

INVERSE PROBLEMS FOR FLUID FLOWS WITH A MOVING BOUNDARY

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ABSTRACT. We consider the classical Hele-Shaw model, describing the flow of an incompressible Newtonian fluid in a gap between two parallel plates. We prove that any compact smooth family $\{D(t)\}$ of domains in \mathbb{R}^2 with analytic boundaries can be obtained from $D(0)$ by action of sources/sinks continuously distributed along closed curves. In the case of the zero-surface-tension boundary condition we find this distribution via solving a special boundary value problem for analytic functions and show how this procedure in many cases allows to explicitly construct the solutions.

INTRODUCTION

Consider a planar flow of an incompressible fluid through a homogeneous porous medium, also modelled by a viscous flow in a narrow gap between two parallel plates (the *Hele-Shaw cell*). One wants to study the evolution of the domain $D(t)$ occupied by fluid at time t evoked by certain prescribed *driving mechanism*, such as action of several sources or sinks. The mathematical model (see §1) is simple: the velocity potential Φ is harmonic in $D(t)$ except for logarithmic singularities at sources/sinks, $\Phi|_{\partial D(t)} = 0$ if the boundary is free (or else one may include the surface tension), and the normal boundary velocity is given by the normal derivative of Φ . To pose a *direct problem* one prescribes an initial domain $D(0)$ and attempts to find $D(t)$ for all t . The first analytic solutions of direct Hele-Shaw problems were obtained more than half a century ago by Polubarinova-Kochina [P-K1], Galin [Gal] and Kufarev with coauthors [K, VK, ABK] due to wide use of complex analytic methods.

The application of another complex analytic approach based on the Schwarz function description of analytic boundaries developed during the last twenty years (see [R1–2, Lac] and a review in [H2]) motivated the idea of posing *inverse problems* [Mil2]: given the evolution of a fluid domain, find a driving mechanism, i.e. the singularities of the fluid flow potential. However, both R. Millar [Mil2] and then S. Howison [H2] stress that those singularities must satisfy certain conditions to be qualified as physically realistic solutions. Thus to deal with inverse problems one needs first to discuss the allowed types of singularities. Following R. Millar, we restrict ourselves to the following two types of driving mechanisms:

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- the discrete set of point sources or sinks, will be referred to as *d-type*;
- the continuous distribution of sources or sinks along a piecewise smooth curve, will be referred to as *c-type*.

The strict formulation of direct and inverse problems is presented in §1; we also define formally the two types of singularities mentioned above, introducing the concept of *singularity function* corresponding to a direct problem, which seems to be very helpful in formalizing the description of the model.

With that in mind, in §2 we review the ideas of Millar, who constructs the flow potential Φ in the fluid domain $D(t)$ from the known boundary values of Φ and $\frac{\partial\Phi}{\partial n}$ in the neighborhood of $\Gamma(t) = \partial D(t)$ and then by analytic continuation finds the singularities of the potential. The numerous examples considered in [Mil2] show that this approach can rarely produce physically realistic solutions.

Our key observation is that in the solutions of c-type the curve carrying the sources formally need not be simply connected; thus the analytic continuation of Φ may not be unique. Moreover, we prove in §2 that it is always possible to find a simple closed contour γ homotopic to $\Gamma(t)$ and lying in $D(t)$ such that Φ could be extended to a function harmonic in $D(t) \setminus \gamma$ with logarithmic singularities at γ . That means c-type solvability of any inverse problem with analytic data. §3 contains several possible generalizations of this result.

§§ 4–7 are devoted to the advantages given by the assumption of the zero-surface-tension boundary condition for the Hele-Shaw model. In this case the connection between the flow potential and the Schwarz function of the moving boundary becomes straightforward [Lac]. We stress in §§ 4 and 5 that the corresponding connection between the Cauchy transform of the fluid domain and the singularity function provides a useful tool for solving inverse problems. In particular, we look at the complicated procedure of constructing the solutions of §2 from a different point of view and express the distribution of sources along γ in terms of a solution of a Hilbert boundary value problem for analytic functions. This is done in §6, and the next section deals with special cases in which this boundary value problem can be easily solved analytically. Finally we illustrate the results obtained by a number of concrete examples.

§1. DIRECT AND INVERSE PROBLEMS

We denote by $D(t)$ the domain in the plane occupied by the viscous fluid at a time t , and call $\Gamma(t)$ the boundary of $D(t)$. We assume that $\Gamma(t)$ is compact, and (except for §3.2) connected. The domain $D(t)$ may or may not be bounded, which gives rise to consideration of either *interior* or *exterior* problems; in the latter case $B(t)$ stands for the complement of the closure of $D(t)$ which will be called a *bubble*. We use the complex coordinate $z = x + iy$.

Let $\Phi(z, t)$ denote the velocity potential of the fluid flow at a point z in $D(t)$, proportional to the pressure p :

$$\Phi = -\frac{b^2}{12\mu}p$$

(here μ is the viscosity of the fluid, b the width of the cell), and let $F(z, t)$ stand

for the complex flow potential, with $\Phi = \operatorname{Re} F$.

We define the *driving mechanism* for the direct problem by prescribing for each t from the fixed interval $[0, T]$ the *singularity set* $L(t)$ and (up to an additive constant) the real *singularity function* $\Sigma(z, t)$ which is harmonic in $D(t) \setminus L(t)$ for each t . If $D(t)$ is unbounded, the set $L(t)$ may contain ∞ . Then we say that the interior (or exterior) *direct problem* is posed if, given some bounded (or unbounded) initial domain D with analytic boundary one wishes to determine the family of domains $D(t)$ with real functions $\Phi(z, t)$ satisfying the following:

$$(1.1) \quad \Phi(z, t) \text{ is harmonic in } D(t) \setminus L(t) \text{ and continuous near } \Gamma(t) = \partial D(t);$$

$$(1.2) \quad \Phi(z, t) - \Sigma(z, t) \text{ is harmonic in } D(t)$$

(if $D(t)$ is unbounded, this implies regularity at ∞);

$$(1.3) \quad \Phi(z, t) = \mp \beta \kappa(z, t), \quad z \in \Gamma(t)$$

(here β is a positive constant proportional to the surface tension coefficient and $\kappa(z, t)$ is the curvature of $\Gamma(t)$ at z ; the upper sign will be always chosen for interior, and the lower for exterior problems);

$$(1.4) \quad v_n(z, t) = \frac{\partial \Phi(z, t)}{\partial n}, \quad z \in \Gamma(t)$$

(here $v_n(z, t)$ is the projection of the velocity of $\Gamma(t)$ at z onto the unit vector n normal to $\Gamma(t)$ at z); and finally, the initial condition

$$(1.5) \quad D(0) = D.$$

Note that the conditions (1.1) and (1.2) can be rewritten in terms of the complex potential F as follows:

$$(1.1C) \quad F(z, t) \text{ is holomorphic in } D(t) \setminus L(t) \text{ and continuous near } \Gamma(t);$$

$$(1.2C) \quad F(z, t) - S(z, t) \text{ is holomorphic in } D(t),$$

where $S(z, t)$ is the *complex singularity function*, $\operatorname{Re} S = \Sigma$.

