BOUNDED ORBITS CONJECTURE AND DIOPHANTINE APPROXIMATION

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Abstract. We describe and generalize S.G. Dani's correspondence between bounded orbits in the space of lattices and systems of linear forms with certain Diophantine properties. The solution to Margulis' Bounded Orbit Conjecture is used to generalize W. Schmidt's theorem on abundance of badly approximable systems of linear forms.

0. Notation

We will denote by $M_{m,n}(\mathbb{R})$ the space or real matrices with m rows and n columns. $I_k \in M_{k,k}(\mathbb{R})$ will stand for the identity matrix. Vectors will be named by lowercase boldface letters, such as $\mathbf{y} = (y_i \mid 1 \le i \le k)$, and, despite the row notation, will always be treated as column vectors. 0 will mean zero vector in any dimension, as well as zero matrix of any size. $\|\cdot\|$ will stand for the norm on $\mathbb{R}^k$ given by $\|\mathbf{y}\| = \max\limits_{1\le i\le k} |y_i|$. For a matrix $L \in M_{m,n}(\mathbb{R})$ and $1 \le i \le n$, we will denote by L_i the linear form $\mathbb{R}^m \to \mathbb{R}$ corresponding to the ith row of L, and by $L^{(i)}$ (resp. $L_{(i)}$) the matrix consisting of first (resp. last) i rows of L. The latter convention will be applied to vectors $\mathbf{y} \in \mathbb{R}^k$ as well, i.e. we will have $\mathbf{y}^{(i)} = (y_j \mid 1 \le j \le i)$ and $\mathbf{y}_{(i)} = (y_j \mid k - i + 1 \le j \le k)$.

We will say that a subset Y of a metric space X is *thick* (in X) if for any nonempty open subset W of X, the Hausdorff dimension of $W \cap Y$ is equal to the Hausdorff dimension of W (i.e. Y has full Hausdorff dimension at any point of X). A subset F of a Lie group G will be called *nonquasiunipotent* if for some $g \in F$ there exists an eigenvalue λ of $\operatorname{Ad} g$ with $|\lambda| \ne 1$.

1. Introduction

The term "Bounded Orbits Conjecture" refers to an assertion made by G.A. Margulis at the ICM-90 on the abundance of bounded orbits of flows on homogeneous spaces. Let us start with the following

Theorem 1. *Let G be a connected semisimple Lie group without compact factors, Γ an irreducible lattice in G, F a one-parameter nonquasiunipotent subgroup of G. Then*

$$\{x \in G/\Gamma \mid Fx \text{ is bounded}\} \text{ is thick in } G/\Gamma\,. \tag{1.1}$$

S.G. Dani [D1, D2], using certain results and methods from the theory of Diophantine approximation, established (1.1) in some special cases. This motivated Margulis to assert [M, Conjecture (A)] that (1.1) holds for any Lie group G and any lattice Γ in G, provided F is nonquasiunipotent. The latter conjecture was settled in a recent paper [KM] by G.A. Margulis and the author, the main step being the proof of the above theorem. The proof is dynamical in its nature, with partial hyperbolicity and mixing of F-action on G/Γ being crucially involved.

We refer the reader to [K1] for a sketch of the main ideas of the proof of Theorem 1 and to [KM] and [K4] for details and generalizations. In this paper we will (§2) briefly review some facts from the classical theory of multi-dimensional Diophantine approximation, and then (§3) follow the approach of Dani [D1], who considered

$$F = \{g_t\} = \{\mathrm{diag}(\underbrace{e^{t/m}, \dots, e^{t/m}}_{m \text{ times}}, \underbrace{e^{-t/n}, \dots, e^{-t/n}}_{n \text{ times}})\} \subset SL_{m+n}(\mathbb{R}) \tag{1.2}$$

and related the dynamics of g_t-action on $SL_{m+n}(\mathbb{R})/SL_{m+n}(\mathbb{Z})$ to Diophantine properties of systems of linear forms. The goal of the paper is to slightly tilt (§4) and then expand (§5) the classical theory by developing structures corresponding to a large family of subgroups $\{g_t\}$ of $SL_{m+n}(\mathbb{R})$. This way one can make a number-theoretic sense (§5, Corollary″) out of the aforementioned result on the abundance of bounded orbits. In the last section we list several open questions, as well as give a necessary and sufficient condition for the validity of (1.1).

