

**ON EFFECTIVE EQUIDISTRIBUTION OF  
EXPANDING TRANSLATES OF CERTAIN  
ORBITS IN THE SPACE OF LATTICES**

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ABSTRACT. We prove an effective version of a result obtained earlier by Kleinbock and Weiss [KW] on equidistribution of expanding translates of orbits of horospherical subgroups in the space of lattices.

1. INTRODUCTION

The motivation for this work is a result obtained recently in [KW]. Fix  $m, n \in \mathbb{N}$ , set  $k = m + n$  and let

$$(1.1) \quad G = \mathrm{SL}_k(\mathbb{R}), \quad \Gamma = \mathrm{SL}_k(\mathbb{Z}), \quad u_Y = \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix}, \quad H = \{u_Y \mid Y \in M_{m,n}\},$$

where  $M_{m,n}$  stands for the space of  $m \times n$  matrices with real entries. Then  $H$  is a unipotent abelian subgroup of  $G$  which is *expanding horospherical* with respect to

$$(1.2) \quad g_t = \mathrm{diag}(e^{t/m}, \dots, e^{t/m}, e^{-t/n}, \dots, e^{-t/n}), \quad t > 0.$$

The latter, by definition, means that the Lie algebra of  $H$  is the span of eigenspaces of  $\mathrm{Ad}(g_t)$ ,  $t > 0$ , with eigenvalues bigger than 1 in absolute value.

The space  $X \stackrel{\mathrm{def}}{=} G/\Gamma$  can be identified with the space of unimodular lattices in  $\mathbb{R}^k$ , on which  $G$  acts by left translations. Denote by  $\pi$  the natural projection  $G \rightarrow X$ ,  $g \mapsto g\Gamma$ , and for any  $z \in X$  let  $\pi_z : G \rightarrow X$  be defined by  $\pi_z(g) = gz$ . Also denote by  $\bar{\mu}$  the  $G$ -invariant probability measure on  $X$  and by  $\mu$  the Haar measure on  $G$  such that  $\pi_*\mu = \bar{\mu}$ . Fix a Haar measure  $\nu$  on  $H$ . Note that the  $H$ -orbit foliation is unstable with respect to the action of  $g_t$ ,  $t > 0$ . It is well known that for any Borel probability measure  $\nu'$  on  $H$  absolutely continuous with respect to  $\nu$  and for any  $z \in X$ ,  $g_t$ -translates of  $(\pi_z)_*\nu'$  become equidistributed, that is, weak-\* converge to  $\bar{\mu}$  as  $t \rightarrow \infty$ . An effective version of this statement was obtained in [KM1, Proposition 2.4.8]. In order to state that result, it will be convenient to introduce the following notation: for  $f \in L^1(H, \nu)$ , a bounded continuous function  $\psi$  on  $X$ ,  $z \in X$  and  $g \in G$  define

$$I_{f,\psi}(g, z) \stackrel{\mathrm{def}}{=} \int_H f(h)\psi(g_thz) d\nu(h).$$

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In other words,  $I_{f,\psi}(g, z)$  is the result of evaluation of the  $g$ -translate of  $(\pi_z)_*\nu'$  at  $\psi$ , where  $d\nu' = f d\nu$ . Then equidistribution of  $g_t$ -translates of  $(\pi_z)_*\nu'$  amounts to the convergence of  $I_{f,\psi}(g_t, z)$  to  $\int_H f \cdot \int_X \psi$  as  $t \rightarrow \infty$  (unless it causes confusion, we will omit measures in the integration notation for the sake of brevity).

The following is a slightly simplified form of [KM1, Proposition 2.4.8]:

**Theorem 1.1.** *There exists  $\gamma > 0$  such that for any  $f \in C_{comp}^\infty(H)$ ,  $\psi \in C_{comp}^\infty(X)$  and for any compact subset  $L$  of  $X$  there exists a constant  $C = C(f, \psi, L)$  such that for all  $z \in L$  and any  $t \geq 0$*

$$(1.3) \quad \left| I_{f,\psi}(g_t, z) - \int_H f \int_X \psi \right| \leq C e^{-\gamma t}.$$

The proof used the exponential decay of correlations of the  $G$ -action on  $X$  (called ‘condition (EM)’ in [KM1]). See §2 for more detail.