We now specify the class of singularity functions to be discussed. In this paper we will mainly consider flows driven by sources and sinks, although other driving mechanisms can also be handled in analogous way¹. That is, the function Σ will

¹For instance, flows driven by multipole singularities were recently studied in [EEK]; the simplest case of the bubble motion in a steady flow was first considered as early as in 1959 [TS].

be chosen in the form of *logarithmic potential*. For interior problems the general expression for such a function is

$$(1.6i) \quad \Sigma(z, t) = \int \log |z - \zeta| d\mu_t(\zeta),$$

where μ_t is a signed measure supported on $L(t)$. The physical meaning of this function is clear: it represents the potential of the one-phase flow driven by sources distributed according to the measure μ_t . Note that Σ is harmonic in $\mathbb{C} \setminus L(t)$ and may have a logarithmic singularity at infinity.

Dealing with exterior problems, to avoid the irrelevant singularity at infinity and also allow the possibility of an infinite source or sink, we assume without loss of generality that the bubble $B(t)$ contains the origin and use the following representation of the singularity function:

$$(1.6e) \quad \Sigma(z, t) = \int \log \left| \frac{z - \zeta}{z} \right| d\mu_t(\zeta) - \frac{q_\infty(t)}{2\pi} \log |z| = \Sigma_0(z, t) - \frac{Q(t)}{2\pi} \log |z|,$$

where $q_\infty(t)$ is the power of a source at infinity,

$$(1.7) \quad Q(t) = 2\pi \int d\mu_t + q_\infty(t) = -\frac{dS(B(t))}{dt}$$

is the total rate of all sources, and

$$(1.8) \quad \Sigma_0(z, t) = \int \log |z - \zeta| d\mu_t(\zeta)$$

is the *reduced singularity function*. Note that in this case $\Sigma(z, t)$ may have a logarithmic singularity at the origin, unlike the function Σ_0 , which is harmonic within the bubble $B(t)$.

Examples. If the flow is produced by discrete sources/sinks at points $z_j(t)$ with prescribed rates $q_j(t)$, $j = 1, \dots, m$, then the measure μ_t is a linear combination of Dirac measures, and the corresponding singularity functions are

$$(1.9i) \quad \Sigma(z, t) = \frac{1}{2\pi} \sum_{j=1}^m q_j(t) \log |z - z_j(t)|$$

for an interior problem, and

$$(1.9e) \quad \Sigma(z, t) = \frac{1}{2\pi} \left(\sum_{j=1}^m q_j(t) \log \left| \frac{z - z_j(t)}{z} \right| - q_\infty(t) \log |z| \right)$$

for an exterior one.

If the sources are continuously distributed along a piecewise smooth curve $\gamma(t)$ (cf. [R3]), then the measure μ_t is supported on $\gamma(t)$ and is absolutely continuous with respect to the arclength measure $ds(\zeta)$ on $\gamma(t)$, the singularity function being a *single layer potential*:

$$(1.10i) \quad \Sigma(z, t) = \frac{1}{2\pi} \int_{\gamma(t)} \rho(\zeta, t) \log |z - \zeta| ds(\zeta)$$

or

$$(1.10e) \quad \Sigma(z, t) = \frac{1}{2\pi} \int_{\gamma(t)} \rho(\zeta, t) \log \left| \frac{z - \zeta}{z} \right| ds(\zeta).$$

The function $\rho(\zeta, t)$ will be called the *density function* of source distribution. It is well known (cf. [Mir]) that such a potential is continuous near $\gamma(t)$ and has the property of “normal derivative jump”:

$$(1.11) \quad \frac{\partial \Sigma(\zeta, t)}{\partial \nu} \Big|_+ - \frac{\partial \Sigma(\zeta, t)}{\partial \nu} \Big|_- = \rho(\zeta, t),$$

where $+$ and $-$ mean two parts of the neighborhood of ζ separated by $\gamma(t)$, so that the unit vector ν , normal to $\gamma(t)$ at ζ and chosen according to the orientation of $\gamma(t)$, is directed from $-$ to $+$.

In the present paper we will mainly consider the singularity functions of types (1.9) and (1.10). In particular, we will show that the latter class of singularities is large enough to guarantee the solvability of a general inverse problem.

Now suppose we are given a smooth family of domains $D(t)$, $0 \leq t \leq T$, with regular analytic boundaries $\Gamma(t)$. R. Millar [Mil2] and S. Howison [H2] formulated inverse problems as follows: find out whether this family can be obtained as a solution of the direct problem with a physically realistic driving mechanism. Using the formalism developed above, we can make a precise definition: to solve the *interior (exterior) inverse problem* is, given a smooth family of Jordan (i.e. simple closed) analytic curves $\Gamma(t)$, to find for each $t \in [0, T]$ a singularity set $L(t)$ and a singularity function $\Sigma(z, t)$ of type (1.6) with μ_t supported on $L(t)$ such that

$$(1.12) \quad \begin{array}{l} \text{the family of bounded (unbounded) domains } D(t) \\ \text{with } \partial D(t) = \Gamma(t) \text{ is the solution of (1.1)–(1.5).} \end{array}$$

The solution is said to be of *discrete type*, or *d-type*, if the functions $\Sigma(z, t)$ are of type (1.9), with finite singularity sets $L(t)$, and of *continuous type*, or *c-type*, if they are of type (1.10), with a family of curves $\gamma(t)$ as singularity sets.

§2. THE EXISTENCE OF SOLUTIONS

Let Γ be a Jordan analytic curve. We denote by $g(z)$ the *Schwarz function* of Γ [D] defined by

$$\bar{z} = g(z), \quad z \in \Gamma.$$

If Γ is a regular analytic curve, this function exists, is unique and holomorphic in the neighborhood of Γ ; it can be obtained by writing an equation determining Γ in terms of z and \bar{z} and solving it for z .

Examples. If Γ is the circle of radius r centered at the point a , its Schwarz function is

$$(2.1) \quad g(z) = \bar{a} + \frac{r^2}{z - a}.$$

If Γ is the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, the Schwarz function is [D, (5.13)]

$$(2.2) \quad g(z) = \frac{a^2 + b^2}{c^2} z - \frac{2ab}{c^2} (z^2 - c^2)^{1/2},$$

where $c^2 = a^2 - b^2$.

Consider a smooth family $\{\Gamma(t) \mid 0 \leq t \leq T\}$ of Jordan analytic curves, and denote by $g(z, t)$ the Schwarz function of $\Gamma(t)$. Many important features of $\Gamma(t)$ can be expressed in terms of $g(z, t)$ and its derivatives. We will mention the formula for the curvature of $\Gamma(t)$ at z [D, (7.17)]:

$$(2.3) \quad \varkappa(z, t) = \frac{i}{2} \frac{g_{zz}(z, t)}{(g_z(z, t))^{3/2}}, \quad z \in \Gamma(t),$$

and the velocity of $\Gamma(t)$ at z in the outward normal direction [Mil2, (3.4)]:

$$(2.4) \quad v_n(z, t) = \frac{i}{2} \frac{g_t(z, t)}{(g_z(z, t))^{1/2}}, \quad z \in \Gamma(t).$$

Suppose now that, given the family $\{\Gamma(t)\}$ as above, one wishes to find a solution of the inverse problem. Observe that for each t the corresponding flow potential $\Phi(z, t)$ must be a solution of the Cauchy problem for the Laplace equation with boundary conditions defined by (1.3) and (1.4), and from (2.3) and (2.4) it follows that this problem has analytic data. By the classical Cauchy-Kovalevskaya theorem, the unique solution exists in a neighborhood of $\Gamma(t)$. That is, for any t one can choose a Jordan curve $\gamma(t)$ lying close to $\Gamma(t)$ so that the function Φ , obtained as a solution of this Cauchy problem, is harmonic in the annulus between $\Gamma(t)$ and $\gamma(t)$.

