SCHMIDT'S GAME, FRACTALS, AND ORBITS OF TORAL ENDOMORPHISMS

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ABSTRACT. Given an integer matrix $M \in \mathrm{GL}_n(\mathbb{R})$ and a point $y \in \mathbb{R}^n/\mathbb{Z}^n$, consider the set

$$\tilde{E}(M,y) \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^n : y \notin \overline{\{M^k\mathbf{x} \bmod \mathbb{Z}^n : k \in \mathbb{N}\}} \right\} .$$

S.G. Dani showed in 1988 that whenever M is semisimple and $y \in \mathbb{Q}^n/\mathbb{Z}^n$, the set $\tilde{E}(M,y)$ is winning in the sense of W. Schmidt (a property implying density and full Hausdorff dimension). In this paper we strengthen this result, extending it to arbitrary $y \in \mathbb{R}^n/\mathbb{Z}^n$ and $M \in \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{M}_{n\times n}(\mathbb{Z})$, and in fact replacing the sequence of powers of M by any lacunary sequence of (not necessarily integer) $m \times n$ matrices. Furthermore, we show that sets of the form $\tilde{E}(M,y)$ and their generalizations always intersect with 'sufficiently regular' fractal subsets of $\mathbb{R}^n$. As an application we strengthen recent results of [2, 22, 32] on badly approximable systems of affine forms.

1. Introduction

Let $\mathbb{T}^n \stackrel{\text{def}}{=} \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Any non-singular $n \times n$ matrix M with integer entries defines a continuous surjective endomorphism f_M of $\mathbb{T}^n$ given by

$$f_M(\mathbf{x} + \mathbb{Z}^n) \stackrel{\text{def}}{=} M\mathbf{x} + \mathbb{Z}^n \quad \forall\, \mathbf{x} \in \mathbb{R}^n ,$$

and any continuous surjective endomoprhism f of $\mathbb{T}^n$ can be obtained this way. Criteria for ergodicity of f (with respect to Haar measure on $\mathbb{T}^n$) are well known, and ergodicity implies that f-orbits of almost all points are dense in $\mathbb{T}^n$. Also in many cases it is known that exceptional sets of points with non-dense orbits are rather big. For example, following the notation used in [14], let us define

$$E(f,y) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{T}^n : y \notin \overline{\{f^k(x) : k \in \mathbb{N}\}} \right\} \tag{1.1}$$

for a fixed $y \in \mathbb{T}^n$ and a self-map f of $\mathbb{T}^n$. In 1988 Dani proved

Theorem 1.1. [5, Theorem 2.1] *For any semisimple $M \in \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{M}_{n\times n}(\mathbb{Z})$ and any $y \in \mathbb{Q}^n/\mathbb{Z}^n$, the set $E(f_M, y)$ is 1/2-winning.*

The above winning property is based on a game, introduced by Schmidt in [27], which is usually referred to as Schmidt's game. This property implies density and full Hausdorff dimension and is stable with respect to countable intersections; see §2 for more detail.

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One of the goals of the present paper is to prove a far-reaching generalization of Theorem 1.1. Namely, we remove the assumptions of M being semisimple and y being rational. Also we are able to intersect sets $E(f, y)$ with many 'sufficiently regular' fractal subsets of $\mathbb{T}^n$. In fact it will be more convenient to lift the problem to $\mathbb{R}^n$: denote by π the quotient map $\mathbb{R}^n \to \mathbb{T}^n$ and, for $M \in \mathrm{M}_{n\times n}(\mathbb{R})$ and $y \in \mathbb{T}^n$, consider

$$\tilde{E}(M, y) \stackrel{\mathrm{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^n : y \notin \overline{\{\pi(M^k \mathbf{x}) : k \in \mathbb{N}\}} \right\} . \tag{1.2}$$

Clearly $\tilde{E}(M, y) = \pi^{-1}\big(E(f_M, y)\big)$ when $M \in \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{M}_{n\times n}(\mathbb{Z})$; however the definition (1.2) makes sense even when M is singular or has non-integer entries.

In §3 we will describe the class of *absolutely friendly measures* on $\mathbb{R}^n$, following a terminology introduced in [16, 26]. Pushing this terminology a little further, we will say that a closed subset K of $\mathbb{R}^n$ is *absolutely friendly* if there exists an absolutely friendly measure μ on $\mathbb{R}^n$ such that $K = \operatorname{supp} \mu$. Lebesgue measure constitutes a trivial example; thus $\mathbb{R}^n$ itself is absolutely friendly. Other, more interesting examples can be found in [16, 30, 34]. For example, limit sets of irreducible families of contracting similarities of $\mathbb{R}^n$, such as the Koch snowflake or the Sierpinski carpet, are absolutely friendly.

It turns out, as was first observed in [12], that absolutely friendly sets provide excellent conditions for playing Schmidt's game. Namely, we will say, following [1], that a subset S of $\mathbb{R}^n$ is *α-winning on K* if $S\cap K$ is α-winning for Schmidt's game played on the metric space K with the metric induced from $\mathbb{R}^n$. From [27] it immediately follows that the intersection of countably many sets α-winning on K is also α-winning on K. We will say that S is *winning on K* if it is α-winning on K for some $\alpha > 0$. Precise definitions are given in §2. It can be shown, see Proposition 3.3 below, that for any absolutely friendly K there exists $\gamma > 0$ such that the Hausdorff dimension of $S \cap K \cap U$ is not less than γ whenever S is winning on K and U is an open set intersecting K. Furthermore, one can often take γ to be equal to the Hausdorff dimension of K, for example when $K = \mathbb{R}^n$ or one of the self-similar sets mentioned above. Here is a generalization of Theorem 1.1:

Theorem 1.2. *For every absolutely friendly $K \subset \mathbb{R}^n$ there exists a positive $\alpha = \alpha(K)$ such that for any $M \in \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{M}_{n\times n}(\mathbb{Z})$ and any $y \in \mathbb{T}^n$, the set $\tilde{E}(M, y)$ is α-winning on K.*

In particular, for any *countable* subset Y of $\mathbb{T}^n$, the set

$$\bigcap_{y\in Y} \ \bigcap_{M\in \mathrm{GL}_n(\mathbb{R})\cap \mathrm{M}_{n\times n}(\mathbb{Z})} \tilde{E}(M, y)$$

intersects with every absolutely friendly K in a set of positive (and often full) Hausdorff dimension. It immediately follows that sets $E(f_M, y)$ discussed in Theorem 1.1 and their countable intersections always intersect those subsets of the torus whose pullbacks to $\mathbb{R}^n$ support absolutely friendly measures. It can also be shown that $\alpha(\mathbb{R}^n) = 1/2$, recovering Dani's result.