Motivated by some questions in simultaneous Diophantine approximation, the first named author and Barak Weiss considered translates of  $H$ -orbits on  $X$  by diagonal elements of  $G$  other than  $g_t$ . Specifically, following [KW], let us denote by  $\mathfrak{a}^+$  the set of  $k$ -tuples  $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$  such that

$$t_1, \dots, t_k > 0 \quad \text{and} \quad \sum_{i=1}^m t_i = \sum_{j=1}^n t_{m+j},$$

and for  $\mathbf{t} \in \mathfrak{a}_+$  define

$$(1.4) \quad g_{\mathbf{t}} \stackrel{\text{def}}{=} \text{diag}(e^{t_1}, \dots, e^{t_m}, e^{-t_{m+1}}, \dots, e^{-t_k}) \in G$$

and

$$[\mathbf{t}] \stackrel{\text{def}}{=} \min_{i=1, \dots, k} t_i$$

(the latter, roughly speaking, measures the distance between  $\mathbf{t}$  and the walls of the cone  $\mathfrak{a}^+ \subset \mathbb{R}^k$ ).

The theorem below is a reformulation of [KW, Theorem 2.2]:

**Theorem 1.2.** *For any  $f \in L^1(H, \nu)$ , any continuous compactly supported  $\psi : X \rightarrow \mathbb{R}$ , any compact subset  $L$  of  $X$  and any  $\varepsilon > 0$  there exists  $T > 0$  such that*

$$\left| I_{f,\psi}(g_{\mathbf{t}}, z) - \int_H f \int_X \psi \right| < \varepsilon$$

for all  $z \in L$  and  $\mathbf{t} \in \mathfrak{a}^+$ ,  $[\mathbf{t}] \geq T$ .

That is,  $g_{\mathbf{t}}$ -translates of  $H$ -orbits become equidistributed as  $[\mathbf{t}] \rightarrow \infty$  uniformly in  $z$  when the latter is restricted to compact subsets of  $X$ . The proof relies on S. G. Dani’s classification of measures invariant under horospherical subgroups and the so-called ‘linearization method’. The purpose of the present paper is to prove an effective version of the above theorem:

**Theorem 1.3.** *There exists  $\tilde{\gamma} > 0$  such that for any  $f \in C_{comp}^\infty(H)$ ,  $\psi \in C_{comp}^\infty(X)$  and for any compact  $L \subset X$  there exists  $\tilde{C} = \tilde{C}(f, \psi, L)$  such that for all  $z \in L$  and all  $\mathbf{t} \in \mathfrak{a}^+$*

$$\left| I_{f, \psi}(g_{\mathbf{t}}, z) - \int_H f \int_X \psi \right| \leq \tilde{C} e^{-\tilde{\gamma} \lfloor \mathbf{t} \rfloor}.$$

Note that the above statement follows from Theorem 1.1 when  $k = 2$ , that is,  $G = \mathrm{SL}_2(\mathbb{R})$ , but is new for  $k > 2$ . The proof uses the ‘exponential mixing’ approach of [KM1, KM3] together with effective nondivergence estimates obtained in [KM2]. We will describe these two parts in §§ 2 and 3 respectively, and then proceed with the proof of Theorem 1.3 in §4. We remark that the method of proof readily extends to the set-up more general than (1.1). Note also that, as observed by N. Shah in [S, Remark 1.0.2], Theorem 1.3 can be used to strengthen one of the main results of [KW], that is, [KW, Theorem 1.4], which constitutes a Diophantine application of Theorem 1.2.

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## 2. EXPONENTIAL MIXING AND $g_t$ -TRANSLATES

**Notation:** We will fix a right-invariant metric ‘dist’ on  $G$ , giving rise to the corresponding metric on  $X$ .  $B(x, r)$  will stand for an open ball of radius  $r$  centered at  $x$ . If a metric space is  $G$  or its subgroups, we will abbreviate  $B(e, r)$  to  $B(r)$ . When necessary, we will use subscripts denoting the ambient metric spaces.  $\|\cdot\|_\ell$  and  $\|\cdot\|_{C^\ell}$  will stand for the  $(2, \ell)$ -Sobolev and  $C^\ell$  norms respectively. We define

$$W^{2, \infty}(X) = \{\psi \in C^\infty(X) : \|\psi\|_\ell < \infty \forall \ell \in \mathbb{N}\};$$

clearly  $C_{comp}^\infty(X) \subset W^{2, \infty}(X)$ . In fact,  $W^{2, \infty}(X)$  coincides with the set of functions  $\psi \in C^\infty(X)$  such that  $D\psi \in L^2(X)$  for any  $D$  from the universal enveloping algebra of  $\mathrm{Lie}(G)$ . We let  $\langle \cdot, \cdot \rangle$  stand for the inner product in  $L^2(X)$ . We also denote by  $\|\psi\|_{\mathrm{Lip}}$  the Lipschitz constant of a function  $\psi$  on  $X$ ,

$$\|\psi\|_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} \sup_{x, y \in X, x \neq y} \frac{|\psi(x) - \psi(y)|}{\mathrm{dist}(x, y)},$$

and let  $\mathrm{Lip}(X) \stackrel{\mathrm{def}}{=} \{\psi : \|\psi\|_{\mathrm{Lip}} < \infty\}$ .