Here we indicate two different directions for further development.

Direction 1. One may, following R. Millar, attempt to continue $\Phi(z, t)$ analytically wherever it is possible, thus obtaining the information about its behavior near singularities. For that one may use the representation of the solution of the Cauchy problem in terms of the Schwarz function [Mil1]:

$$(2.5) \quad \Phi(z, t) = \frac{1}{2} \left(\mp \beta \operatorname{Re} \left(\frac{ig_{zz}(z, t)}{(g_z(z, t))^{3/2}} \right) + \frac{1}{2} \int_{g(z, t)}^z g_t(\zeta, t) d\zeta \right).$$

But in general, as was mentioned in [Mil2], one can hardly hope to obtain the singularities like (1.9) and (1.10), and even like (1.6), so this approach rarely produces physically realistic solutions in the meaning formalized in the previous section.

Direction 2. One may as well stop at the contour $\gamma(t)$ and try to find an analytic continuation of Φ with the singularities of c-type at this contour. In other words, find a function $\tilde{\Phi}$ harmonic in the interior (exterior) $\Delta(t)$ of $\gamma(t)$ and continuous near $\gamma(t)$ with the appropriate boundary conditions. Having in mind the continuity property of the potential (1.10), one can easily see that it suffices to solve the following Dirichlet problem for $\tilde{\Phi}$:

$$(2.6) \quad \begin{cases} \Delta\tilde{\Phi}(z, t) = 0, & z \in \Delta(t) \\ \tilde{\Phi}(z, t) = \Phi(z, t), & z \in \gamma(t) \end{cases}$$

(for an exterior problem we imply that $\tilde{\Phi}$ is bounded at infinity).

The solution of (2.6) always exists and is unique. Moreover, the contour $\gamma(t)$ can obviously be chosen to be analytic. The only condition restricting one's choice is that this contour has to bound all the singularities of the function (2.5). If one takes $\gamma(t)$ sufficiently close to the boundary curve $\Gamma(t)$ for all t and the family $\{\Gamma(t)\}$ is smooth, one can obtain a smooth family of analytic curves $\gamma(t)$ as singularity sets.

Choosing the anticlockwise orientation with the corresponding normal direction $\nu(\zeta, t)$ on all the curves $\gamma(t)$ and using (1.11), one can then easily write the expression for the density function ρ :

$$\rho(\zeta, t) = \frac{\partial\Phi(\zeta, t)}{\partial\nu} - \frac{\partial\tilde{\Phi}(\zeta, t)}{\partial\nu},$$

and see that the function $\rho(\zeta, t)$ is differentiable with respect to both ζ and t . We now summarize the results obtained in the following

Theorem 1. *Any inverse problem with the smooth family of analytic curves $\Gamma(t)$, $0 \leq t \leq T$, admits a c-type solution (1.10) with a smooth family $\{\gamma(t)\}$ of Jordan analytic curves and a smooth family $\rho(\zeta, t)$ of real analytic functions on $\gamma(t)$. Any family $\{\gamma(t)\}$ such that the function (2.5) has no singularities in the annulus between $\Gamma(t)$ and $\gamma(t)$ gives such a solution, with density function uniquely determined by the choice of $\gamma(t)$.*

Remark. One can notice the analogy between the proof of Theorem 1 and the solution for the problem of balayage out of a regular domain, cf. [Lan]. However this theorem does not follow directly from the classical balayage theory, since the function (2.5), which has to be swept out of $\Delta(t)$ to its boundary, is not a priori defined in the form of logarithmic or any other potential.

In general singularity sets obtained by the procedure described above may move with time. But Theorem 1 also gives a simple criterion for existence of solutions with fixed location of sources/sinks.

Theorem 2. *The inverse problem (1.12) admits a solution with fixed singularity set $L(t) \equiv L$ iff there exists a Jordan analytic curve γ lying in $\bigcap_{t \in [0, T]} D(t)$ such that for each $t \in [0, T]$ the function (2.5) has no singularities in the annulus between $\Gamma(t)$ and γ .*

Proof. Assume that the family of domains $D(t)$ is the solution of (1.1)–(1.5) with fixed $L(t) \equiv L \subset D(t)$, $0 \leq t \leq T$. The function $\text{dist}(L, \Gamma(t))$ must attain its

minimum on $[0, T]$. Hence there exists $\delta > 0$ such that the δ -neighborhood of L is contained in $D(t)$ for all t . Clearly one can then surround L by an analytic curve γ without leaving this neighborhood. Conversely, if such a curve γ exists, it can be chosen as a singularity set by Theorem 1. \square

§3. SOME GENERALIZATIONS

3.1. The kinetic undercooling effect. There exist different approaches to regularization of direct Hele-Shaw moving boundary problems by adding extra terms, which penalize large curvatures or large normal velocities, to the free boundary condition (see review in [T] and also in [H³OL]). The most acceptable corrections are to include a Gibbs-Thomson term proportional to the curvature like in (1.3) or a kinetic undercooling term proportional to the normal velocity of the moving boundary, namely

$$(3.1) \quad \Phi(z, t) = \mp \varepsilon v_n(z, t), \quad z \in \Gamma(t),$$

where ε is a positive constant.

If in the setting of the direct problem one replaces (1.3) by (3.1), one gets a model in which the surface-tension boundary condition is replaced by a “kinetic undercooling” condition which was proposed for the Stefan problem by [SG]; Hele-Shaw flows with this condition were first studied in [Ro] (see also a review in [S]). It is known [HR] that the direct problem with the kinetic undercooling condition at the moving boundary has for a small period of time a unique classical solution, and the fluid domain remains simply connected and has analytic boundary.

Dealing with inverse problems, one can easily see that equations (1.4) and (3.1) completely determine the Cauchy boundary data for $\Phi(z, t)$; the representation (2.5) transforms (cf. [H2]) into

$$\Phi(z, t) = \frac{1}{2} \left(\mp \varepsilon \operatorname{Re} \left(\frac{ig_t(z, t)}{(g_z(z, t))^{1/2}} \right) + \frac{1}{2} \int_{g(z, t)}^z g_t(\zeta, t) d\zeta \right).$$

Clearly Theorems 1 and 2 remain valid for this case.

3.2. Multiply connected fluid domains. Direct Hele-Shaw problems with “holes” in fluid domains $D(t)$ have been studied since 1950s [ABK]. In the formulation of such problems one assumes that the boundary $\Gamma(t)$ of the fluid domain consists of several connected components $\Gamma_1(t), \dots, \Gamma_m(t)$, and instead of (1.3) one has

$$(3.2) \quad \Phi(z, t) = \Phi_j(t) \mp \beta \varkappa(s, t), \quad z \in \Gamma_j(t),$$

where $\Phi_j(t)$ is the prescribed boundary value of the flow potential at $\Gamma_j(t)$.