2. Diophantine approximation

We start with the basic question of Diophantine approximation:

1 Given $\alpha \in \mathbb{R}$, how small can be $|\alpha q + p|$, $p, q \in \mathbb{Z}$, in terms of the value of $|q|$?

The above question gives rise to the one-dimensional theory. In order to build a multi-dimensional generalization, one may view α as a linear operator from $\mathbb{R}$ to $\mathbb{R}$, and then change it to a linear operator A from $\mathbb{R}^n$ to $\mathbb{R}^m$. This gives

2 Given $m, n \in \mathbb{N}$ and $A \in M_{m,n}(\mathbb{R})$ (interpreted as a system of m linear forms A_i on $\mathbb{R}^n$), how small can be $\|A\mathbf{q} + \mathbf{p}\|$, $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n$, in terms of the value of $\|\mathbf{q}\|$?

The whole theory starts from a positive result of Dirichlet, namely

Theorem 2. *Let $A \in M_{m,n}(\mathbb{R})$. Then there exist infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$ with arbitrarily large $\|\mathbf{q}\|$ such that $\|A\mathbf{q} + \mathbf{p}\|^m \|\mathbf{q}\|^n < 1$.*

The original proof (1842) involved the famous Dirichlet's pigeon-hole principle. However we will make use of another

Proof. (Minkowski 1896) For any $b > 0$, the set

$$\left\{ \mathbf{y} \in \mathbb{R}^{m+n} \;\middle|\; \begin{array}{ll} |y_i| < (1/b)^{1/m}, & 1 \le i \le m \\ |y_{m+j}| \le b^{1/n}, & 1 \le j \le n \end{array} \right\} \tag{2.1}$$

has volume 2^{m+n}. Hence, by a theorem of Minkowski, it contains a nonzero vector $\mathbf{y} = \begin{pmatrix} \mathbf{p} + A\mathbf{q} \\ \mathbf{q} \end{pmatrix}$ from the lattice $L_A \mathbb{Z}^{m+n}$, where $L_A \stackrel{\text{def}}{=} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}$. Then the inequalities in (2.1) can be written as $\|A\mathbf{q}+\mathbf{p}\|^m < 1/b$ and $\|\mathbf{q}\|^n \le b$, and one can multiply the latter inequalities by each other and then let $b \to \infty$ to obtain the claim. (More precisely, the claim is obvious if $A\mathbf{q}+\mathbf{p} = 0$ for some nonzero vector $(\mathbf{p},\mathbf{q})$; otherwise, for fixed $\mathbf{q}$, the inequality $\|A\mathbf{q}+\mathbf{p}\|^m < 1/b$ can only hold for small enough b, thus, as $b \to \infty$, one obtains infinitely many solutions with different values of $\mathbf{q}$.) $\square$

Definition. A system of linear forms given by $A \in M_{m,n}(\mathbb{R})$ is said to be *badly approximable* if 1 in Theorem 2 cannot be replaced by an arbitrarily small constant. I.e. if

$$\liminf_{(\mathbf{p},\mathbf{q})\in\mathbb{Z}^{m+n},\ \mathbf{q}\to\infty} \|A\mathbf{q}-\mathbf{p}\|^m\|\mathbf{q}\|^n > 0\,.$$

It has been known since 1920s that the set of badly approximable $A \in M_{m,n}(\mathbb{R})$ is infinite (Perron 1921) and of zero Lebesgue measure in $M_{m,n}(\mathbb{R})$ (Khinchin 1926). In 1969 W. Schmidt [S] used the technique of (α,β)-games to show that

$$\{A \in M_{m,n}(\mathbb{R}) \mid A \text{ is badly approximable}\} \text{ is thick in } M_{m,n}(\mathbb{R})\,. \tag{2.2}$$

3. Dani's correspondence.

S.G. Dani deduced one of his results on bounded orbits from the following

Theorem 3. [D1] $A \in M_{m,n}(\mathbb{R})$ *is badly approximable iff the trajectory*

$$\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}\,, \tag{3.1}$$

with g_t *as in* (1.2) *and* L_A *as in the proof of Theorem 2, is bounded in the space* $\Omega_{m+n} \cong SL_{m+n}(\mathbb{R})/SL_{m+n}(\mathbb{Z})$ *of unimodular lattices in* $\mathbb{R}^{m+n}$.