The following property of the  $G$ -action on  $X$  is deduced in [KM3] from the spectral gap on  $L^2(X)$ :

**Theorem 2.1** [KM3, Corollary 3.5]. *There exist  $\gamma > 0$  and  $\ell \in \mathbb{N}$  such that for any two functions  $\varphi, \psi \in W^{2, \infty}(X)$  and for any  $t \geq 0$  one has*

$$\left| \langle g\varphi, \psi \rangle - \int_X \varphi \int_X \psi \right| \ll \|\varphi\|_\ell \|\psi\|_\ell \cdot e^{-\gamma \mathrm{dist}(g, e)}.$$

Here and hereafter the implicit constants in  $\ll$  depend only on the dimensions of the corresponding spaces and the choices of the metric. Taking  $g = g_t$  as in (1.2), it follows that

$$(2.1) \quad \left| \langle g_t \varphi, \psi \rangle - \int_X \varphi \int_X \psi \right| \ll \|\varphi\|_\ell \|\psi\|_\ell \cdot e^{-\gamma t}.$$

An estimate analogous to (2.1) was used in [KM1] to derive Theorem 1.1. In this section we apply Theorem 2.1 to prove a statement similar to Theorem 1.1, providing some information as to how  $C$  in (1.3) depends on  $f$  and  $L$ . The argument follows the lines of the proof in [KM1]; in fact, the statement below is basically an intermediate step in the proof of [KM1, Proposition 2.4.8]. However we have decided to include details for the sake of making this paper self-contained.

To pass from  $\langle g_t \varphi, \psi \rangle$  to  $I_{f,\psi}(g_t, z)$ , we need to thicken  $f$  into a suitable function  $\varphi$  on  $X$ . To explain this process, we need to introduce some more notation. Let

$$H^- = \left\{ \begin{pmatrix} I_m & 0 \\ Y & I_n \end{pmatrix} \middle| Y \in M_{n,m} \right\}$$

and

$$H^0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in \mathrm{GL}_m(\mathbb{R}), B \in \mathrm{GL}_n(\mathbb{R}), \det(A) \det(B) = 1 \right\}.$$

The product map  $H^- \times H^0 \times H \rightarrow G$  is a local diffeomorphism; we will choose  $r_0$  so that the inverse of this map is well defined on  $B_G(r_0)$ . Note that  $H^-$  is expanding horospherical with respect to  $g_{-t}$ ,  $t > 0$ , while  $H^0$  is centralized by  $\{g_t\}$ . Thus, the inner automorphism  $\Phi_t$  of  $G$  given by  $\Phi_t(g) \stackrel{\mathrm{def}}{=} g_t h(g_t)^{-1}$  is non-expanding on the group

$$\tilde{H} \stackrel{\mathrm{def}}{=} H^- H^0 = \left\{ \begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \right\};$$

in fact, one has

$$(2.2) \quad \forall r > 0 \forall t > 0 \quad \Phi_t(B_{\tilde{H}}(r)) \subset B_{\tilde{H}}(r).$$

Let us choose Haar measures  $\nu^-, \nu^0$  on  $H^-, H^0$  respectively, normalized so that  $\mu$  is locally almost the product of  $\nu^-, \nu^0$  and  $\nu$ . By the latter, in view of [B, Ch. VII, §9, Proposition 13], we mean that  $\mu$  can be expressed via  $\nu^-, \nu^0$  and  $\nu$  in the following way: for any  $\varphi \in L^1(G)$

$$(2.3) \quad \int_{H^- H^0 H} \varphi(g) d\mu(g) = \int_{H^- \times H^0 \times H} \varphi(h^- h^0 h) \Delta(h^0) d\nu^-(h^-) d\nu^0(h^0) d\nu(h),$$

where  $\Delta$  is the modular function of (the non-unimodular group)  $\tilde{H}$ .

The ‘thickening’ will be based on the following properties of the Sobolev norm, cf. [KM1, Lemma 2.4.7]:

**Lemma 2.2.** (a) For any  $r > 0$ , there exists a nonnegative function  $\theta \in C_{comp}^\infty(\mathbb{R}^N)$  such that  $\text{supp}(\theta)$  is inside  $B(r)$ ,  $\int_{\mathbb{R}^N} \theta = 1$ , and  $\|\theta\|_\ell \ll r^{-(\ell+N/2)}$ .