As before, the information contained in (1.4) and (3.2) is sufficient to obtain a solution of the Cauchy problem in the neighborhood of each component of the boundary. Then to solve the inverse problem one needs to choose contours $\gamma_j(t)$, $j = 1, \dots, m$, and then consider the Dirichlet problem in the multiply connected domain with boundary $\gamma(t) = \bigcup_{j=1}^m \gamma_j(t)$. Again the solution exists and is unique, so the statements similar to Theorems 1 and 2 are true.

3.3. Multidimensional flows. The eager interest to moving boundary problems was first caused by their applications to the flows in porous media, and the first analytic solutions of direct problems [Gal, K1, P-K1] were based on the two-dimensional Darcy model for seepage flows in homogeneous porous media, mathematically identical to the Hele-Shaw model described in §1. Consider now the same model for flows in \mathbb{R}^n , $n > 2$ (see [BG1]; the case $n = 3$ is especially important for applications). All the ideas developed in §2 can be applied to this model with a slight modification (e.g. with singularity functions built up from the Riesz kernel:

$$\Sigma(\mathbf{x}, t) = \int \frac{1}{|\mathbf{x} - \mathbf{y}|^{n-2}} d\mu_t(\mathbf{y}),$$

and with appropriate multidimensional generalization of the notion of the Schwarz function [Sh]). The existence theorems for the Cauchy and Dirichlet boundary problems hold for any dimensionality, thus any smooth family of analytic $(n - 1)$ -hypersurfaces admits a realization as boundaries of a family of n -dimensional fluid domains, driven by a continuous $(n - 1)$ -dimensional distribution of sources or sinks.

3.4. Flows in inhomogeneous porous media. In applications to underground fluid mechanics, inhomogeneity of the porous medium is often crucial [P-K2]; dealing with the Hele-Shaw cell, one can reflect this factor by varying the width of the cell. In the corresponding mathematical model the condition (1.1) is replaced by a second-order elliptic equation, its analytic consideration being much more elaborate (see e.g. [BG2] or [GS]).

Still, the approach developed above again leads to the proof of solvability of any inverse problem with analytic data. For that one has to replace the logarithmic or Riesz kernel by a fundamental solution of the corresponding equation (the Levi function) and express the singularity function in the form of generalized single layer potential [Mir]. Such a potential is continuous and satisfies the analogue of (1.11), so the density function can be constructed by solving the Cauchy and Dirichlet boundary value problems for that elliptic equation. Moreover, in the two-dimensional case the Schwarz function of $\Gamma(t)$ can still be used, according to [Mil1], for explicit representation of solutions.

§4. THE CAUCHY TRANSFORM AND THE SCHWARZ FUNCTION

It is well known that neglecting surface tension effects at the fluid-gas interface makes life much easier in the investigation of direct problems and generates many nice properties of the model. It is natural to expect something of that kind from inverse problems. We show in §§ 4–7 how these expectations come true. For that we need to review some concepts of the potential theory related to direct problems.

Let Γ be a Jordan analytic curve with the interior D^+ and the exterior D^- . Without loss of generality we can assume that D^+ contains the origin. The *Cauchy transform* h_{D^+} of D^+ is the function

$$h_{D^+}(w) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{D^+} \frac{dx dy}{w - z}, \quad z = x + iy, \quad w \in \mathbb{C}.$$

It is holomorphic within the domain D^- , vanishes at infinity and for large enough $|w|$ has the expansion

$$(4.1) \quad h_{D^+}(w) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{M_n(D^+)}{w^{n+1}},$$

where

$$(4.2) \quad M_n(D^+) \stackrel{\text{def}}{=} \int_{D^+} z^n dx dy, \quad n \geq 0,$$

are the *moments* of D^+ . We will consider the analytic continuation of (4.1) and refer to this analytic function with some singularities in D^+ as the Cauchy transform of D^+ ; this ambiguous notation will not lead to a confusion.

Define also the Cauchy transform of an unbounded domain D^- by

$$(4.3) \quad h_{D^-}(w) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{U_R \cap D^-} \frac{dx dy}{w - z}$$

where U_R is a large disk of the radius R centered at the origin. The integral does not depend on R for $R \rightarrow \infty$, and the function (4.3) is holomorphic in D^+ . Moreover, in some disk centered at the origin one has

$$(4.4) \quad h_{D^-}(w) = -\frac{1}{\pi} \sum_{n=1}^{\infty} M_n(D^-) w^{n-1},$$

where

$$M_n(D^-) \stackrel{\text{def}}{=} \int_{U_R \cap D^-} z^{-n} dx dy, \quad n \geq 1,$$

are the *moments* of the unbounded domain D^- . The expansion (4.4) does not contain the value of the 0th moment of D^- , defined by $M_0(D^-) = -S(D^+)$. Again the analytic continuation of (4.4) with possible singularities in D^- will be referred to as the Cauchy transform of D^- .

These functions are related to the Schwarz function $g(z)$ of Γ in a simple way: the equality

$$(4.5) \quad h_{D^+}(z) + h_{D^-}(z) = g(z)$$

holds when $z \in \Gamma$, and hence wherever all the above functions can be analytically continued. Thus one has a decomposition of the Schwarz function $g(z)$ as a sum of two analytic functions, one of which is holomorphic in the exterior, and the other in the interior of Γ .

Examples. Formulae (2.1) and (2.2) together with (4.5) immediately allow one to write the Cauchy transforms of the interior and exterior of the circle $\{|z - a| = r\}$:

$$(4.6) \quad h_{D^-}(z) = \bar{a}, \quad h_{D^+}(z) = \frac{r^2}{z - a},$$

and the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$:

$$(4.7) \quad h_{D^-}(z) = \frac{a-b}{a+b}z, \quad h_{D^+}(z) = \frac{2ab}{c^2}((z^2 - c^2)^{1/2} - z).$$

§5. THE CAUCHY TRANSFORM AND THE SINGULARITY FUNCTION

The following theorem shows how the concepts discussed above are related to Hele-Shaw problems. As before, $g(z, t)$ denotes the Schwarz function of $\Gamma(t) = \partial D(t)$ and $F(z, t)$ is the complex potential of fluid flow.

Theorem 3. *If the family of domains $D(t)$ is the solution of (1.1)–(1.5) with $\beta = 0$, the equality*

$$(5.1) \quad \frac{\partial F(z, t)}{\partial z} = \frac{1}{2} \frac{\partial g(z, t)}{\partial t}$$

holds wherever both parts can be analytically continued.

This fact is implicit in the work of Richardson [R1–2] and was first obtained by Lacey [Lac]; it also follows directly from (2.5), and a simple proof of this statement is given in [H2]. Still for the reader's convenience we include a short geometric

*Proof*². Note that $\overline{g(z, t)}$ is the Schwarz reflection of a point z along $\Gamma(t)$ [D, Chapter 6]. Hence $\partial \overline{g(z, t)} / \partial t$ is the velocity of the *mirror image* of z under this reflection, which for $z \in \Gamma(t)$ is twice the velocity of $\Gamma(t)$ at z . From (1.4) and the zero-surface-tension boundary condition it follows that the velocity of the boundary at z equals $\overline{\partial F(z, t) / \partial z}$; after conjugation and analytic continuation one gets (5.1). \square

The geometric interpretation of $\partial \overline{g(z, t)} / \partial t$ mentioned above also makes obvious the following statement, which will help in studying inverse problems.

