** (1978), 276–284.

[F] K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Math., vol. 85, Cambridge Univ. Press, Cambridge and New York, 1986.

[H] R. Howe, *A notion of rank for unitary representations of the classical groups*, Harmonic Analysis and Group Representations, Liguori, Naples, 1982, pp. 223–331.

[KS] A. Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank Abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 131–156.

[Mag] W. Magnus, *Noneuclidean tesselations and their groups*, Academic Press, New York and London, 1974.

[Mc] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), 329–342.

[Mo1] C. C. Moore, *Ergodicity of flows on homogeneous spaces*, Amer. J. Math. **88** (1966), 154–178.

[Mo2] ———, *Exponential decay of correlation coefficients for geodesic flows*, Group Representations, Ergodic Theory, Operator Algebras and Mathematical Physics, Math. Sci. Res. Inst. Publ., vol. 6, Springer-Verlag, Berlin and New York, 1987, pp. 163–181.

[Mrc] B. Marcus, *Unique ergodicity of the horocycle flow: variable negative curvature case*, Israel J. Math. **21** (1975), 133–144.

[Mrg1] G. A. Margulis, *On some problems in the theory of U-systems*, Ph.D. Thesis, Moscow State University, 1970. (Russian)

[Mrg2] ———, *Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 193–215.

[Mrs] J. M. Marstrand, *The dimension of Cartesian product sets*, Proc. Camb. Phil. Soc. **50** (1954), 198–202.

[PW] Ya. Pesin and H. Weiss, *On the dimension of deterministic and random Cantor-like sets*, Math. Res. Lett. **1** (1994), 519–529.

[Rag] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, Berlin and New York, 1972.

[Rat1] M. Ratner, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory Dynamical Systems **7** (1987), 267–288.

[Rat2] ———, *Raghunathan's topological conjecture and distribution of unipotent flows*, Duke Math. J. **63** (1991), 235–280.

[S1] W. M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. **123** (1966), 178–199.

[S2] ———, *Badly approximable systems of linear forms*, J. Number Theory **1** (1969), 139–154.

[U] M. Urbanski, *The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system*, Nonlinearity **2** (1991), 385–397.

[Weg] H. Wegmann, *Die Hausdorff-Dimension von kartesischen Produktmengen in metrischen Räumen*, J. Reine Angew. Math. **234** (1969), 163–171.

D. Y. Kleinbock, Department of Mathematics, Yale University, New Haven, CT 06520

E-mail address: `kleinboc@math.yale.edu`

G. A. Margulis, Department of Mathematics, Yale University, New Haven, CT 06520

E-mail address: `margulis@math.yale.edu`