(b) Given  $\theta_1 \in C_{comp}^\infty(\mathbb{R}^N)$ ,  $\theta_2 \in C_{comp}^\infty(\mathbb{R}^N)$ , define  $\theta \in C_{comp}^\infty(\mathbb{R}^N)$  by  $\theta(x) = \theta_1(x)\theta_2(x)$ . Then  $\|\theta\|_\ell \ll \|\theta_1\|_\ell \|\theta_2\|_\ell$ .

(c) Given  $\theta_1 \in C_{comp}^\infty(\mathbb{R}^{N_1})$ ,  $\theta_2 \in C_{comp}^\infty(\mathbb{R}^{N_2})$ , define  $\theta \in C_{comp}^\infty(\mathbb{R}^{N_1+N_2})$  by  $\theta(x_1, x_2) = \theta_1(x_1)\theta_2(x_2)$ . Then  $\|\theta\|_\ell \ll \|\theta_1\|_\ell \|\theta_2\|_\ell$ .

We will apply the above lemma to functions supported on small enough balls centered at identity elements in  $G$ ,  $H$ ,  $H^0$ ,  $H^-$ .

**Theorem 2.3.** Let  $f \in C_{comp}^\infty(H)$ ,  $0 < r < r_0/2$  and  $z \in X$  be such that

(i)  $\text{supp} f \subset B_H(r)$ , and

(ii)  $\pi_z$  is injective on  $B_G(2r)$ .

Then for any  $\psi \in W^{2,\infty}(X) \cap \text{Lip}(X)$  with  $\int_X \psi = 0$  there exists  $E = E(\psi)$  such that for any  $t \geq 0$  one has

$$(2.4) \quad |I_{f,\psi}(g_t, z)| \leq E \left( r \int_H |f| + r^{-(2\ell+N/2)} \|f\|_\ell e^{-\gamma t} \right),$$

where  $\gamma$  and  $\ell$  are as in Theorem 2.1 and  $N = m^2 + mn + n^2 - 1 = \dim \tilde{H}$ .

*Proof.* Using Lemma 2.2, one can choose nonnegative functions  $\theta^- \in C_{comp}^\infty(H^-)$ ,  $\theta^0 \in C_{comp}^\infty(H^0)$  with

$$(2.5) \quad \int_{H^-} \theta^- = \int_{H^0} \theta^0 = 1$$

such that

$$(2.6) \quad \text{supp}(\theta^-) \cdot \text{supp}(\theta^0) \subset B_{\tilde{H}}(r),$$

and at the same time

$$(2.7) \quad \|\tilde{\theta}\|_\ell \ll r^{(2\ell+N/2)},$$

where  $\tilde{\theta} \in C_{comp}^\infty(\tilde{H})$  is defined by

$$\tilde{\theta}(h^-h^0) \stackrel{\text{def}}{=} \theta^-(h^-)\theta^0(h^0)\Delta(h^0)^{-1}.$$

Also define  $\varphi \in C_{comp}^\infty(X)$  by  $\varphi(h^-h^0hz) = \tilde{\theta}(h^-h^0)f(h)$ ; the definition makes sense because of (2.6) and assumptions (i), (ii) of the theorem. Then  $I_{f,\psi}(g_t, z)$  can be reasonably well approximated by  $\langle g_t\varphi, \psi \rangle = \langle \varphi, g_{-t}\psi \rangle$ :

$$\begin{aligned} & |I_{f,\psi}(g_t, z) - \langle \varphi, g_{-t}\psi \rangle| \\ & \stackrel{(2.3)}{=} \left| \int_H f(h)\psi(g_t hx) d\nu(h) - \int_G \tilde{\theta}(h^-h^0)f(h)\psi(g_t h^-h^0 hx) d\mu(h^-h^0 h) \right| \\ & \stackrel{(2.5)}{=} \left| \int_G \theta^-(h^-)\theta^0(h^0)f(h) \left( \psi(g_t hx) - \psi(\Phi_t(h^-h^0)g_t hx) \right) \Delta(h^0)^{-1} d\mu(h^-h^0 h) \right| \\ & \stackrel{(2.2), (2.6)}{\leq} \sup_{g \in B_{\tilde{H}}(r), y \in X} |\psi(gy) - \psi(y)| \int_G |\theta^-(h^-)\theta^0(h^0)f(h)\Delta(h^0)^{-1}| d\mu(h^-h^0 h) \\ & \stackrel{(2.3)}{\leq} \|\psi\|_{\text{Lip}} \cdot r \cdot \int_H |f|. \end{aligned}$$