Theorem 4. *For any smooth family of analytic curves $\Gamma(t)$ there exist functions $F(z, t)$ holomorphic in a neighborhood of $\Gamma(t)$ and satisfying (1.3) and (1.4) with $\beta = 0$ and $\Phi = \operatorname{Re} F$.*

Proof. One can consider functions

$$F(z, t) = \int_{z_0}^z \frac{1}{2} \frac{\partial g(\zeta, t)}{\partial t} d\zeta + C.$$

For any such function and $z \in \Gamma(t)$ the gradient of its real part

$$\nabla \Phi(z, t) = \overline{\partial F(z, t) / \partial z} = \frac{1}{2} \overline{\partial g(\zeta, t) / \partial t}$$

will be perpendicular to $\Gamma(t)$ and equal $v_n(z, t)$, so one needs only to choose the constant C so that (1.3) holds. \square

Theorem 3 clearly provides a starting point for approaching Hele-Shaw problems (cf. [H2]), since it connects physical characteristics of the flow with geometric properties of the moving boundary. However, it cannot be applied as written to direct problems, since the function $F(z, t)$, and hence $\partial F(z, t) / \partial z$, is unknown – one is only aware of its singularities inside the fluid domain. But in many cases the information about the singularities of the Schwarz function of $\Gamma(t)$ inside $D(t)$ turns out to be sufficient to reconstruct $D(t)$ (this was first stated in an implicit form by Kufarev [K], see also [R1–3, Lac, H1–2]). In other words, one only needs to worry about singularities of both parts of (5.1). Using the notion of the singularity function, we can formalize this discussion as follows.

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Theorem 5i. *If a family of bounded domains $D(t)$ is the solution of (1.1)–(1.5) with the zero-surface-tension boundary condition, then*

$$(5.2) \quad \frac{\partial S(z, t)}{\partial z} = \frac{1}{2} \frac{\partial h_{D(t)}(z)}{\partial t}$$

(here S is the complex singularity function) holds wherever both parts can be analytically continued.

Proof. One has, by (1.2C):

$$(5.3) \quad F(z, t) = S(z, t) + G(z, t),$$

where $G(z, t)$ is holomorphic in $D(t)$. Differentiation of (5.3) and (4.5) with $D^+ = D(t)$ and substitution in (5.1) gives

$$(5.4) \quad \frac{\partial S(z, t)}{\partial z} - \frac{1}{2} \frac{\partial h_{D(t)}(z)}{\partial t} = \frac{\partial G(z, t)}{\partial z} - \frac{1}{2} \frac{\partial h_{D^-(t)}(z)}{\partial t}$$

(here $D^-(t)$ denotes the exterior of $\Gamma(t)$). The l.h.s. of (5.4) is holomorphic outside $D(t)$ and vanishes at ∞ , the r.h.s. is holomorphic in $D(t)$. Therefore, by Liouville's theorem, both must be identically zero. \square

It is not difficult to modify this result for the exterior problem.

Theorem 5e. *If a family of unbounded domains $D(t)$ with bounded complements $B(t)$ is the solution of (1.1)–(1.5) with the zero-surface-tension boundary condition, then*

$$(5.5) \quad \frac{\partial S_0(z, t)}{\partial z} = \frac{1}{2} \frac{\partial h_{D(t)}(z)}{\partial t}$$

(here S_0 is the reduced complex singularity function, with $\operatorname{Re} S_0 = \Sigma_0$) holds wherever both parts can be analytically continued.

Proof. In this case one has, by (1.2C) and (1.8),

$$F(z, t) = S_0(z, t) - \frac{Q(t)}{2\pi} \log z + G(z, t),$$

where G is holomorphic in $D(t)$; thus the same Liouville argument can be applied to the equality

$$\frac{\partial S_0(z, t)}{\partial z} - \frac{1}{2} \frac{\partial h_{D(t)}(z)}{\partial t} = \frac{\partial G(z, t)}{\partial z} - \frac{Q(t)}{2\pi z} + \frac{1}{2} \frac{\partial h_{B(t)}(z)}{\partial t},$$

showing both sides to be identically zero. \square

Remark. Using expressions (1.6) for the singularity functions, we can restate both of these theorems in the form

$$(5.6) \quad \int \frac{d\mu_t(\zeta)}{z - \zeta} = \frac{1}{2} \frac{\partial h_{D(t)}(z)}{\partial t},$$

which connects the Cauchy transform of the domain $D(t)$ and the Cauchy transform [Gar] of the measure μ_t determining the distribution of sources.

When solving the direct problem (1.1)–(1.5), equations (5.2) and (5.5) help to calculate Cauchy transforms of unknown domains $D(t)$, which, as follows from (4.5), accumulate all the information about singularities of the Schwarz function of $\Gamma(t)$ in $D(t)$ (in exterior problems, to reconstruct $D(t)$ one needs also to know the bubble area, which can be found from (1.7)). But these results can be also applied to inverse problems, and, as an immediate application, one has the following criterion.

Theorem 6. *The interior (exterior) inverse problem (1.12) has a d-type solution iff for all $t \in [0, T]$ the functions $\partial h_{D(t)}(z)/\partial t$ are rational and have only poles of first order with real residues. If a d-type solution exists, it is unique.*

Proof. If a d-type solution of an interior problem exists, the complex singularity function has the form

$$S(z, t) = \frac{1}{2\pi} \sum_{j=1}^m q_j(t) \log(z - z_j(t)),$$

and the claim follows from Theorem 5. Conversely, if $\partial h_{D(t)}(z)/\partial t$ looks like

$$\frac{\partial h_{D(t)}(z)}{\partial t} = \frac{1}{\pi} \sum_{j=1}^m \frac{q_j(t)}{z - z_j(t)}, \quad q_j \in \mathbb{R},$$

then for all t the functions $F(z, t)$ constructed in Theorem 4 in addition to (1.3) and (1.4) satisfy (1.1) and (1.2) with $L(t) = \{z_1, \dots, z_m\}$ and $\Sigma(z, t)$ of the form (1.9). Hence a d-type solution exists, and the sources/sinks z_j and rates q_j are uniquely determined. The same proof works for exterior problems if one substitutes S_0 for S and allows a source or sink at infinity. \square

§6. REDUCTION TO HILBERT BOUNDARY VALUE PROBLEM

The purpose of this section is to show how Theorem 5 helps one to make easier the construction of c-type solutions described in §2. We recall that a solution of the interior (exterior) inverse problem (1.12) can be given in the form

$$(6.1) \quad S(z, t) \quad (\text{or } S_0(z, t)) \quad = \frac{1}{2\pi} \int_{\gamma(t)} \rho(\zeta, t) \log(z - \zeta) ds(\zeta),$$

where $\gamma(t)$ is an analytic curve lying in $D(t)$ such that $\partial g(z, t)/\partial t$ has no singularities in the annulus between $\Gamma(t)$ and $\gamma(t)$. If $\nu(\zeta, t)$ denotes the outward unit vector normal to $\gamma(t)$ at ζ , then differentiation of (6.1) yields

$$(6.2) \quad \frac{\partial S(z, t)}{\partial z} \text{ (or } \frac{\partial S_0(z, t)}{\partial z} \text{)} = \frac{1}{2\pi} \int_{\gamma(t)} \rho(\zeta, t) \frac{ds(\zeta)}{z - \zeta} = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{\rho(\zeta, t)}{\nu(\zeta, t)} \frac{d\zeta}{z - \zeta}.$$