Corollary. ([D1], special case of Theorem 1) *For* $\{g_t\}$ *of the form* (1.2), *lattices* $\Lambda \in \Omega_{m+n}$ *such that* $\{g_t\Lambda \mid t \in \mathbb{R}_+\}$ *is bounded form a thick set.*

Proof. Any lattice $\Lambda \in \Omega_{m+n}$ can be written in the form $\Lambda = \begin{pmatrix} B & 0 \\ C & D\end{pmatrix} L_A\mathbb{Z}^{m+n}$, and one has, for $t \ge 0$,

$$g_t\Lambda = g_t \begin{pmatrix} B & 0 \\ C & D\end{pmatrix} g_{-t}\cdot g_t L_A \mathbb{Z}^{m+n} \subset K\cdot g_t L_A\mathbb{Z}^{m+n}\,, \tag{3.2}$$

where K is some bounded subset of $SL_{m+n}(\mathbb{R})$. Therefore $\{g_t\Lambda \mid t \in \mathbb{R}_+\}$ is bounded iff $\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$ is bounded, and one can apply Theorem 3 and (2.2) to get the claim. $\square$

We remark that one can use Theorem 3 and the above argument to deduce (2.2) from Theorem 1. Thus one obtains a new (dynamical in its nature) proof of Schmidt's result.

We now present a

Proof of Theorem 3, which will basically be a rephrasing of the original proof of Dani, but in the form which seems to be more conceptual and will be replicated later on. The proof is based upon the classical description of bounded subsets of Ω_{m+n} given by

Mahler's Compactness Criterion. *A subset K of Ω_{m+n} is bounded iff there exists a neighborhood U of $0 \in \mathbb{R}^{m+n}$ such that $\Lambda \cap U = \{0\}$ for any $\Lambda \in K$.*

Define a function $\delta_{m,n} : \Omega_{m+n} \to \mathbb{R}_+$ by

$$\delta_{m,n}(\Lambda) \stackrel{\text{def}}{=} \inf_{\mathbf{y}\in\Lambda\smallsetminus\{0\}} \max(\|\mathbf{y}^{(m)}\|^m, \|\mathbf{y}_{(n)}\|^n)\,. \tag{3.3}$$

From Mahler's Criterion it follows that a sequence Λ_k tends to infinity in Ω_{m+n} iff $\delta_{m,n}(\Lambda_k) \to 0$. Observe that (1.2) and (3.3) give

$$\delta_{m,n}(g_t L_A \mathbb{Z}^{m+n}) = \inf_{(\mathbf{p},\mathbf{q})\in\mathbb{Z}^{m+n}\smallsetminus\{0\}} \max(e^t\|\mathbf{p} + A\mathbf{q}\|^m, e^{-t}\|\mathbf{q}\|^n)\,.$$

Therefore the orbit $\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$ is unbounded iff there exists sequences $t_k \to +\infty$, $\mathbf{p}_k \in \mathbb{Z}^m$ and $\mathbf{q}_k \in \mathbb{Z}^n$ such that

$$\max(e^{t_k}\|\mathbf{p}_k + A\mathbf{q}_k\|^m, e^{-t_k}\|\mathbf{q}_k\|^n) \to 0 \text{ as } k \to \infty\,. \tag{3.4}$$

On the other hand, A is well approximable iff there exists sequences $\mathbf{p}_k \in \mathbb{Z}^m$ and $\mathbf{q}_k \in \mathbb{Z}^n$ such that $\|\mathbf{q}_k\| \to \infty$ and

$$\|\mathbf{p}_k + A\mathbf{q}_k\|^m\|\mathbf{q}_k\|^n \to 0 \text{ as } k \to \infty\,. \tag{3.5}$$