The one-dimensional case of Theorem 1.2, that is, a result describing orbits of maps

$$f_b : \mathbb{T} \to \mathbb{T}, \quad x \mapsto bx \bmod 1\,,$$

where b is an integer, appeared recently in [1], and also[1], independently and for $K = \mathbb{R}$, in [10]. In other words, the sets

$$\tilde{E}(b, y) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R} : y \notin \overline{\{\pi(b^k x) : k \in \mathbb{N}\}}\right\}. \tag{1.3}$$

are shown to be winning on K for any absolutely friendly $K \subset \mathbb{R}$, any integer $b > 1$ and any $y \in \mathbb{T}$. However, the main result of [1] applies to much more general situations. Namely, b in (1.3) does not have to be an integer, and one can replace the sequence of powers of b by an arbitrary lacunary sequence t_k of real numbers; see also earlier results [6, 24, 25] due to Pollington and de Mathan. (We recall that (t_k) is called *lacunary* if $\inf_{k\in\mathbb{N}} \frac{t_{k+1}}{t_k} > 1$.) Furthermore, instead of $y \in \mathbb{T}$ in (1.3) one can consider an arbitrary sequence of points (y_k), and the condition $y \notin \overline{\{\pi(b^k x)\}}$ can be replaced by

$$\inf_{k\in\mathbb{N}} d\big(\pi(t_k x), y_k\big) > 0 \iff \inf_{k\in\mathbb{N}} d\big(t_k x, \pi^{-1}(y_k)\big) > 0$$

(here and hereafter $d(\cdot,\cdot)$ stands for the Euclidean distance on $\mathbb{R}^n$ and the induced distance on $\mathbb{T}^n$).

We now describe an analogous generalization of Theorem 1.2, which is the main result of the present paper. We are going to fix $m, n \in \mathbb{N}$, consider a sequence $\mathcal{M} = (M_k)$ of $m \times n$ matrices and a sequence $\mathcal{Z} = (Z_k)$ of subsets of $\mathbb{R}^m$, and define

$$\tilde{E}(\mathcal{M}, \mathcal{Z}) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : \inf_{k\in\mathbb{N}} d(M_k\mathbf{x}, Z_k) > 0\}\,. \tag{1.4}$$

The sets $\tilde{E}(M, y)$ defined in (1.2) constitute a special case, with $m = n$, $\mathcal{M} = (M^k)$ and $Z_k = \pi^{-1}(y)$.

Some assumptions on $\mathcal{M}$ and $\mathcal{Z}$ are in order. We will say that a sequence $\mathcal{M}$ of nonzero $m \times n$ matrices is *lacunary* if so is the sequence $(\|M_k\|_{op})$ of the values of their operator norms. A subset Z of $\mathbb{R}^n$ will be called *δ-uniformly discrete* if $\inf_{\mathbf{x},\mathbf{y}\in Z, x\neq y} d(\mathbf{x}, \mathbf{y}) > \delta$. With some abuse of terminology, we say that a sequence $\mathcal{Z} = (Z_k)$ is δ-uniformly discrete if Z_k is δ-uniformly discrete for every $k \in \mathbb{N}$, and that $\mathcal{Z}$ is *uniformly discrete* if it is δ-uniformly discrete for some $\delta > 0$. For example, for an arbitrary sequence (y_k) of points of $\mathbb{T}^m$, the sequence of sets $Z_k = \pi^{-1}(y_k) \subset \mathbb{R}^m$ is 1-uniformly discrete.

We can now formulate our main result.

Theorem 1.3. *For every absolutely friendly $K \subset \mathbb{R}^n$ there exists a positive $\alpha = \alpha(K)$ such that if $\mathcal{Z}$ is a uniformly discrete sequence of subsets of $\mathbb{R}^m$ and $\mathcal{M}$ is a lacunary sequence of $m \times n$ matrices with real entries, then $\tilde{E}(\mathcal{M}, \mathcal{Z})$ is α-winning on K.*

[1] Earlier in [31] it was proved that sets $E(f_b, y)$ are α-winning without a bound on α uniform in b; see [1] for other references.

An important special case is $m = n$ and $\mathcal{M} = (M^k)$, where M is an $n\times n$ matrix with operator norm strictly greater than 1 (not necessarily invertible and not necessarily with integer entries); this is used to derive Theorem 1.2 from Theorem 1.3, see §4. In fact we are going to prove a more precise result (see Lemma 3.4 and Theorem 4.1), showing how α depends on constants C, γ, D involved in the definition of absolutely friendly sets and measures. Note that the paper [2] studied a special case when

$$\begin{gathered}\mathcal{M} \text{ is a lacunary sequence of } 1\times n \text{ integer matrices}\\ \text{and } \mathcal{Z} = (Z_k),\ \text{where } Z_k = \mathbb{Z} = \pi^{-1}(0)\ \forall\, k \in \mathbb{N}\,.\end{gathered} \tag{1.5}$$

Under these assumptions it was proved in [2] that $\dim\big(\tilde{E}(\mathcal{M},\mathcal{Z}) \cap K\big) = \dim(K)$ whenever K satisfies certain conditions which are more restrictive than the absolute friendliness (see a remark after Definition 3.2 for more detail). Also [22, Lemma 1] establishes that in the case (1.5) sets $E(\mathcal{M},\mathcal{Z})$ are 1/2-winning; we explain in §5 that the method of [22] can be used to verify that in Theorem 1.3 $\alpha(\mathbb{R}^n)$ can also be taken to be equal to 1/2.

It was observed both in [2] and in [22] that the aforementioned results in the case (1.5) can be used to prove the abundance of badly approximable systems of affine forms; in fact this was the main application of those papers. Recall that a pair $(A, \mathbf{x})$, where $A \in \mathrm{M}_{n\times m}(\mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^n$, is said to be *badly approximable* if

$$\inf_{\mathbf{q}\in\mathbb{Z}^m\smallsetminus\{0\}} \|\mathbf{q}\|^{m/n} d(A\mathbf{q} - \mathbf{x}, \mathbb{Z}^n) > 0\,.$$