Recall that the function $\partial h_{D(t)}(z)/\partial t$ is holomorphic outside $\gamma(t)$ and vanishes at infinity when $D(t)$ is bounded, and is holomorphic inside $\gamma(t)$ in the exterior case. Let $\Delta(t)$ denote the interior (exterior) of $\gamma(t)$. Then one has, by Cauchy integral formula,

$$(6.3) \quad \frac{\partial h_{D(t)}(z)}{\partial t} = \pm \frac{1}{2\pi i} \int_{\gamma(t)} \frac{\partial h_{D(t)}(\zeta)}{\partial t} \frac{d\zeta}{z - \zeta}, \quad z \notin \Delta(t)$$

(as before, the upper sign is chosen for the interior problems).

Substituting (6.2) and (6.3) into (5.2) or (5.5) (Theorem 5), one can conclude that

$$\frac{1}{2\pi i} \int_{\gamma(t)} \left(\frac{\rho(\zeta, t)}{\nu(\zeta, t)} \mp \frac{1}{2} \frac{\partial h_{D(t)}(\zeta)}{\partial t} \right) \frac{d\zeta}{z - \zeta} = 0 \quad \forall z \notin \Delta(t).$$

Hence the function

$$\varphi(\zeta, t) \stackrel{\text{def}}{=} 2 \frac{\rho(\zeta, t)}{\nu(\zeta, t)} \mp \frac{\partial h_{D(t)}(\zeta)}{\partial t}$$

admits analytic continuation in $\Delta(t)$. Moreover, the imaginary part of

$$(6.4) \quad \rho(\zeta, t) = \frac{1}{2} \left(\pm \frac{\partial h_{D(t)}(\zeta)}{\partial t} + \varphi(\zeta, t) \right) \nu(\zeta, t)$$

must vanish for $\zeta \in \gamma(t)$.

Let $\alpha(\zeta, t)$ stand for the argument of $\nu(\zeta, t)$, and denote

$$c(\zeta, t) \stackrel{\text{def}}{=} \text{Im} \left(\frac{\partial h_{D(t)}(\zeta)}{\partial t} \nu(\zeta, t) \right).$$

Then the equality

$$(6.5) \quad \text{Im} \left(\left(\pm \frac{\partial h_{D(t)}(\zeta)}{\partial t} + \varphi(\zeta, t) \right) \nu(\zeta, t) \right) = 0$$

is equivalent to the condition

$$(6.6) \quad \sin \alpha(\zeta, t) \text{Re} \varphi(\zeta, t) + \cos \alpha(\zeta, t) \text{Im} \varphi(\zeta, t) = \pm c(\zeta, t), \quad \zeta \in \gamma(t).$$

The problem of finding a function φ such that the conditions

$$(6.7) \quad \varphi(\zeta, t) \text{ is holomorphic in } \Delta(t)$$

and (6.6) hold is called the *Hilbert boundary value problem* [Gak]; its *index*

$$\chi \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma(t)} (\sin \alpha(\zeta, t) + i \cos \alpha(\zeta, t)) d\zeta$$

is equal to -1 . Here we must consider the two cases separately.

Interior problems. Since $\chi = -1$, the problem (6.6)–(6.7) cannot have more than one solution [Gak, 29.2]. On the other hand, if $\varphi(\zeta, t)$ is the solution of (6.6)–(6.7) and $\rho(\zeta, t)$ is given by (6.4), then the equality

$$(6.8i) \quad \frac{\partial g(z, t)}{\partial t} = \frac{1}{\pi} \int_{\gamma(t)} \rho(\zeta, t) \frac{ds(\zeta)}{z - \zeta} + \frac{\partial h_{D^-(t)}(z)}{\partial t} - \varphi(z, t)$$

takes place in the exterior of $\gamma(t)$. Thus the function $F(z, t)$ constructed from the r.h.s. of (6.8i) as in Theorem 4 will satisfy (1.1)–(1.4) with singularity sets $\gamma(t)$ and singularity functions (6.1). We have proved the following

Theorem 7i. *The function (6.1) is a solution of the interior inverse problem (1.12) with the zero-surface-tension boundary condition iff the density function $\rho(\zeta, t)$ is given by (6.4), where $\varphi(\zeta, t)$ is the solution of the boundary value problem (6.6)–(6.7).*

Exterior problems. The exterior Hilbert boundary value problem with index $\chi = -1$ must have a one-parameter family of solutions [Gak, 29.4]. It is inevitable, since in this case $h_{D(t)}$ does not contain all the information needed to reconstruct $D(t)$. Each of those solutions will do if one allows a source or sink at infinity; and each of them will make the equality

$$(6.8e) \quad \frac{\partial g(z, t)}{\partial t} = -\frac{1}{\pi} \int_{\gamma(t)} \rho(\zeta, t) \frac{ds(\zeta)}{z - \zeta} + \frac{\partial h_{B(t)}(z)}{\partial t} + \varphi(z, t)$$

hold in the interior of $\gamma(t)$. To obtain a pure c-type solution one should derive the additional condition from the calculation of the bubble area:

$$\begin{aligned} -\frac{dS(B(t))}{dt} &= \int_{\gamma(t)} \rho(\zeta, t) ds(\zeta) = \frac{1}{2} \int_{\gamma(t)} \left(\frac{\partial h_{D(t)}(\zeta)}{\partial t} + \varphi(\zeta, t) \right) \nu(\zeta, t) ds(\zeta) \\ &= \frac{1}{2i} \int_{\gamma(t)} \left(\frac{\partial h_{D(t)}(\zeta)}{\partial t} + \varphi(\zeta, t) \right) d\zeta = \frac{1}{2i} \int_{\gamma(t)} \varphi(\zeta, t) d\zeta. \end{aligned}$$

The last equality is true since $\partial h_{D(t)}(\zeta)/\partial t$ is holomorphic in the interior of $\gamma(t)$. Thus the following statement holds.

Theorem 7e. *The function*

$$(6.9) \quad S(z, t) = \frac{1}{2\pi} \int_{\gamma(t)} \rho(\zeta, t) \log \frac{z - \zeta}{z} ds(\zeta),$$

is a solution of the exterior inverse problem (1.12) with the zero-surface-tension boundary condition iff the density function $\rho(\zeta, t)$ is given by (6.4), where $\varphi(\zeta, t)$ is the solution of the exterior Hilbert boundary value problem (6.6)–(6.7) with the condition

$$(6.10) \quad -\frac{1}{\pi} \frac{dS(B(t))}{dt} = \frac{1}{2\pi i} \int_{\gamma(t)} \varphi(\zeta, t) d\zeta = \lim_{\zeta \rightarrow \infty} \zeta \varphi(\zeta, t).$$

Unfortunately, in general case there is no simple procedure for solving Hilbert boundary value problems. In the next section we describe a special case when Theorem 7 helps to easily construct explicit solutions. But it also can help one to move a little further in the direction indicated in Theorem 2.