We immediately see that (3.5) follows from (3.4). To finish the proof, first exclude the trivial case when $A\mathbf{q} + \mathbf{p} = 0$ for some nonzero $(\mathbf{p}, \mathbf{q})$ (such A is clearly well approximable and the trajectory (3.1) diverges). Then it is clear that (3.4) can only hold if $\|\mathbf{q}_k\| \to \infty$, i.e. unboundedness of the trajectory (3.1) forces A to be well approximable. On the other hand, for well approximable A one can define $e^{t_k} \stackrel{\text{def}}{=} \sqrt{\|\mathbf{q}_k\|^n/\|\mathbf{p}_k + A\mathbf{q}_k\|^m}$ and check that $t_k \to +\infty$ and (3.4) is satisfied. □

4. Another approach to Diophantine approximation.

Note that the passage from A to L_A has helped us twice: first in the proof of Theorem 2, then in the dynamical interpretation of Diophantine properties of A (Theorem 3). It is quite tempting to pick an arbitrary element $L = \begin{pmatrix} L^{(m)} \\ L_{(n)} \end{pmatrix}$ of $SL_{m+n}(\mathbb{R})$, not necessarily of the form L_A, and play similar games with it. Specifically, instead of $\boxed{2}$ we will be interested in

$\boxed{2'}$ Given $m, n \in \mathbb{N}$ and $L \in SL_{m+n}(\mathbb{R})$, how small can be $\|L^{(m)}\mathbf{x}\|$, $\mathbf{x} \in \mathbb{Z}^{m+n}$, in terms of the value of $\|L_{(n)}\mathbf{x}\|$?

This way one can lift all the results from the preceding two sections to the level of $SL_{m+n}(\mathbb{R})$, Theorem 2 being transformed into

Theorem 2′. *Let $L \in SL_{m+n}(\mathbb{R})$. Then there exist infinitely many $\mathbf{x} \in \mathbb{Z}^{m+n}$ with arbitrarily large $\|L_{(n)}\mathbf{x}\|$ such that* $\|L^{(m)}\mathbf{x}\|^m\|L_{(n)}\mathbf{x}\|^n < 1$.

The proof is exactly the same as that of Theorem 2, modulo the substitutions $L_A \to L$, $(\mathbf{p}, \mathbf{q}) \to \mathbf{x}$, $A\mathbf{q}+\mathbf{p} \to L^{(m)}\mathbf{x}$ and $\mathbf{q} \to L_{(n)}\mathbf{x}$. This motivates the following

Definition′. Say that $L \in SL_{m+n}(\mathbb{R})$ is $(m,n,+)$-*loose* if 1 in Theorem 2′ cannot be replaced by an arbitrarily small constant. I.e. if

$$\liminf_{\mathbf{x}\in\mathbb{Z}^{m+n},\, L_{(n)}\mathbf{x}\to\infty} \|L^{(m)}\mathbf{x}\|^m \|L_{(n)}\mathbf{x}\|^n > 0\,. \tag{4.1}$$

Clearly L_A is $(m,n,+)$-loose iff A is badly approximable. Moreover, the proof of Theorem 3 can be applied verbatim (after the above substitutions) to establish

Theorem 3′. *$L \in SL_{m+n}(\mathbb{R})$ is $(m,n,+)$-loose iff the trajectory*

$$\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}\,, \tag{4.2}$$

with $\{g_t\}$ of the form (3.1)*, is bounded in Ω_{m+n}.*

Corollary′. *$(m,n,+)$-loose matrices form a thick subset of $SL_{m+n}(\mathbb{R})$.*

Remark. The use of "+" in the notation is justified by "+"in (4.2) and also by the asymmetry of the above definition. One can interchange $L^{(m)}$ and $L_{(n)}$ in (4.1) to get the definition of $(m,n,-)$-*loose* matrices, corresponding to bounded trajectories of the form $\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_-\}$.

5. Diophantine approximation with weights

We now want to apply Theorem 1 in a situation that is a little bit more general than that of Theorems 3 and 3′. Specifically, take a generic nonquasiunipotent one-parameter subgroup of $SL_{m+n}(\mathbb{R})$ with real eigenvalues. In a suitable basis, it has the form

$$g_t = \operatorname{diag}(e^{r_1 t},\dots,e^{r_m t},e^{-s_1 t},\dots,e^{-s_n t})\,, \tag{5.1}$$

where $\mathbf{r} = (r_i \mid 1 \le i \le m)$ and $\mathbf{s} = (s_j \mid 1 \le j \le n)$ are such that

$$r_i, s_j > 0 \quad \text{and} \quad \sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j\,. \tag{5.2}$$

Note that the choice (1.2) of the subgroup $\{g_t\}$ corresponds to the case

$$\mathbf{r} = \mathbf{m} \overset{\text{def}}{=} (\tfrac{1}{m},\dots,\tfrac{1}{m}) \quad \text{and} \quad \mathbf{s} = \mathbf{n} \overset{\text{def}}{=} (\tfrac{1}{n},\dots,\tfrac{1}{n})\,. \tag{5.3}$$

A natural question to ask is whether there is some Diophantine meaning of boundedness of orbits relative to the action of subgroups of the above type.