This is an inhomogeneous analog of the notion of badly approximable systems of linear forms, see [28, 29]. It was proved in [15] that the set $\mathbf{Bad}(n,m)$ of badly approximable pairs $(A, \mathbf{x})$ has full Hausdorff dimension. Then a much easier proof was found in [2], where, for fixed $A \in \mathrm{M}_{n\times m}(\mathbb{R})$, the sets

$$\mathbf{Bad}_A(n,m) = \big\{\mathbf{x} \in \mathbb{R}^n : (A,\mathbf{x}) \in \mathbf{Bad}(n,m)\big\}$$

were considered, and it was shown that $\dim\big(\mathbf{Bad}_A(n,m)\big) = n$ for any $A \in \mathrm{M}_{n\times m}(\mathbb{R})$. The latter result was strengthened by Tseng in the case $m = n = 1$: he proved [32] that $\mathbf{Bad}_a(1,1) \subset \mathbb{R}$ is 1/8-winning for any $a \in \mathbb{R}$, and announced that a similar result for arbitrary m, n is to appear in a forthcoming work of Einsiedler and Tseng [7]. Shortly thereafter, Moshchevitin concluded [22] that the sets $\mathbf{Bad}_A(n,m)$ are 1/2-winning for any m, n and any $A \in \mathrm{M}_{n\times m}(\mathbb{R})$. Our main theorem can be easily used to deduce the following common generalization of the results of [2] and [22]:

Corollary 1.4. *Le $K \subset \mathbb{R}^n$ be absolutely friendly and let α be as in Theorem 1.3. Then for any $A \in \mathrm{M}_{n\times m}(\mathbb{R})$, $\mathbf{Bad}_A(n,m)$ is α-winning on K.*

We derive this corollary in §4. At the end of the paper a remark is made explaining how all the results of this paper can be strengthened to replace 'winning' by 'strong winning', a property introduced recently in [10, 11, 21].

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2. Schmidt's game

In this section we describe the game, first introduced by Schmidt in [27]. Let (X, d) be a complete metric space. Consider $\Omega \stackrel{\text{def}}{=} X \times \mathbb{R}_+$, and define a partial ordering

$$(x_2, \rho_2) \le_s (x_1, \rho_1) \text{ if } \ \rho_2 + d(x_1, x_2) \le \rho_1.$$

We associate to each pair (x, ρ) a ball in (X, d) via the 'ball' function $B(\cdot)$ as in (2.1). Note that $(x_2, \rho_2) \le_s (x_1, \rho_1)$ clearly implies (but is not necessarily implied by) $B(x_2, \rho_2) \subset B(x_1, \rho_1)$. However the two conditions are equivalent when X is a Euclidean space.

Schmidt's game is played by two players, whom we will call Alice and Bob, following a convention used previously in [18, 1]. The two players are equipped with parameters α and β respectively, satisfying $0 < \alpha, \beta < 1$. Choose a subset S of X (a target set). The game starts with Bob picking $x_1 \in X$ and $\rho > 0$, hence specifying a pair $\omega_1 = (x_1, \rho)$. Alice and Bob then take turns choosing $\omega'_k = (x'_k, \rho'_k) \le_s \omega_k$ and $\omega_{k+1} = (x_{k+1}, \rho_{k+1}) \le_s \omega'_k$ respectively satisfying

$$\rho'_k = \alpha\rho_k \text{ and } \rho_{k+1} = \beta\rho'_k\,. \tag{2.1}$$

As the game is played on a complete metric space and the diameters of the nested balls

$$B(\omega_1) \supset B(\omega'_1) \supset \ldots \supset B(\omega_k) \supset B(\omega'_k) \supset \ldots$$

tend to zero as $k \to \infty$, the intersection of these balls is a point $x_\infty \in X$. Call Alice the winner if $x_\infty \in S$. Otherwise Bob is declared the winner. A strategy consists of specifications for a player's choices of centers for his or her balls given the opponent's previous moves.

If for certain α, β and a target set S Alice has a winning strategy, i.e., a strategy for winning the game regardless of how well Bob plays, we say that S is an (α, β)-*winning set*. If S and α are such that S is an (α, β)-winning set for all possible β's, we say that S is an α-*winning* set. Call a set *winning* if such an α exists.

Intuitively one expects winning sets to be large. Indeed, every such set is clearly dense in X; moreover, under some additional assumptions on the metric space winning sets can be proved to have positive, and even full, Hausdorff dimension. For example, the fact that a winning subset of $\mathbb{R}^n$ has Hausdorff dimension n is due to Schmidt [27, Corollary 2]. Another useful result of Schmidt [27, Theorem 2] states that the intersection of countably many α-winning sets is α-winning.

Schmidt himself used the machinery of the game he invented to prove that certain subsets of $\mathbb{R}$ or $\mathbb{R}^n$ are winning, and hence have full Hausdorff dimension. Now let K be a closed subset of X. Following an approach initially introduced in [12], we will say that a subset S of X is (α, β)-*winning on* K (resp., α-*winning on* K, *winning on* K) if $S \cap K$ is (α, β)-winning (resp., α-winning, winning) for Schmidt's game played on the metric space K with the metric induced from (X, d). In the present paper we let $X = \mathbb{R}^n$ and take K to be the support of an absolutely decaying measure (defined in the next section). In other words, since the metric is induced, playing the game on K amounts to choosing balls in $\mathbb{R}^n$ according to the rules of a game played on

$\mathbb{R}^n$, but with an additional constraint that the centers of all the balls lie in K. Since the first appearance of this approach in [12], where it was used to show that sufficiently regular fractals meet with a countable intersection of non-singular affine images of the set of badly approximable vectors in $\mathbb{R}^n$, it has been utilized in [13, 20], and most recently in [1], of which the present paper is a sequel and a generalization.

3. Absolutely friendly sets

We first describe the class of absolutely friendly measures, whose supports provide a hospitable playground for Schmidt's game. The class of friendly measures was first introduced in [16], while the term 'absolutely friendly' was coined in [26] where stronger assumptions on the measures were needed.

Definition 3.1. *Let μ be a locally finite Borel measure on $\mathbb{R}^n$ and let $C, \gamma, D > 0$. We say that μ is (C,γ)-absolutely decaying if there exists $\rho_0 > 0$ such that*

$$\mu\big(B(\mathbf{x},\rho)\cap\mathcal{L}^{(\varepsilon\rho)}\big) < C\varepsilon^\gamma\mu\big(B(\mathbf{x},\rho)\big) \quad \text{for any affine hyperplane } \mathcal{L}\subset\mathbb{R}^n \text{ and any } \mathbf{x}\in\operatorname{supp}\mu,\ 0<\rho<\rho_0,\ 0<\varepsilon<1\,. \tag{3.1}$$

(Here $\mathcal{L}^{(t)} \stackrel{\text{def}}{=} \{\mathbf{x}\in\mathbb{R}^n : d(\mathbf{x},\mathcal{L})\le t\}$ is the closed t-neighborhood of $\mathcal{L}$.)