On the other hand, in view of Lemma 2.2 and  $\pi_z$  being a local isometry,

$$\|\varphi\|_\ell = \|\tilde{\theta} \cdot f\|_\ell \ll \|\tilde{\theta}\|_\ell \|f\|_\ell \stackrel{(2.7)}{\ll} r^{-(2\ell+N/2)} \|f\|_\ell,$$

hence by (2.1)

$$|\langle g_t \varphi, \psi \rangle| \ll r^{(2\ell+N/2)} \|f\|_\ell \|\psi\|_\ell e^{-\gamma t},$$

finishing the proof.  $\square$

**Remark 2.4.** In order to derive Theorem 1.1 from Theorem 2.3 it suffices to choose  $r = e^{-\beta t}$  for some suitable  $\beta$ . The same trick will help us in the proof of Theorem 1.3. Note that  $t$  needs to be taken large enough so that condition (ii) of Theorem 2.3 is satisfied for all  $z \in L$ . The latter is possible because, in view of the compactness of  $L$  and discreteness of  $\Gamma$  in  $G$ , the value

$$r(L) \stackrel{\text{def}}{=} \inf_{z \in L} \sup\{r > 0 \mid \pi_z : G \rightarrow X \text{ is injective on } B(r)\}$$

is positive; we will call it the *injectivity radius of  $L$* .

**Remark 2.5.** It is worthwhile to point out that  $H$  being the expanding horospherical subgroup relative to  $g_t$ ,  $t > 0$ , was crucially important for the proof. When  $g_t$  is replaced with  $g_{\mathbf{t}}$  where  $\mathbf{t}$  is an arbitrary element of  $\mathfrak{a}^+$ , one can still talk about  $\Phi_{\mathbf{t}}$ , the inner automorphism of  $H$  given by

$$(2.8) \quad \Phi_{\mathbf{t}}(h) \stackrel{\text{def}}{=} g_{\mathbf{t}} h (g_{\mathbf{t}})^{-1}.$$

It is expanding on  $H$ , since the latter is contained in the expanding horospherical subgroup relative to  $g_{\mathbf{t}}$ ; however it is not non-expanding on  $\bar{H}$  in the sense of (2.2), thus there is no guarantee that  $I_{f,\psi}(g_{\mathbf{t}}, z)$  is close to  $\langle \varphi, g_{-\mathbf{t}} \psi \rangle$  for  $\varphi$  constructed as in the above proof. We bypass this difficulty by means of an additional step, based on the nondivergence phenomenon, to be described in the next section.

### 3. QUANTITATIVE NONDIVERGENCE

For any  $\varepsilon > 0$  consider

$$K_\varepsilon \stackrel{\text{def}}{=} \pi(\{g \in G \mid \|g\mathbf{v}\| \geq \varepsilon \quad \forall \mathbf{v} \in \mathbb{Z}^k \setminus \{0\}\}).$$

In other words,  $K_\varepsilon$  consists of lattices in  $\mathbb{R}^k$  with no nonzero vector of length less than  $\varepsilon$ . These sets are compact by virtue of Mahler's Compactness Criterion (see [R, Corollary 10.9] or [BM]). Here  $\|\cdot\|$  can be any norm on  $\mathbb{R}^k$  which we will from now on take to be the standard Euclidean norm.

It was proved in [KM2], refining previous work on non-divergence of unipotent flows [M, D], that certain polynomial maps from balls in Euclidean spaces to  $X$  cannot take values outside of  $K_\varepsilon$  on a set of big measure. Namely, the following is a special case of [BKM, Theorem 6.2] (see also [KLW, KT, K] for further generalizations):

**Theorem 3.1.** For  $d \in \mathbb{N}$ , let  $\varphi$  be a map  $\mathbb{R}^d \rightarrow \mathrm{GL}_k(\mathbb{R})$  such that

(i) all coordinates (matrix elements) of  $\varphi(\cdot)$  are affine (degree 1 polynomials), and let a ball  $B \subset \mathbb{R}^d$  and  $0 < \rho \leq 1$  be such that

(ii) for any  $j = 1, \dots, k-1$  and any  $\mathbf{w} \in \bigwedge^j(\mathbb{Z}^k) \setminus \{0\}$  one has

$$\|\varphi(x)\mathbf{w}\| \geq \rho \quad \text{for some } x \in B.$$

Then for any positive  $\varepsilon \leq \rho$  one has

$$(3.1) \quad \lambda(\{x \in B \mid \pi(\varphi(x)) \notin K_\varepsilon\}) \ll \left(\frac{\varepsilon}{\rho}\right)^{1/d(k-1)} \lambda(B).$$

Here  $\lambda$  is Lebesgue measure on  $\mathbb{R}^d$ , and the Euclidean<sup>1</sup> norm  $\|\cdot\|$  is naturally extended from  $\mathbb{R}^k$  to its exterior powers. We remark that the way assumption (i) is used in the proof is by verifying that all the functions  $x \mapsto \|\varphi(x)\mathbf{w}\|$ , where  $\mathbf{w} \in \bigwedge^j(\mathbb{Z}^k)$ , are  $(C, \alpha)$ -good on  $\mathbb{R}^d$ , with some fixed  $C = C(d, k) > 0$  and  $\alpha = 1/d(k-1)$ , the exponent appearing in (3.1). See [KM2] for more detail.