It turns out that in order to get an adequate answer to this question, one has to dig in from the very beginning of the theory, i.e. from Theorem 2 (or a more general Theorem 2′). Indeed, the strategy is to replace all the occurences of $1/m$ (resp. $1/n$) by r_i (resp. s_j). Thus one replaces the domain (2.1) by

$$\left\{ \mathbf{y} \in \mathbb{R}^{m+n} \;\middle|\; \begin{array}{ll} |y_i| < (1/b)^{r_i}, & 1 \le i \le m \\ |y_{m+j}| \le b^{s_j}, & 1 \le j \le n \end{array} \right\}. \tag{5.4}$$

In view of (5.2), it also has volume 2^{m+n}, hence contains a nonzero vector $\mathbf{y} = \begin{pmatrix} L^{(m)}\mathbf{x} \\ L_{(n)}\mathbf{x} \end{pmatrix}$ from the lattice $L\mathbb{Z}^{m+n}$. Moreover, the inequalities in (5.4) can be written as $\max_{i=1,\dots,m} \left(|L_i(\mathbf{x})|^{1/r_i}\right) < 1/b$ and $\max_{j=1,\dots,n} \left(|L_{m+j}(\mathbf{x})|^{1/s_j}\right) \le b$.

It is now clear that to proceed further one needs to substitute the standard norms $\|\cdot\|$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ with certain *quasinorms*, dependent on parameters r_i and s_j. More precisely, for a k-tuple $\mathbf{w} = (w_1, \dots, w_k)$ with positive components, define the $\mathbf{w}$-*quasinorm* $\|\cdot\|_\mathbf{w}$ on $\mathbb{R}^k$ by $\|\mathbf{y}\|_\mathbf{w} \stackrel{\text{def}}{=} \max_{1\le i\le k} |y_i|^{1/w_i}$. One may think of components of $\mathbf{w}$ as weights assigned to the coordinates of $\mathbf{y}$, so that some of them are treated as more important than the others. However, when weights are chosen to be equal, one returns to the situation of §§2–4, i.e. the choice $\mathbf{w} = \mathbf{k} \stackrel{\text{def}}{=} (\frac{1}{k}, \dots, \frac{1}{k})$ gives $\|\cdot\|_\mathbf{k} = \|\cdot\|^k$.

With this notation, it is clear that we have just proved

Theorem 2″. *Let $L \in SL_{m+n}(\mathbb{R})$, an m-tuple $\mathbf{r}$ and an n-tuple $\mathbf{s}$ satisfying* (5.2) *be given. Then there exist infinitely many $\mathbf{x} \in \mathbb{Z}^{m+n}$ with arbitrarily large $\|L_{(n)}\mathbf{x}\|$ such that* $\|L^{(m)}\mathbf{x}\|_\mathbf{r}\|L_{(n)}\mathbf{x}\|_\mathbf{s} < 1$.

Obviously Theorem 2′ is a special case, with $\mathbf{r}$ and $\mathbf{s}$ as in (5.3). In fact, one can view the basic question $\boxed{2'}$ as a special case of

$\boxed{2''}$ Given $m, n \in \mathbb{N}$, $\mathbf{r}$, $\mathbf{s}$ with (5.2) and $L \in SL_{m+n}(\mathbb{R})$, how small can be $\|L^{(m)}\mathbf{x}\|_\mathbf{r}$, $\mathbf{x} \in \mathbb{Z}^{m+n}$, in terms of the value of $\|L_{(n)}\mathbf{x}\|_\mathbf{s}$?