We say that μ is D-Federer if there exists $\rho_0 > 0$ such that

$$\mu\big(B(\mathbf{x},2\rho)\big) < D\mu\big(B(\mathbf{x},\rho)\big) \quad \forall\,\mathbf{x}\in\operatorname{supp}\mu,\ \forall\,0<\rho<\rho_0\,. \tag{3.2}$$

A measure μ will be called (C,γ,D)-absolutely friendly if it is both (C,γ)-absolutely decaying and D-Federer, i.e. if there exists $\rho_0 > 0$ such that both (3.1) *and* (3.2) *hold.*

We will say that μ is *absolutely friendly* (resp. *absolutely decaying*, *Federer*) if it is (C,γ,D)-absolutely friendly (resp. (C,γ)-absolutely decaying, D-Federer) for some values of constants C,γ,D. If μ is (C,γ,D)-absolutely friendly, we will denote by $\rho_{C,\gamma,D}(\mu)$ the supremum of ρ_0 for which the two properties in Definition 3.1 hold.

Many examples of absolutely friendly measures can be found in [16, 17, 34, 30]. The Federer (also called doubling) condition is very well studied; it obviously holds when μ satisfies a *power law*, i.e. there exist positive δ, c_1, c_2, ρ_0 such that

$$c_1\rho^\delta \le \mu\big(B(\mathbf{x},\rho)\big) \le c_2\rho^\delta \quad \forall\,\mathbf{x}\in\operatorname{supp}\mu,\ \forall\,0<\rho<\rho_0\,. \tag{3.3}$$

Such measures are often referred to as *δ-Ahlfors regular*. However it is not hard to construct absolutely friendly measures not satisfying a power law, see [17] for an example. Also, when $n = 1$ the Federer property is implied by the absolute decay, which in its turn is implied by a power law (see [1] for a thorough discussion of equivalent definitions of absolute friendliness in the one-dimensional case). However these implications fail to hold in higher dimensions. In particular, the volume measures on smooth k-dimensional submanifolds of $\mathbb{R}^n$ obviously satisfy a k-power law but are not absolutely decaying unless $k = n$.

The goal of the current work, as well as in several earlier papers [17, 19, 12, 13, 20], is to use measures in order to construct points in their supports with prescribed (dynamical or Diophantine) properties. Our attention will therefore be focused on sets which support absolutely friendly measures, and it will be convenient to introduce the following terminology:

Definition 3.2. *Let K be a closed subset of $\mathbb{R}^n$ and let $C,\gamma,D>0$. Say that K is (C,γ,D)-absolutely friendly (resp., absolutely friendly) if there exists a (C,γ,D)-absolutely friendly (resp., absolutely friendly) measure μ on $\mathbb{R}^n$ with $K=\operatorname{supp}\mu$.*

For example, $\mathbb{R}^n$ itself is absolutely friendly, as well as any limit set of an irreducible family of contracting self-similar [16] or self-conformal [34] transformations of $\mathbb{R}^n$. More examples can be found in [17, 30]. Note that the paper [2] established full Hausdorff dimension of $\tilde{E}(\mathcal{M},\mathcal{Z})\cap K$ for $\mathcal{M},\mathcal{Z}$ as in (1.5) and under an assumption that $K\subset\mathbb{R}^n$ supports an absolutely decaying, δ-Ahlfors regular measure with $\delta>n-1$. It is not hard to show by an elementary covering argument that (3.3) with $\delta>n-1$ implies (3.1) with $\gamma=\delta-n+1$; hence the sets considered in [2] are absolutely friendly.

If $K\subset\mathbb{R}^n$ is absolutely friendly, playing Schmidt games on K has many advantages. In particular, sets which are winning on K must have positive Hausdorff dimension. Recall that the *lower pointwise dimension* of a measure μ at $\mathbf{x}\in\operatorname{supp}\mu$ is defined as

$$\underline{d}_\mu(\mathbf{x})\stackrel{\text{def}}{=}\liminf_{\rho\to 0}\frac{\log\mu(B(\mathbf{x},\rho))}{\log\rho}.$$

For an open U with $\mu(U)>0$ let

$$\underline{d}_\mu(U)\stackrel{\text{def}}{=}\inf_{\mathbf{x}\in\operatorname{supp}\mu\cap U}\underline{d}_\mu(\mathbf{x})\,. \tag{3.4}$$

It is well known, see e.g. [9, Proposition 4.9], that (3.4) constitutes a lower bound for the Hausdorff dimension of $\operatorname{supp}\mu\cap U$ (where this bound is sharp when μ satisfies a power law). It is also easy to see that $\underline{d}_\mu(\mathbf{x})\ge\gamma$ for every $\mathbf{x}\in\operatorname{supp}\mu$ whenever μ is (C,γ,D)-absolutely friendly: indeed, take $\rho<\rho_0<\rho_{C,\gamma,D}(\mu)$ and $\mathbf{x}\in\operatorname{supp}\mu$; then, letting $\varepsilon=\frac{\rho}{\rho_0}$ and noting that $B(\mathbf{x},\rho)\subset\mathcal{L}^{(\rho)}$ for some hyperplane $\mathcal{L}$, one has

$$\mu\big(B(\mathbf{x},\rho)\big)<C\left(\frac{\rho}{\rho_0}\right)^\gamma\mu\big(B(\mathbf{x},\rho_0)\big).$$

Thus, for $\rho<1$,

$$\frac{\log\mu\big(B(\mathbf{x},\rho)\big)}{\log\rho}\ge\gamma+\frac{\log C-\gamma\log\rho_0+\log\mu\big(B(\mathbf{x},\rho_0)\big)}{\log\rho},$$

and the claim follows.

The following proposition [18, Proposition 5.1] makes it possible to estimate the Hausdorff dimension of sets winning on supports of Federer measures:

Proposition 3.3. *Let K be the support of a Federer measure μ on $\mathbb{R}^n$,and let S be winning on K. Then for any open $U \subset \mathbb{R}^n$ with $\mu(U) > 0$ one has*

$$\dim(S \cap K \cap U) \geq \underline{d}_\mu(U)\,.$$

In particular, in the above proposition one can replace $\underline{d}_\mu(U)$ with γ if K is (C, γ, D)-absolutely friendly, sand with $\dim(K)$ if μ satisfies a power law. Note that this generalizes estimates for the Hausdorff dimension of winning sets due to Schmidt [27] for μ being Lebesgue on $\mathbb{R}^n$, and to Fishman [12, §5] for measures satisfying a power law.