Our plan is to apply Theorem 3.1 with  $\varphi : M_{m,n} \rightarrow G$  given by

$$(3.2) \quad \varphi(Y) = g_{\mathbf{t}} u_Y g$$

for some  $g \in G$  and  $\mathbf{t} \in \mathfrak{a}^+$ . It is clear that assumption (i) holds. As for (ii), we will need to have uniformity in  $\mathbf{t} \in \mathfrak{a}^+$  and in  $g$  such that  $\pi(g)$  belongs to a compact subset of  $X$ . This can be extracted from the next lemma, which is immediate from [KW, Proposition 2.4] applied to the representations of  $G$  on  $\bigwedge^j(\mathbb{R}^k)$ ,  $j = 1, \dots, k-1$ :

**Lemma 3.2.** *There exists  $\alpha > 0$  with the following property. Let  $B$  be a ball centered at 0 in  $M_{m,n}$ . Then one can find  $b > 0$  such that for any  $j = 1, \dots, k-1$ , any  $\mathbf{w} \in \bigwedge^j(\mathbb{R}^k)$  and any  $\mathbf{t} \in \mathfrak{a}^+$  one has*

$$\sup_{Y \in B} \|g_{\mathbf{t}} u_Y \mathbf{w}\| \geq b e^{\alpha \lfloor \mathbf{t} \rfloor} \|\mathbf{w}\|.$$

**Corollary 3.3.** *Let  $B$  be a neighborhood of 0 in  $M_{m,n}$  and let  $L \subset X$  be compact. Then there exists  $b > 0$  such that for any  $j = 1, \dots, k-1$ , any  $\mathbf{w} \in \bigwedge^j(\mathbb{Z}^k) \setminus \{0\}$ , any  $g \in \pi^{-1}(L)$  and any  $\mathbf{t} \in \mathfrak{a}^+$  one has*

$$\sup_{Y \in B} \|g_{\mathbf{t}} u_Y g \mathbf{w}\| \geq b e^{\alpha \lfloor \mathbf{t} \rfloor}.$$

*Proof.* Apply the above lemma with  $\mathbf{w}$  replaced by  $g\mathbf{w}$ ; it follows from the compactness of  $L$  and discreteness of  $\bigwedge^j(\mathbb{Z}^k)$  in  $\bigwedge^j(\mathbb{R}^k)$  that

$$\inf \{ \|g\mathbf{w}\| \mid \pi(g) \in L, \mathbf{w} \in \bigwedge^j(\mathbb{Z}^k) \setminus \{0\}, j = 1, \dots, k-1 \}$$

is positive.  $\square$

<sup>1</sup>In [KM2] the statement of Theorem 5.2 involved the sup norm instead of the Euclidean one, which resulted in a restriction for  $\rho$  to be not greater than  $1/k$ ; thus we chose to refer to [BKM] for the Euclidean norm version.

**Corollary 3.4.** *Let  $L \subset X$  be compact and let  $B \subset H$  be a ball centered at  $e \in H$ . Then there exists  $T = T(B, L)$  such that for every  $0 < \varepsilon < 1$ , any  $z \in L$  and any  $\mathbf{t} \in \mathfrak{a}^+$  with  $\lfloor \mathbf{t} \rfloor \geq T$  one has*

$$\nu(\{h \in B \mid g_{\mathbf{t}}hz \notin K_\varepsilon\}) \ll \varepsilon^{\frac{1}{mn(k-1)}} \nu(B).$$

*Proof.* Define  $T$  by  $be^{\alpha T} = 1$ , where  $\alpha$  is given by Lemma 3.2 and  $b$  by Corollary 3.3 applied to  $\log(B) \subset M_{m,n}$  and  $L$ . (Note that the exponential map from  $M_{m,n}$  to  $H$  is an isometry.) Take  $\varphi$  as in (3.2) with  $g \in \pi^{-1}(L)$ . Then, in view of Corollary 3.3, assumption (ii) of Theorem 3.1, with  $d = mn$ , will be satisfied with  $\rho = 1$  as long as  $\lfloor \mathbf{t} \rfloor \geq T$ .  $\square$