Theorem 8. *The interior (exterior) inverse problem (1.12) with $\beta = 0$ admits a solution with fixed singularity set $L(t) \equiv L$ and singularity function $\Sigma(z, t) \equiv \Sigma(z)$ iff*

(i) *there exists a simple closed analytic curve γ lying in $\bigcap_{t \in [0, T]} D(t)$ such that for each $t \in [0, T]$ the function $\partial g(z, t)/\partial t$ has no singularities in the annulus between $\Gamma(t)$ and γ , and*

(ii) *the function $\partial h_{D(t)}(z)/\partial t$ (with the quantity $dS(B(t))/dt$) is independent on time.*

Proof. If the singularities do not change with time, the function $\partial S(z, t)/\partial z$ ($\partial S_0(z, t)/\partial z$ together with $dS(B(t))/dt$) is independent on time, and the claim follows from Theorem 5. Conversely, if the contour γ is chosen as a singularity set, then the data for the Hilbert problem (and the condition (6.10)) does not depend on time, and neither does the function $\rho(\zeta, t) \equiv \rho(\zeta)$, expressible in terms of $\partial h_{D(t)}(\zeta)/\partial t$ and the solution of that problem. \square

§7. EXPLICIT SOLUTION OF NICE PROBLEMS

Suppose a smooth family of Jordan analytic curves $\Gamma(t)$, $0 \leq t \leq T$, is the data for an interior (exterior) inverse problem (1.12) with the zero-surface-tension boundary condition. Call this family (and the corresponding problem) *nice* if for any $t \in [0, T]$ all the singularities of $\partial h_{D(t)}(\zeta)/\partial t$ are contained in the interior (exterior) of some circle of radius $R(t)$ centered at a point $a(t)$ and lying inside (outside) $\Gamma(t)$. If one can choose $a(t) = 0$ for all t , the problem, and the family of curves as well, will be called *extremely nice*.

Let $D(t)$ be a nice family of domains. Then it can be generated by a distribution of sources along the circle $\gamma(t) \stackrel{\text{def}}{=} \{|z - a(t)| = R(t)\}$; to calculate the density function it suffices to solve the appropriate Hilbert boundary value problems. When the boundary curve is a circle, the solution of (6.6)–(6.7) can be expressed in terms of the Schwarz integral [Gak, 29.3]; however the specific form of the condition (6.6) allows one to find an easier way. For the sake of the reader's convenience, in the following calculations we omit the variable t , although all the functions and sets still may be time-dependent. We also employ the notation $h_t(z) = \partial h_{D(t)}(z)/\partial t$.

To construct a solution of the Hilbert boundary value problem, we observe that in the nice case

$$(7.1) \quad \nu(\zeta) = \frac{\zeta - a}{R},$$

and rewrite (6.5) in the form

$$(7.2) \quad \overline{h_t(\zeta)(\zeta - a)} \mp \varphi(\zeta)(\zeta - a) \in \mathbb{R}, \quad \zeta \in \gamma.$$

But clearly $\overline{\zeta - a} = \frac{R^2}{\zeta - a}$ when $\zeta \in \gamma$, so (7.2) is equivalent to

$$(7.3) \quad \pm \frac{R^2}{\zeta - a} h_t^* \left(\bar{a} + \frac{R^2}{\zeta - a} \right) = \varphi(\zeta)(\zeta - a) + r(\zeta), \quad \zeta \in \gamma,$$

where $h_t^*(z) \stackrel{\text{def}}{=} \overline{h_t(\bar{z})}$ is the function with conjugate Taylor coefficients, and $r(\zeta)$ is some real function on γ .

Here again the separate consideration of the two cases is needed.

Interior problems. Since the function h_t is holomorphic outside γ , the l.h.s. of (7.3) is holomorphic inside γ except maybe the point a ; moreover, one can use the expansion (4.2) to verify that it tends to S_t/π as $\zeta \rightarrow a$ (here $S_t \equiv dS(D(t))/dt$). Therefore to satisfy (7.3) one can choose $r(\zeta) \equiv \text{const} = S_t/\pi$.

Exterior problems. The regularity of h_t inside γ implies that the l.h.s. of (7.3) is holomorphic outside γ and vanishes at infinity. Hence one can substitute any real constant instead of $r(\zeta)$ to obtain a solution of (6.6)–(6.7). However, the condition (6.9) still has to be satisfied, for which one again needs this constant to equal S_t/π , where S_t stands now for $dS(B(t))/dt$.

Thus in both cases the function $\varphi(\zeta)$ can be written in the form

$$\varphi(\zeta) = \pm \frac{R^2}{(\zeta - a)^2} h_t^* \left(\bar{a} + \frac{R^2}{\zeta - a} \right) - \frac{S_t}{\pi(\zeta - a)}.$$

Hence, using (6.4) and (7.1), one can express the density function as follows:
(7.4)

$$\rho(\zeta) = \frac{\pm h_t(\zeta)(\zeta - a) \pm \overline{h_t(\zeta)(\zeta - a)} - S_t/\pi}{2R} = \frac{\pm \text{Re}(h_t(\zeta)(\zeta - a)) - S_t/2\pi}{R},$$

or

$$(7.5) \quad \rho(\theta) = \pm \text{Re}(e^{i\theta} h_t(a + Re^{i\theta})) - \frac{S_t}{2\pi R}$$

if one uses the natural parametrization of γ .

Finally, for extremely nice problems it is easy to obtain an extremely nice formula for $\rho(\theta)$ in terms of moments of $D(t)$. We denote

$$A_n + iB_n = \frac{d}{dt} M_n(D(t)), \quad A_n, B_n \in \mathbb{R}$$

(observe that $B_0 = 0$ and $A_0 = \pm S_t$ in our notation). Then short calculation based on (4.1) and (4.4) gives the equivalent version of (7.5) with $a = 0$:

$$(7.6) \quad \rho(\theta) = \frac{1}{\pi R} \left(\frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n R^{\mp n} \cos n\theta \pm B_n R^{\mp n} \sin n\theta) \right),$$

i.e. a Fourier series for the density function.

We now specify several particular cases where the above formulae can be applied.

7.1. Polynomial domains. Let us say that a bounded domain D containing the origin is *polynomial of degree n* if the conformal mapping from the unit disk onto D leaving the origin fixed is a polynomial of degree n . Similarly, a domain D with bounded complement is *polynomial of degree n* , if the conformal mapping from the unit disk onto D sending the origin to infinity has the form $p(\zeta)/\zeta$, where $p(\zeta)$ is a polynomial of degree n .