The next lemma exhibits a crucial feature of absolutely friendly sets, namely the fact that while playing Schmidt's game on such a set, Alice can distance herself from hyperplanes 'efficiently'. This observation is the cornerstone of the proof of our main theorem. The argument is adapted from one in [22], where the case $K = \mathbb{R}^n$ is proved with $\alpha = \frac{1}{2}$ (see §5.1 for more detail).

Lemma 3.4. *For every $C, D, \gamma > 0$ and*

$$\alpha < \frac{1}{2}\left(\frac{1}{CD}\right)^{1/\gamma} \tag{3.5}$$

there exists $\varepsilon = \varepsilon(C, D, \gamma, \alpha) \in (0, 1)$ such that if K is the support of a (C, γ, D)-absolutely friendly measure μ on $\mathbb{R}^n$, $0 < \rho < \rho_{C,\gamma,D}(\mu)$, $\mathbf{x}_1 \in K$, $N \in \mathbb{N}$, and $\mathcal{L}_1, \ldots, \mathcal{L}_N$ are hyperplanes in $\mathbb{R}^n$, there exists $\mathbf{x}_2 \in K$ with

$$B(\mathbf{x}_2, \alpha\rho) \subset B(\mathbf{x}_1, \rho) \tag{3.6}$$

and

$$d(B(\mathbf{x}_2, \alpha\rho), \mathcal{L}_i) > \alpha\rho \quad \textit{for at least } \lceil \varepsilon N \rceil \textit{ of the hyperplanes } \mathcal{L}_i. \tag{3.7}$$

Proof. Let $A_i = B\big(\mathbf{x}_1, (1-\alpha)\rho\big) \smallsetminus \mathcal{L}_i^{(2\alpha)}$. By Definition 3.1, for each $1 \leq i \leq N$,

$$\frac{\mu(A_i)}{\mu\big(B(\mathbf{x}_1, \rho)\big)} > \frac{1}{D} - C(2\alpha)^\gamma \stackrel{\text{def}}{=} \varepsilon > 0.$$

We claim there exist $j_1, \ldots, j_k$, where $k \geq \lceil \varepsilon N \rceil$, such that $K \cap \bigcap_{i=1}^k A_{j_i} \neq \varnothing$. To see this, let $f(\mathbf{x}) = \sum_{i=1}^N \chi_{A_i}(\mathbf{x})$. Then

$$\int_{B(\mathbf{x}_1,\rho)} f(\mathbf{x})\, d\mu(\mathbf{x}) \geq N\varepsilon\mu\big(B(\mathbf{x}_1, \rho)\big),$$

so clearly there exists some $\mathbf{x}_2 \in K$ with $f(\mathbf{x}_2) \geq N\epsilon$. Since $f(\mathbf{x}_2) \in \mathbb{Z}$, there must exist $j_1, \ldots, j_k$ as above. Hence, $\mathbf{x}_2$ satisfies (3.6) and (3.7). □

We will also need the following corollary of the above lemma:

Corollary 3.5. *Let K be a (C, γ, D)-absolutely friendly subset of $\mathbb{R}^n$, let α be as in* (3.5)*, let $S \subset \mathbb{R}^n$ be α-winning on K, and let $S' \subset S$ be a countable union of hyperplanes. Then $S \smallsetminus S'$ is also α-winning on K.*

Proof. In view of the countable intersection property, it suffices to show that for any hyperplane $\mathcal{L} \subset \mathbb{R}^n$, the set $\mathbb{R}^n \smallsetminus \mathcal{L}$ is (α, β)-winning on K for any β. Let μ be a (C, γ, D)-absolutely friendly measure with $K = \operatorname{supp} \mu$. We let Alice play arbitrarily until the radius of a ball chosen by Bob is less than

$\rho_{C,\gamma,D}(\mu)$. Then apply Lemma 3.4 with $N = 1$ and $\mathcal{L}_1 = \mathcal{L}$, which yields a ball disjoint from $\mathcal{L}$. Afterwards she can keep playing arbitrarily, winning the game. □

4. Proofs

Let us now state a more precise version of Theorem 1.3:

Theorem 4.1. *Let K be a (C, γ, D)-absolutely friendly subset of $\mathbb{R}^n$, and let α be as in (3.5). Then for any uniformly discrete sequence $\mathcal{Z}$ of subsets of $\mathbb{R}^m$ and any lacunary sequence $\mathcal{M}$ of $m \times n$ real matrices, the set $\tilde{E}(\mathcal{M}, \mathcal{Z})$ is α-winning on K.*

Proof. Write $\mathcal{M} = (M_k)$, let $t_k \overset{\text{def}}{=} \|M_k\|_{op}$ and let $\mathbf{v}_k$ be a unit vector satisfying

$$\|M_k \mathbf{v}_k\| = t_k.$$

Take $\delta > 0$ such that $\mathcal{Z}$ is δ-uniformly discrete, and let

$$\inf_k \frac{t_{k+1}}{t_k} = Q > 1\,. \tag{4.1}$$

Now pick an arbitrary $0 < \beta < 1$, take ε as in Lemma 3.4, and choose N large enough that

$$(\alpha\beta)^{-r} \le Q^N, \text{ where } r = \lfloor \log_{\frac{1}{1-\varepsilon}} N \rfloor + 1\,. \tag{4.2}$$

We will denote by $M_k^{-1}(Z)$ the preimage of a set $Z \subset \mathbb{R}^n$ under M_k. Notice that for each $k \in \mathbb{N}$, $M_k^{-1}(Z_k)$ is contained in a countable union of hyperplanes, so applying Corollary 3.5 a finite number of times, we may assume that $t_1 \ge 1$.

By playing arbitrary moves if needed, we may assume without loss of generality that $B(\omega_1)$ has radius

$$\rho < \min\left(\frac{\alpha\beta\delta}{4}, \rho_{C,\gamma,D}\right)\,. \tag{4.3}$$

Now let

$$c = \min\left(\rho(\alpha\beta)^{2r-1}, \frac{\delta}{4}\right)\,. \tag{4.4}$$

We will describe a strategy for Alice to play the (α, β)-game on K and to ensure that for all $j \in \mathbb{N}$, for all $\mathbf{x} \in B(\omega'_{r(j+1)})$ and for all k with $1 \le t_k < (\alpha\beta)^{-rj}$, one has $d(M_k\mathbf{x}, Z_k) > c$. This will imply that $\bigcap_k B(\omega'_k) \in \tilde{E}(\mathcal{M}, \mathcal{Z}) \cap K$, finishing the proof.

