From dklein@math.ias.edu Fri Sep 20 18:33:59 1996 Received: from IAS.EDU (mailgate.ias.edu) b CAL.MATH.YALE.EDU via SMTP; Fri, 20 Sep 1996 18:33:58 -0400 Received: from sevilla.math (sevilla.math.ias.edu [192.108.106.26]) by IAS.EDU (8.6.12/8.6.12) with ESMTP id SAA10733 for boc@math.yale.edu¿; Fri, 20 Sep 1996 18:35:52 -0400 Received: from monaco.math.ias.edu (monaco.m [192.108.106.37]) by sevilla.math.ias.edu (8.6.12/8.6.12) with ESMTP id SAA23592 for ¡kleinboc@mat Fri, 20 Sep 1996 18:35:53 -0400 From: Dmitry Y Kleinbock ¡dklein@IAS.EDU¿ Received: (dklein@loc by monaco.math.ias.edu (8.6.12/8.6.12) id SAA25912 for kleinboc@math.yale.edu; Fri, 20 Sep 1996 1 -0400 Date: Fri, 20 Sep 1996 18:35:53 -0400 Message-Id: ¡199609202235.SAA25912@monaco.math.i To: kleinboc@PASCAL.MATH.YALE.EDU Status: R

FLOWS ON HOMOGENEOUS SPACES AND DIOPHANTINE PROPERTIES OF MATRICES

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August 13, 1996

Abstract. We generalize the notions of badly approximable (resp. singular) systems of m linear forms in n variables, and relate these generalizations to certain bounded (resp. divergent) trajectories in the space of lattices in $\mathbb{R}^{m+n}$.

Introduction

Notation. We will denote by $M_{m,n}(\mathbb{R})$ the space or real matrices with m rows and n columns. $I_k \in M_{k,k}(\mathbb{R})$ will stand for the identity matrix. Vectors will be named by lowercase boldface letters, such as $\mathbf{x} = (x_i \mid 1 \le i \le k)$, and, despite the row notation, will always be treated as column vectors. 0 will mean zero vector in any dimension, as well as zero matrix of any size. For a matrix $L \in M_{m,n}(\mathbb{R})$ and $1 \le i \le n$, we will denote by L_i the linear form $\mathbb{R}^n \to \mathbb{R}$ corresponding to the ith row of L, and by $L^{(i)}$ (resp. $L_{(i)}$) the matrix consisting of first (resp. last) i rows of L.

Any statement involving "$\pm$" will stand for two statements, one for each choice of the sign. Hausdorff dimension of a subset Y of a metric space X will be denoted by $\dim(Y)$, and we will say that Y is *thick* (in X) if for any nonempty open subset W of X, $\dim(W \cap Y) = \dim(W)$ (i.e. Y has full Hausdorff dimension at any point of X).

Throughout the sequel we will fix two positive integers m, n, denote by G the group $\{L \in GL_{m+n}(\mathbb{R}) \mid \det(L) = \pm 1\}$, and by $\Omega \cong SL_{m+n}(\mathbb{R})/SL_{m+n}(\mathbb{Z}) \cong G/GL_{m+n}(\mathbb{Z})$ the space of unimodular lattices in $\mathbb{R}^{m+n}$.

History. A system $(A_1, \dots, A_m)$ of linear forms in n variables is called *badly approximable* if there exists a constant $c > 0$ such that for every $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$

$$\max\left(|A_1(\mathbf{q}) - p_1|^m, \dots, |A_m(\mathbf{q}) - p_m|^m\right) \cdot \max\left(|q_1|^n, \dots, |q_n|^n\right) > c\,. \qquad (1)$$

It was proven by W. Schmidt in 1969 [S3] that matrices $A \in M_{m,n}(\mathbb{R})$ such that the system $(A_1, \dots, A_n)$ is badly approximable form a thick subset of $M_{m,n}(\mathbb{R})$.