And the next natural step is of course

Definition″. Say that $L \in SL_{m+n}(\mathbb{R})$ is $(\mathbf{r}, \mathbf{s}, +)$-*loose* if 1 in Theorem 2″ cannot be replaced by an arbitrarily small constant. I.e. if

$$\liminf_{\mathbf{x}\in\mathbb{Z}^{m+n},\, L_{(n)}\mathbf{x}\to\infty} \|L^{(m)}\mathbf{x}\|_\mathbf{r}\|L_{(n)}\mathbf{x}\|_\mathbf{s} > 0\,.$$

Then one immediately gets

Theorem 3″. *$L \in SL_{m+n}(\mathbb{R})$ is $(\mathbf{r}, \mathbf{s}, +)$-loose iff the trajectory* (4.2)*, with g_t as in* (5.1)*, is bounded in Ω_{m+n}.*

The proof of Theorem 3′ applies verbatim modulo the substitution $\|\cdot\|^m \to \|\cdot\|_\mathbf{r}$ and $\|\cdot\|^n \to \|\cdot\|_\mathbf{s}$. In particular, the function $\delta_{m,n}$ gets replaced by

$$\delta_{\mathbf{r},\mathbf{s}}(\Lambda) \stackrel{\text{def}}{=} \inf_{\mathbf{x}\in\Lambda\smallsetminus\{0\}} \max(\|\mathbf{x}^{(m)}\|_\mathbf{r}, \|\mathbf{x}_{(n)}\|_\mathbf{s})\,.$$

Finally, one can apply Theorem 1 to obtain

Corollary″. *$(\mathbf{r}, \mathbf{s}, +)$-loose matrices form a thick subset of $SL_{m+n}(\mathbb{R})$.*

Note that this set has zero Haar measure in $SL_{m+n}(\mathbb{R})$ by ergodicity of $\{g_t\}$-action on Ω_{m+n}.

Remarks. One can also

- consider the set of $(\mathbf{r},\mathbf{s})$-*loose* matrices (corresponding to two-sided bounded orbits) which is also thick in $SL_{m+n}(\mathbb{R})$;
- derive quantitative versions of Theorems 3, $3'$, $3''$, i.e. express "sizes" of trajectories of lattices in Diophantine terms;
- further generalize Dani's correspondence by relating the *rate of approximation* of A (or the *tightness* of L) with the rate of growth of the corresponding orbit in the space of lattices.

See [K3] and [K4, Chapter VII] for these and other extensions.

6. Concluding remarks and open questions

6.1. A logical conclusion to the developments of the previous section would be to transform the question $\boxed{2}$ into

$\boxed{2'''}$ Given $A \in M_{m,n}(\mathbb{R})$, an m-tuple $\mathbf{r}$ and an n-tuple $\mathbf{s}$ satisfying (5.2), how small can be $\|A\mathbf{q}-\mathbf{p}\|_{\mathbf{r}}$, $\mathbf{p}\in\mathbb{Z}^m$, $\mathbf{q}\in\mathbb{Z}^n$, in terms of the value of $\|\mathbf{q}\|_{\mathbf{s}}$?

Here different weights are assigned to the forms A_i and their arguments q_j. Similarly one gets

Definition$'''$. Say that $A \in M_{m,n}(\mathbb{R})$ is $(\mathbf{r},\mathbf{s})$-*badly approximable* if L_A is $(\mathbf{r},\mathbf{s},+)$-loose, i.e. if

$$\liminf_{(\mathbf{p},\mathbf{q})\in\mathbb{Z}^{m+n},\ \mathbf{q}\to\infty} \|A\mathbf{q}-\mathbf{p}\|_{\mathbf{r}}\|\mathbf{q}\|_{\mathbf{s}} > 0\,.$$

In particular, one can speak about an $\mathbf{r}$-badly approximable m-tuple (the case $n=1$) or an $\mathbf{s}$-badly approximable linear form ($m=1$).