To satisfy the above goal, Alice can choose ω'_i arbitrarily for $i < r$. Now fix $j \in \mathbb{N}$. By (4.1) and (4.2), there are at most N indices $k \in \mathbb{N}$ for which

$$(\alpha\beta)^{-r(j-1)} \le t_k < (\alpha\beta)^{-rj}. \tag{4.5}$$

Let k be one of these indices. For any $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| \ge \frac{1}{t_k}\|M_k(\mathbf{x})\|$. Thus, if $\mathbf{y}_1, \mathbf{y}_2$ are two different points in Z_k, then by (4.3) and (4.5)

$$d\Big(M_k^{-1}\big(B(\mathbf{y}_1, c)\big), M_k^{-1}\big(B(\mathbf{y}_2, c)\big)\Big) \ge \frac{\delta - 2c}{t_k} \ge \frac{\delta}{2t_k} > \frac{\delta}{2}(\alpha\beta)^{rj} \ge 2\rho(\alpha\beta)^{rj-1}; \tag{4.6}$$

therefore $B(\omega_{rj})$ intersects with at most one set of the form $M_k^{-1}\big(B(\mathbf{y},c)\big)$, where $\mathbf{y}\in Z_k$. Hence, for each k satisfying (4.5),

$$B(\omega_{rj})\cap M_k^{-1}(Z_k^{(c)})\subset M_k^{-1}\big(B(\mathbf{y},c)\big)\text{ for some }\mathbf{y}\in Z_k. \tag{4.7}$$

We will now show that the preimage of such a ball is contained in a 'small enough' neighborhood of some hyperplane, so that we can apply the decay condition. Toward this end, let $V\subset\mathbb{R}^m$ be the hyperplane perpendicular to $M_k\mathbf{v}_k$ and passing through $\mathbf{0}$. Then

$$W\stackrel{\text{def}}{=}M_k^{-1}(V)$$

is a hyperplane in $\mathbb{R}^n$ passing through $\mathbf{0}$.

If $\mathbf{x}\notin W^{(c/t_k)}$, then $\mathbf{x}=\mathbf{w}+\eta\mathbf{v}_k$ for some $\eta>c/t_k$ and $\mathbf{w}\in W$, thus

$$\|M_k\mathbf{x}\|=\|M_k\mathbf{w}+M_k\eta\mathbf{v}_k\|\geq\eta\|M_k\mathbf{v}_k\|=t_k\eta>c\,.$$

Hence, $M_k^{-1}\big(B(\mathbf{0},c)\big)\subset W^{(c/t_k)}$, which clearly implies that for each $\mathbf{y}\in Z_k$, $M_k^{-1}\big(B(\mathbf{y},c)\big)\subset\mathcal{L}^{(c/t_k)}$ for some hyperplane $\mathcal{L}\subset\mathbb{R}^n$. By (4.4) and (4.5),

$$\frac{c}{t_k}\leq(\alpha\beta)^{r(j+1)-1}\rho\stackrel{\text{def}}{=}\zeta\,.$$

Therefore, by (4.7),

$$\bigcup_{t_k\text{ satisfies (4.5)}}B(\omega_{rj})\cap M_k^{-1}\big(Z_k^{(c)}\big)\subset\bigcup_{i=1}^{N}\mathcal{L}_i^{(\zeta)}, \tag{4.8}$$

where $\mathcal{L}_i$ are hyperplanes. Noticing that by (4.2) $(1-\varepsilon)^rN<1$, Alice can utilize Lemma 3.4 r times to distance herself by ζ from each of the hyperplanes $\mathcal{L}_i$ after r turns. Thus for k satisfying (4.5), it holds that

$$B(\omega'_{r(j+1)})\cap M_k^{-1}\big(Z_k^{(c)}\big)=\varnothing\,.$$

We conclude that $d(M_k\mathbf{x},Z_k)\geq c$ for any $\mathbf{x}\in B(\omega'_{r(j+1)})$, which implies the desired statement. □

Proof of Theorem 1.2. Recall that we are given $M\in\mathrm{GL}_n(\mathbb{R})\cap\mathrm{M}_n(\mathbb{Z})$. If all the eigenvalues of M have modulus less than or equal to 1, then obviously every eigenvalue of M must have modulus 1. By a theorem of Kronecker [20], they must be roots of unity, so there exists an $N\in\mathbb{N}$ such that the only eigenvalue of M^N is 1. Let $J=L^{-1}M^NL$ be the Jordan normal form of M^N, and let $\mathbf{v}_i=L\mathbf{e}_i$, $i=1,\ldots,n$, be the Jordan basis for M^N. Then, since M^N is an integer matrix, we have $\mathbf{v}_i\in\mathbb{Q}^n$ for each $1\leq i\leq n$. Hence, letting $V=\mathrm{span}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})$, $V+\mathbb{Z}^n$ is a union of positively separated parallel hyperplanes. Since J fixes the last coordinate of any vector, if $a_1,\ldots,a_n\in\mathbb{R}$, then

$$M^N\left(\sum_{i=1}^{n}a_i\mathbf{v}_i\right)\in a_n\mathbf{v}_n+V.$$

Therefore, for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ with $\mathbf{x}-\mathbf{y}\notin V+\mathbb{Z}^n$ and any $k\in\mathbb{N}$ one has

$$d(M^{Nk}\mathbf{x},\mathbf{y}+\mathbb{Z}^n)\geq c_0d(\mathbf{x}-\mathbf{y},V+\mathbb{Z}^n)>0,$$

where c_0 is a positive constant depending only on $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Hence, for any $y \in \mathbb{T}^n$,

$$\tilde{E}(M^N, y) \supset \mathbb{R}^n \smallsetminus (\pi^{-1}(y) + V) = \mathbb{R}^n \smallsetminus (\mathbf{y} + V + \mathbb{Z}^n),$$

where $\mathbf{y}$ is an arbitrary vector in $\pi^{-1}(y)$. Thus $\tilde{E}(M^N, y)$ is α-winning on K by Corollary 3.5. Hence $\tilde{E}(M^N, z)$ is α-winning on K whenever $z \in f_M^{-i}(y)$, where $0 \le i < N$. Thus the intersection

$$\tilde{E}(M, y) = \bigcap_{i=0}^{N-1} \bigcap_{z \in f_M^{-i}(y)} \tilde{E}\big(M^N, f_M^{-i}(y)\big)$$

is also α-winning on K.