Galín [Gal] observed that a polynomial domain remains polynomial of the same degree during the evolution under the Hele-Shaw flow. In terms of the moments of domains it means [R1] that all but finite number of them are equal to zero. Thus the Cauchy transform of a polynomial domain has just one singularity, either at zero or at infinity. This means that any smooth family of polynomial domains is extremely nice, hence can be produced by the distribution of sources according to (7.6). When the degrees of all the domains in the family are uniformly bounded, the density function (7.6) is a trigonometric polynomial. For example, for a family of circular bubbles $\{|z - a(t)| > r(t)\}$ substitution of (4.6) into (7.5) gives

$$\rho(\theta, t) = -\frac{r(t)r'(t)}{R} - (\operatorname{Re} a'(t) \cos \theta + \operatorname{Im} a'(t) \sin \theta),$$

and for a family of elliptic bubbles $\{\frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} > 1\}$ one can, using (4.7), write the density function in the form

$$\rho(\theta, t) = -\frac{1}{2R} \frac{d}{dt} (a^2(t) + b^2(t)) - R \frac{d}{dt} \left(\frac{a(t) - b(t)}{a(t) + b(t)} \right) \cos 2\theta,$$

where R is large enough for the circle of radius R to accommodate all the curves $\Gamma(t)$, $t \in [0, T]$.

7.2. Balayage of logarithmic singularities. Let D be any bounded domain containing the origin, and suppose that the family $\{D(t)\}$ is obtained from D by injection/suction of fluid at the origin. Then by Theorem 5i the functions h_t have the only singularity at the origin, hence the family $\{D(t)\}$ is extremely nice. Moreover, since all the moments except M_0 are invariant ([R1], and also (5.2) and (4.1)), the density function (7.6) defines uniform distribution of sources along any circle centered at zero and contained in D . (This proves once again that uniform injection of fluid through a circular opening can serve as a good approximation of a point source/sink.) Similarly, any family of bubbles obtained from the initial one by the action of a source/sink at infinity is extremely nice, and can be realized via uniform distribution of sources along some large circle.

More generally, fix an interior (exterior) Δ of some circle centered at the origin and contained in a bounded (unbounded) domain D , and suppose the family $\{D(t)\}$ is obtained from D by injection of fluid through some sources lying in Δ . Then this family is clearly extremely nice. Let the measure μ_t define the distribution of sources. Then substitution of (5.6) into (7.5) yields the solution of the balayage problem for the circle (see Remark after Theorem 1). In particular, if Δ is a disk

$\{|z| < R\}$, and the injection through a single point a , $0 < a < R$ with the fixed rate $q = 1$ takes place, then (7.5) expresses the density function in the form of a Poisson kernel:

$$\rho(\theta) = \frac{1}{2\pi R} \frac{R^2 - a^2}{R^2 - 2aR \cos \theta + a^2},$$

the measure $\rho(\theta)R d\theta$ being the harmonic measure at the point a .

7.3. Translation of the fluid domain. Let D be a bounded domain with the Cauchy transform $h(z)$, and Δ a disk $\{|z| < R\}$. Denote by W the set of complex numbers w such that h is holomorphic in the set $(D + w) \setminus \Delta$. Then clearly the family of domains $D(t) = D + f(t)$, where f is any smooth function from $[0, T]$ to W , is extremely nice. The Cauchy transform of $D(t)$ equals

$$(7.7i) \quad h_{D(t)}(z) = h(z - f(t)).$$

One can then substitute (7.7i) and $S_t \equiv 0$ in (7.4) to find that sources can be distributed along $\{|z| = R\}$ with the density function

$$\rho(\zeta, t) = -\frac{1}{R} \operatorname{Re} (\zeta f'(t) h'(\zeta - f(t))).$$

For example, one can choose $D = \{z < 2R\}$, $W = \Delta$ and $f(t) = t$ for $t \in [-R + \varepsilon, R - \varepsilon]$, $\varepsilon > 0$, and, using (4.6), generate the translation of the disk D with the distribution

$$\rho(\zeta, t) = 4R \cdot \operatorname{Re} \left(\frac{\zeta}{(\zeta - t)^2} \right).$$

In the same fashion one can treat the translation of any bubble, with the formula (7.7i) replaced by

$$(7.7e) \quad h_{D(t)}(z) = h(z - f(t)) + \overline{f(t)}.$$

7.4. Rotation of the fluid domain. Let D be a bounded (unbounded) domain such that all the singularities of its Cauchy transform $h(z)$ are contained inside (outside) some circle of radius R centered at the origin, and this circle is contained in D . Let $D(t)$ be the result of rotation of D : $D(t) = e^{it}D$. Then the Cauchy transform of $D(t)$ is

$$h_{D(t)}(z) = e^{-it} h(ze^{-it}),$$

and its derivative with respect to t is

$$h_t(z, t) = -ie^{-it} (h(ze^{-it}) + ze^{-it} h'(ze^{-it})).$$

In this case again $S_t \equiv 0$, and from (7.5) it follows, with $\gamma = \{|z| = R\}$, that

$$\rho(\theta, t) = \pm \operatorname{Im} \left(e^{i(\theta-t)} (h(Re^{i(\theta-t)}) + Re^{i(\theta-t)} h'(Re^{i(\theta-t)})) \right) = \rho_0(\theta - t),$$

where $\rho_0(\theta)$ is the density function for $t = 0$:

$$(7.8) \quad \rho_0(\zeta) = \pm \frac{1}{R} \operatorname{Im} \left(\zeta \frac{d}{d\zeta} (\zeta h(\zeta)) \right).$$

Hence one can realize the initial source distribution according to (7.8) and then rotate the circle bearing these sources, generating the rotation of the whole fluid domain.

Similarly one can generate families $D(t) = e^{\lambda t} D$ for arbitrary $\lambda \in \mathbb{C}$ (i.e. combine rotation of D with contraction or dilation). It suffices to replace i by λ in the above formulae and add an extra term reflecting the area changes.

CONCLUSION

Hele-Shaw moving boundary problems have a reach history (see [H2] for a review). Most of the work in this field has been related to direct problems, where the driving mechanism (i.e. singularities of the fluid velocity field) is prescribed, and the evolution of initial fluid domain is defined as the solution of the problem. The main issue discussed in this paper is finding such a driving mechanism when the evolution of a fluid domain is given.

As it was earlier mentioned by Millar [Mil2], a discrete driving mechanism can rarely produce physically realistic solutions. However, a generalization of the concept of driving mechanism for sources/sinks distributed in accordance with more general measure supported within the fluid domain allows one to prove a global solvability of inverse Hele-Shaw problems with analytic data³. More precisely, we prove that any inverse problem with a compact smooth family $\{D(t)\}$ of domains with analytic boundaries as the initial data admits solutions given by continuous distribution of sources along closed analytic curves. Moreover, in the zero-surface-tension case one can find this distribution via solving a Hilbert boundary value problem for analytic functions. In many cases the latter problem can be solved explicitly, which leads to explicit expressions for the source distribution.

An analogous approach can be carried out using other types of singularities, such as multipoles (e.g. double layer potentials) or measures of the form $f(z) dx dy$ with $f(z) > 0$ for all $z \in D(t)$ (these arise e.g. in squeeze film problems, cf. [HLO]).

Since inverse problems described above always have solutions, one may want to impose additional restrictions. For example, one may restrict oneself to finding a driving mechanism with a *simply connected* singularity set $L(t)$ or with a *circular* singularity set $L(t) = \{|z| = R\}$ – then clearly there exist unsolvable problems.

³Recall that for direct problems there is only local solvability in time [VK, RW] for the zero-surface-tension case, and in general the problem is ill-posed in a variety of respects, see a discussion in [H³OL].

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