Theorem $3''$ immediately implies that $A \in M_{m,n}(\mathbb{R})$ is $(\mathbf{r},\mathbf{s})$-badly approximable iff the trajectory (3.1), with g_t as in (5.1), is bounded in Ω_{m+n}. Unfortunately, the existence of objects defined this way does not immediately follow from Theorem 1. The problem lies in the fact that elements $\begin{pmatrix} B & 0 \\ C & D\end{pmatrix}$ may get expanded when conjugated by g_t of the form (5.1) (in other words, the expanding horospherical subgroup corresponding to g_t is in general bigger than $\{L_A \mid A\in M_{m,n}(\mathbb{R})\}$), and the argument in (3.2) fails. Still, we propose the following

Conjecture. *$(\mathbf{r},\mathbf{s})$-badly approximable systems of m linear forms in n variables form a thick subset of $M_{m,n}(\mathbb{R})$.*

6.2. Say that a subset Z of the one-point compactification $(G/\Gamma)^* \stackrel{\text{def}}{=} (G/\Gamma)\cup\{\infty\}$ of G/Γ is *escapable relative to F* (or, briefly, *F-escapable*) if

$$\{x\in G/\Gamma \mid \overline{Fx}\cap Z=\varnothing\} \text{ is thick in } G/\Gamma\,,$$

with the closure taken in the topology of $(G/\Gamma)^*$. Then the conclusion (1.1) of Theorem 1 can be read as "$\{\infty\}$ is F-escapable". In fact, it was proved in [KM] that, under the assumptions of Theorem 1, Z is F-escapable whenever $\overline{FZ}$ has zero Haar measure. This raised a question of finding other conditions sufficient for escapability. It was shown in [K2] that a modification of methods from [KM] can yield certain results in this direction. In particular, it is proved there that finite subsets of G/Γ are F-escapable for any unimodular Lie group G, a discrete subgroup Γ of G and a nonquasiunipotent one-parameter subgroup F of G.

It seems natural to expect that under the assumptions of Theorem 1 any finite subset of $(G/\Gamma)^*$ is F-escapable (this is in fact one of the statements of Conjecture (B) from [M]). However the proof of this assertion seems beyond our reach.

6.3. Finally we return to the Bounded Orbits Conjecture in its original form. Let G be a Lie group and Γ a lattice in G. Our goal is to describe all the one-parameter subgroups F of G for which $\{\infty\} \subset (G/\Gamma)^*$ is F-escapable.

It is easy to see that the restriction on F to be nonquasiunipotent is essential. In fact, from Ratner's work on closures of unipotent orbits ([R], see also [DM, Theorem 3]) it follows that if F is a (not relatively compact) unipotent one-parameter subgroup of a connected simple Lie group G, then there exist countably many proper submanifolds X_i of G/Γ such that the orbit Fx for $x \notin \cup_i X_i$ is uniformly distributed and in particular dense in G/Γ. Consequently, there are no nonempty escapable subsets of G/Γ. Similarly, $\{\infty\}$ is not escapable if G/Γ is noncompact.

On the other hand, take $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, and $F = \{g_t\}$, where $g_t = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right)$. Then G/Γ is a direct product of two copies Ω_2' and Ω_2'' of $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, and from the above argument it follows that for any $x_1 \in \Omega_2'$, the set of points of $\{x_1\} \times \Omega_2''$ with bounded orbits is not thick in $\{x_1\} \times \Omega_2''$. Therefore $\{\infty\} \subset (G/\Gamma)^*$ is not F-escapable, even though F is clearly nonquasiunipotent.

It was shown in [KM] that all the possible counterexamples to the Bounded Orbits Conjecture look more or less like the one described above. More precisely, one has

Theorem 4 [KM, Theorem 5.2]**.** *Let G be a Lie group, Γ a lattice in G, F a one-parameter subgroup of G. Then* (1.1) *is equivalent to*

- (Q) *for any connected normal subgroup $N \subset G^\circ$ with the quotient map $p : G^\circ \to G' \overset{\text{def}}{=} G/N$ such that G' is semisimple without compact factors and $p(\Gamma \cap G^\circ)$ is an irreducible lattice in G', at least one of the following three conditions is satisfied:*
- (Q1) $p(\Gamma \cap G^\circ)$ *is cocompact in G';*
- (Q2) $\mathrm{Ad}\, p(F)$ *is relatively compact;*
- (Q3) $p(F)$ *is not quasiunipotent.*

The proof of (1.1)$\Rightarrow$(Q) follows the lines of the above discussion of the $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$-counterexample, while the opposite direction is done by reduction to the case of Theorem 1 using certain properties of lattices in Lie groups.

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