In the case where at least one of the eigenvalues is of absolute value strictly greater than 1, we will show that the sequence $(\|M^k\|_{op})$ is a finite union of lacunary sequences, which will clearly imply that $\tilde{E}\big((M^k), \mathcal{Z}\big)$ is α-winning on K. Let $J = L^{-1}ML$ be the Jordan normal form of M. Since the operator norm of M as a real transformation is equal to its operator norm as a complex transformation and

$$\|J^k\|_{op} \le \|L\|_{op}\|L^{-1}\|_{op}\|M^k\|_{op} \quad \text{and} \quad \|M^k\|_{op} \le \|L\|_{op}\|L^{-1}\|_{op}\|J^k\|_{op},$$

letting $c = \|L\|_{op}\|L^{-1}\|_{op}$, we have $\frac{1}{c}\|M^k\|_{op} \le \|J^k\|_{op} \le c\|M^k\|_{op}$ for all $k \in \mathbb{N}$. Hence, if $(\|J^k\|_{op})$ is eventually lacunary, then there exists $\ell, N \in \mathbb{N}$ and $Q > 1$ such that, for all $k \ge N$,

$$\frac{\|M^{k+\ell}\|_{op}}{\|M^k\|_{op}} \ge \frac{1}{c^2}\frac{\|J^{k+\ell}\|_{op}}{\|J^k\|_{op}} \ge Q.$$

Thus it will suffice to show that $(\|J^k\|_{op})$ is eventually lacunary.

Let B be an $m \times m$ block of J associated to an eigenvalue λ and write $B^k = \big(b_{ij}(k)\big)$. Direct computation shows that, for $0 \le j - i \le k$,

$$b_{ij}(k) = \binom{k}{j-i}\lambda^{k-(j-i)}, \tag{4.9}$$

and $b_{ij}(k) = 0$ otherwise. Since $|b_{ij}(k)| = o(|b_{1m}(k)|)$ as functions of k for all $(i, j) \ne (1, m)$,

$$\lim_{k\to\infty} \frac{\|B^k\|_{op}}{|b_{1m}(k)|} = 1. \tag{4.10}$$

Hence,

$$\lim_{k\to\infty} \frac{\|B^{k+1}\|_{op}}{\|B^k\|_{op}} = |\lambda|, \tag{4.11}$$

so clearly if $|\lambda| > 1$ then $(\|B^k\|_{op})$ is eventually lacunary. Write $J = B_1 \oplus \cdots \oplus B_s$, where $s \in \mathbb{N}$ and B_i are the Jordan blocks, with associated eigenvalues λ_i. Let $\lambda_{max} = \max|\lambda_i|$, and let B_{max} be a block with associated eigenvalue having absolute value λ_{max} and of maximal dimension among such blocks. By (4.9) and (4.10), for any i,

$$\lim_{k\to\infty} \frac{\|B_{max}^k\|_{op}}{\|B_i^k\|_{op}} \ge 1.$$

Hence, by (4.11),

$$\lim_{k\to\infty}\frac{\|J^{k+1}\|_{op}}{\|J^k\|_{op}}=\lim_{k\to\infty}\frac{\|B^{k+1}_{max}\|_{op}}{\|B^k_{max}\|_{op}}=\lambda_{max}.$$

Since by assumption M (and therefore J) has an eigenvalue with absolute value greater than 1, $(\|J^k\|_{op})$ is eventually lacunary. □

In the remaining part of this section we apply Theorem 1.3 to badly approximable systems of affine forms.

Proof of Corollary 1.4. Recall that we need to fix $A \in \mathrm{M}_{n\times m}(\mathbb{R})$ and study the set

$$\mathbf{Bad}_A(n,m)=\left\{\mathbf{x}\in\mathbb{R}^n : \inf_{\mathbf{q}\in\mathbb{Z}^m\smallsetminus\{0\}}\|\mathbf{q}\|^{m/n}d(A\mathbf{q}-\mathbf{x},\mathbb{Z}^n)>0\right\}.$$

First observe that the above set is easy to understand in the 'rational' case when there exists a nonzero $\mathbf{u}\in\mathbb{Z}^n$ such that $A^T\mathbf{u}\in\mathbb{Z}^m$ (or equivalently, when the rank of the group $A^T\mathbb{Z}^n+\mathbb{Z}^m$ is strictly smaller than $m+n$). In this case, by a theorem of Kronecker, see [4, Ch. III, Theorem IV], $\inf_{\mathbf{q}\in\mathbb{Z}^m} d(A\mathbf{q}-\mathbf{x},\mathbb{Z}^n)$ is positive if and only if the value of $\mathbf{u}\cdot\mathbf{x}$ is not an integer. Therefore

$$\mathbf{Bad}_A(n,m)\supset\{\mathbf{x}\in\mathbb{R}^n : \mathbf{u}\cdot\mathbf{x}\notin\mathbb{Z}\}.$$

Since the right-hand side is the complement of a countable union of hyperplanes, in view of Corollary 3.5 $\mathbf{Bad}_A(n,m)$ is α-winning on K whenever K is absolutely friendly and α is as in Theorem 1.3.

In the more interesting 'irrational' case when $\mathrm{rank}(A^T\mathbb{Z}^n+\mathbb{Z}^m)=m+n$, one can utilize the theory of best approximations to A as developed by Cassels [4, Ch. III] and recently made more precise by Bugeaud and Laurent [3]. In [2, §§5–6], using results from [3], it is shown that if $\mathrm{rank}(A^T\mathbb{Z}^n+\mathbb{Z}^m)=m+n$, then there exists a lacunary sequence of vectors $\mathbf{y}_k\in\mathbb{Z}^n$ (a subsequence of the sequence of best approximations to A) such that whenever $\mathbf{x}\in\mathbb{R}^n$ satisfies

$$\inf_{k\in\mathbb{N}} d(\mathbf{y}_k\cdot\mathbf{x},\mathbb{Z})>0\,,$$

it follows that $\mathbf{x}\in\mathbf{Bad}_A(n,m)$. In other words,

$$\tilde{E}(\mathcal{Y},\mathcal{Z})\subset\mathbf{Bad}_A(n,m)\,,$$

where $\mathcal{Y}\stackrel{\mathrm{def}}{=}(\mathbf{y}_k)$ and $\mathcal{Z}=(Z_k)$ with $Z_k=\mathbb{Z}$ for each k. (See also [22, §2] for an alternative exposition.) Therefore in this case $\mathbf{Bad}_A(n,m)$ is α-winning on K by Theorem 1.3. □

5. Concluding remarks

5.1. **Playing on $\mathbb{R}^n$ with $\alpha=1/2$.** As was mentioned before, the special case $K=\mathbb{R}^n$ of our main theorem is essentially contained in [22]. In fact, arguing as in §4 and using [22, Lemma 2] (the analogue of our Lemma 3.4) and [22, Lemma 3] (Schmidt's escaping lemma, cf. [29, Ch. 3, Lemma 1B]), one can show that for $\mathcal{Z}$ and $\mathcal{M}$ as in Theorem 1.3 and any $\alpha,\beta>0$ with $1+\alpha\beta-2\alpha>0$, the sets $\tilde{E}(\mathcal{M},\mathcal{Z})$ are (α,β)-winning. In particular, this shows that one can take $\alpha(\mathbb{R}^n)=1/2$ in Theorems 1.2 and 1.3.