We conclude this section by an estimate of the injectivity radius of  $K_\varepsilon$ , to make it possible to combine the above corollary with Theorem 2.3. Observe that any lattice  $\Lambda \in K_\varepsilon$  can be generated by vectors of norm  $\ll 1/\varepsilon^{k-1}$ ; if  $g\Lambda = \Lambda$  and  $g \neq e$ , then for one of those vectors  $\mathbf{v}$  one has  $\|g\mathbf{v} - \mathbf{v}\| \geq \varepsilon$ . This implies that  $\text{dist}(e, g) \gg \varepsilon^k$ . We arrive at

**Proposition 3.5.** *There exists positive  $c = c(k)$  such that  $r(K_\varepsilon) \geq c \cdot \varepsilon^k \forall \varepsilon > 0$ .*

#### 4. PROOF OF THEOREM 1.3

Our goal in this section will be to find  $\tilde{\gamma} > 0$  such that for any  $f \in C_{comp}^\infty(H)$ ,  $\psi \in W^{2,\infty}(X) \cap \text{Lip}(X)$  with  $\int_X \psi = 0$  and compact  $L \subset X$  there exists  $\tilde{C} > 0$  such that for all  $z \in L$  and all  $\mathbf{t} \in \mathfrak{a}^+$  one has

$$(4.1) \quad |I_{f,\psi}(g_{\mathbf{t}}, z)| \leq \tilde{C}e^{-\tilde{\gamma}\lfloor \mathbf{t} \rfloor}.$$

Then Theorem 1.3 will follow by applying (4.1) with  $\psi$  replaced by  $\psi - \int_X \psi$ . Note also that, by increasing  $\tilde{C}$  if needed, it is enough to prove (4.1) for  $\mathbf{t}$  with large enough  $\lfloor \mathbf{t} \rfloor$ .

Given  $\mathbf{t} \in \mathfrak{a}^+$ , define  $t \stackrel{\text{def}}{=} \lfloor \mathbf{t} \rfloor / 2$ , and let

$$(4.2) \quad \mathbf{u} = \mathbf{u}(\mathbf{t}) \stackrel{\text{def}}{=} \mathbf{t} - \left(\frac{t}{m}, \dots, \frac{t}{m}, \frac{t}{n}, \dots, \frac{t}{n}\right).$$

Note that  $\mathbf{u} \in \mathfrak{a}^+$ ,  $\lfloor \mathbf{u} \rfloor \geq \lfloor \mathbf{t} \rfloor / 2 = t$ , and  $g_{\mathbf{t}} = g_{\mathbf{t}}g_{\mathbf{u}}$  (here  $g_{\mathbf{t}}$  and  $g_{\mathbf{u}}$  are defined via (1.4), and  $g_t$  is as in (1.2)).

Take a function  $\theta$  supported on  $B_H(r)$  as in Lemma 2.2(a), with  $r = e^{-\beta t}$  where  $\beta$  is to be specified later; since  $\int_H \theta = 1$  and  $\nu$  is translation-invariant, one can write

$$\begin{aligned} I_{f,\psi}(g_{\mathbf{t}}, z) &= \int_H f(h)\psi(g_{\mathbf{t}}hz) d\nu(h) \int_H \theta(y) d\nu(y) \\ &= \int_H \int_H f(\Phi_{\mathbf{u}}^{-1}(y)h)\theta(y)\psi(g_{\mathbf{t}}g_{\mathbf{u}}\Phi_{\mathbf{u}}^{-1}(y)hz) d\nu(y) d\nu(h) \\ &= \int_H \int_H f(\Phi_{\mathbf{u}}^{-1}(y)h)\theta(y)\psi(g_t y g_{\mathbf{u}} h z) d\nu(y) d\nu(h). \end{aligned}$$

Note that  $\Phi_{\mathbf{u}}^{-1}$  is a contracting automorphism of  $H$ , in fact, one has

$$\text{dist}(e, \Phi_{\mathbf{u}}^{-1}(h)) \leq e^{-2\lfloor \mathbf{u} \rfloor} \text{dist}(e, h) \leq e^{-2t} \text{dist}(e, h)$$

for any  $h \in H$ . Choose  $B = B(r)$  containing  $\text{supp } f$ . Then the supports of all functions of the form  $h \mapsto f(\Phi_{\mathbf{u}}^{-1}(y)h)$  are contained in

$$\tilde{B} \stackrel{\text{def}}{=} B(r + e^{-(2+\beta)t}).$$

By taking  $t$  large enough it is safe to assume that

$$(4.3) \quad e^{-\beta t} < r_0/2,$$

$\nu(\tilde{B}) \leq 2\nu(B)$ , and also that  $t > T \stackrel{\text{def}}{=} T(\tilde{B}, L)$  as in Corollary 3.4. Now define  $\varepsilon$  by

$$(4.4) \quad \varepsilon = \left(\frac{2}{c}e^{-\beta t}\right)^{1/k},$$

where  $c$  is from Proposition 3.5, and denote

$$A \stackrel{\text{def}}{=} \{h \in \tilde{B} \mid g_{\mathbf{u}}hz \notin K_\varepsilon\}.$$

Then for any  $\mathbf{u} \in \mathfrak{a}^+$  with  $\lfloor \mathbf{u} \rfloor \geq T$  and any  $z \in L$  one knows, in view of Corollary 3.4, that

$$\nu(A) \ll \varepsilon^{\frac{1}{mn(k-1)}} \nu(\tilde{B}).$$

Hence the absolute value of

$$\int_A \int_H f(\Phi_{\mathbf{u}}^{-1}(y)h) \theta(y) \psi(g_t y g_{\mathbf{u}} h z) d\nu(y) d\nu(h)$$

is

$$\ll \varepsilon^{\frac{1}{mn(k-1)}} \nu(\tilde{B}) \sup |f| \sup |\psi| \int_H \theta \ll \sup |f| \sup |\psi| \nu(B) \cdot e^{-\frac{\beta}{mnk(k-1)}t}.$$

Now let us assume that  $h \in \tilde{B} \setminus A$ , and apply Theorem 2.3 with  $r = e^{-\beta t}$ ,  $g_{\mathbf{u}}hz$  in place of  $z$  and

$$f_h(y) \stackrel{\text{def}}{=} f(\Phi_{\mathbf{u}}^{-1}(y)h) \theta(y)$$

in place of  $f$ . Clearly condition (i) follows from (4.3), and, since  $g_{\mathbf{u}}hz \in K_\varepsilon$  whenever  $h \notin A$ , condition (ii) is satisfied in view of Proposition 3.5 and (4.4). Also, because  $\Phi_{\mathbf{u}}^{-1}|_H$  is contracting, partial derivatives of  $f$  will not increase after precomposition with  $\Phi_{\mathbf{u}}^{-1}$ , and thus

$$(4.5) \quad \|f_h\|_\ell \stackrel{\text{Lemma 2.2(b)}}{\ll} \|f\|_{C^\ell} \|\theta\|_\ell \stackrel{\text{Lemma 2.2(a)}}{\ll} r^{-(\ell+mn/2)} \|f\|_{C^\ell}.$$

This way one gets:

$$\begin{aligned} \left| \int_{\tilde{B} \setminus A} \int_H f(\Phi_{\mathbf{u}}^{-1}(y)h) \theta(y) \psi(g_t y g_{\mathbf{u}} h z) d\nu(y) d\nu(h) \right| &\leq \int_{\tilde{B} \setminus A} |I_{f_h, \psi}(g_t, g_{\mathbf{u}} h z)| d\nu(h) \\ &\stackrel{(2.4)}{\leq} E(\psi) \left( r \int_H |f_h| + r^{-(2\ell+N/2)} \|f_h\|_\ell e^{-\gamma t} \right) \nu(\tilde{B}) \\ &\stackrel{(4.5)}{\ll} E(\psi) \left( \sup |f| \cdot e^{-\beta t} + \|f\|_{C^\ell} \cdot e^{-(\gamma-(2\ell+N/2)\beta)t} \right) \nu(B). \end{aligned}$$

Combining the two estimates above, one can conclude that

$$\begin{aligned} |I_{f,\psi}(g_{\mathbf{t}}, z)| &\ll C_1 e^{-\frac{\beta}{mnk(k-1)}t} + C_2 e^{-\beta t} + C_3 e^{-(\gamma-(2\ell+N/2)\beta)t} \\ &\leq \max(C_1, C_2) e^{-\frac{\beta}{mnk(k-1)}t} + C_3 e^{-(\gamma-(2\ell+N/2)\beta)t}, \end{aligned}$$

where  $C_i$ ,  $i = 1, 2, 3$ , depend on  $f$ ,  $\psi$  and  $L$ . An elementary computation shows that choosing  $\beta$  equalizing the two exponents above will produce

$$\tilde{\gamma} = \frac{\gamma}{1 + mnk(k-1)(2\ell + N/2)}$$

such that (4.1) will hold with  $\tilde{C} \ll \max(C_1, C_2, C_3)$ .  $\square$

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