5.2. **Strong winning.** Recently in [10, 11] and independently in [21] a modification of Schmidt's game has been introduced, where condition (2.1) is replaced by

$$\rho'_k \geq \alpha\rho_k \text{ and } \rho_{k+1} \geq \beta\rho'_k\,. \tag{5.1}$$

Following [21], a subset S of a metric space X is said to be (α, β)-*strong winning* if Alice has a winning strategy in the game defined by (5.1). Analogously, one defines α-strong winning and strong winning sets. It is not hard to verify that strong winning implies winning (see [11] for a proof), and that a countable intersection of α-strong winning sets is α-strong winning. Furthermore, this class has stronger invariance properties, e.g. it is proved in [21] that strong winning subsets of $\mathbb{R}^n$ are preserved by quasisymmetric homeomorphisms.

It is not hard to modify the proofs given above to show that in Theorem 1.3 (and therefore in all its corollaries), α-winning may be replaced by α-strong winning. This is done by adding 'dummy moves' in order to accommodate the possibly slower decrease in radii of the chosen balls. Details will appear elsewhere.

References

[1] R. Broderick, Y. Bugeaud, L. Fishman, D. Kleinbock and B. Weiss, *Schmidt's game, fractals, and numbers normal to no base*, Preprint, `arXiv:0909.4251`.

[2] Y. Bugeaud, S. Harrap, S. Kristensen and S. Velani, *On shrinking targets for $\mathbb{Z}^m$ actions on tori*, Preprint, `arXiv:0807.3863`.

[3] Y. Bugeaud and M. Laurent, *On exponents of homogeneous and inhomogeneous Diophantine approximation*, Moscow Math. J. **5**, no. 4, 747–766, 972.

[4] J.W.S. Cassels, *An introduction to diophantine approximation*, Cambridge University Press, Cambridge, UK, 1957.

[5] S.G. Dani, *On orbits of endomorphisms of tori and the Schmidt game*, Ergod. Th. Dynam. Syst. **8** (1988), 523–529.

[6] B. de Mathan, *Numbers contravening a condition in density modulo* 1, Acta Math. Acad. Sci. Hungar. **36** (1980), 237–241.

[7] M. Einsiedler and J. Tseng, *Badly approximable systems of affine forms, fractals, and Schmidt games*, in preparation.

[8] R. Esdahl-Schou and S. Kristensen, *On badly approximable complex numbers*, Glasg. Math. J., to appear.

[9] K. Falconer, *Fractal geometry. Mathematical foundations and applications*, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[10] D. Färm, *Simultaneously Non-dense Orbits Under Different Expanding Maps*, Preprint, `arXiv:0904.4365v1`.

[11] D. Färm, T. Persson and J. Schmeling, *Dimension of Countable Intersections of Some Sets Arising in Expansions in Non-Integer Bases*, preprint (2009).

[12] L. Fishman, *Schmidt's game on fractals*, Israel J. Math. **171** (2009), no. 1, 77–92.

[13] ———, *Schmidt's game, badly approximable matrices and fractals*, J. Number Theory **129**, no. 9 (2009), 2133–2153.

[14] D. Kleinbock, *Nondense orbits of flows on homogeneous spaces*, Ergodic Theory Dynam. Systems **18** (1998), 373-396.

[15] ———, *Badly approximable systems of affine forms*, J. Number Theory **79** (1999), no. 1, 83–102.

[16] D. Kleinbock, E. Lindenstrauss and B. Weiss, *On fractal measures and diophantine approximation*, Selecta Math. **10** (2004), 479–523.

[17] D. Kleinbock and B. Weiss, *Badly approximable vectors on fractals*, Israel J. Math. **149** (2005), 137-170.

[18] ———, *Modified Schmidt games and Diophantine approximation with weights*, Advances in Math., to appear.

[19] S. Kristensen, R. Thorn and S.L. Velani, *Diophantine approximation and badly approximable sets*, Advances in Math. **203** (2006), 132–169.

[20] L. Kronecker, *Zwei Sätse über Gleichungen mit ganzzahligen Coefcienten*, J. Reine Angew. Math. **53** (1857), 173–175; see also Werke, Vol. 1, 103–108, Chelsea Publishing Co., New York, 1968.

[21] C. McMullen, *Winning sets, quasiconformal maps and Diophantine approximation*, preprint (2009).

[22] N.G. Moshchevitin, *A note on badly approximable affine forms and winning sets*, Preprint, `arXiv:0812.3998v2`.

[23] D. Mauldin and M. Urbanski, *The doubling property of conformal measures of infinite iterated function systems*, J. Number Theory **102** (2003), 23–40.

[24] A.D. Pollington, *On nowhere dense Θ-sets*, Groupe de travail d'analyse ultramtrique. **10** (1982-1983), no. 2, Exp. No. 22, 2 p.

[25] ———, *On the density of sequence* $\{\eta_k\xi\}$, Illinois J. Math. **23** (1979), no. 4, 511–515.

[26] A.D. Pollington and S.L. Velani, *Metric Diophantine approximation and 'absolutely friendly' measures*, Selecta Math. **11** (2005) 297–307.

[27] W.M. Schmidt, *On badly approximable numbers and certain games*, Trans. A.M.S. **123** (1966), 27–50.

[28] ———, *Badly approximable systems of linear forms*, J. Number Theory **1** (1969), 139–154.

[29] ———, *Diophantine approximation*, Lecture Notes in Mathematics, vol. 785, Springer-Verlag, Berlin, 1980.

[30] B. Stratmann and M. Urbanski, *Diophantine extremality of the Patterson measure*, Math. Proc. Cambridge Phil. Soc. **140** (2006), 297–304.

[31] J. Tseng, *Schmidt games and Markov partitions*, Nonlinearity **22** (2009), no. 3, 525–543.

[32] ———, *Badly approximable affine forms and Schmidt games*, J. Number Theory **129** (2009), 3020–3025.

[33] M. Urbanski, *The Hausdorff dimension of the set of points with non-dense orbit under a hyperbolic dynamical system*, Nonlinearity **4** (1991), 385–397.

[34] ———, *Diophantine approximation of self-conformal measures*, J. Number Theory **110** (2005), 219–235.

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