Supported in part by Alfred P. Sloan Graduate Dissertation Fellowship.

In 1986, S.G. Dani exhibited a correspondence between badly approximable systems of linear forms and certain bounded trajectories in Ω. His result [D1, Theorem 2.20] can be restated as follows: for $A \in M_{m,n}(\mathbb{R})$, consider the lattice $\Lambda = \begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n} \in \Omega$ and the one-parameter subgroup of G of the form

$$g_t = \operatorname{diag}(e^{t/m}, \dots, e^{t/m}, e^{-t/n}, \dots, e^{-t/n})\,. \tag{2}$$

Then a system of linear forms given by A is badly approximable iff the trajectory $\{g_t\Lambda \mid t \in \mathbb{R}_+\}$ is bounded in Ω. This and the aforementioned result of W. Schmidt allowed Dani to conclude [D1, Corollary 2.21] that the set of lattices in Ω with bounded g_t-trajectories is thick.

It was suggested by Dani [D2] and then conjectured by G.A. Margulis [Ma, Conjecture (A)] that the abundance of bounded orbits is a general feature of nonquasiunipotent (see §4.1 for the definition) flows on homogeneous spaces of Lie groups. In a recent paper [KM] by G.A. Margulis and the author, Margulis' conjecture was settled. In particular, the results of that paper imply that for any nonquasiunipotent one-parameter subgroup $\{g_t\}$ of G, the set $\big\{\Lambda \in \Omega \mid \{g_t\Lambda \mid t \in \mathbb{R}\}$ is bounded$\big\}$ is thick.

Outline. The present paper is an attempt to make a number-theoretic sense out of the above result. We take a generic nonquasiunipotent one-parameter subgroup of G with real eigenvalues. In a suitable basis, it has the form

$$g_t = \operatorname{diag}(e^{r_1 t}, \dots, e^{r_m t}, e^{-s_1 t}, \dots, e^{-s_n t})\,, \tag{3}$$

where $\mathbf{r} = (r_i \mid 1 \le i \le m)$ and $\mathbf{s} = (s_j \mid 1 \le j \le n)$ are such that

$$r_i, s_j > 0 \quad \text{and} \quad \sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j \tag{4}$$

(the choice (2) of the subgroup $\{g_t\}$ corresponds to $\mathbf{r} = \mathbf{m} \overset{\text{def}}{=} (\frac{1}{m}, \dots, \frac{1}{m})$ and $\mathbf{s} = \mathbf{n} \overset{\text{def}}{=} (\frac{1}{n}, \dots, \frac{1}{n})$). One of the goals of the paper is to show that the sets

$$\big\{L \in G \mid \{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_\pm\} \text{ is bounded}\big\}\,,$$

as well as their intersection consisting of matrices $L \in G$ with bounded orbits $\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}\}$, admit natural Diophantine description, generalizing the definition of badly approximable systems of linear forms.

We start §1 by looking at the motivation for the aforementioned definition, namely, the fact that the quantity in the left hand side of (1) is for any $A \in M_{m,n}(\mathbb{R})$ less than 1 for infinitely many $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ (see Corollary 1.3). One of the proofs of this fact (cf. [C] or [S4]) is based upon Minkowski's Linear Forms Theorem (see Theorems 1.1 and 1.2). It is not hard to notice that

- the proof of Theorem 1.2 uses the matrix $L_A = \begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix}$, which is exactly the one that naturally arises in Dani's dynamical interpretation of Diophantine properties of A;

- the argument of the proof goes without changes for any $L \in G$ instead of L_A, the expressions $A_i(\mathbf{q}) - p_i$, $1 \le i \le m$, and q_j, $1 \le j \le n$, being replaced by the rows of L applied to the vector $\mathbf{x} \in \mathbb{Z}^{m+n}$;
- the proof can be modified to allow arbitrary exponents $1/r_1, \dots, 1/r_m$, $1 \le i \le m$ (resp. $1/s_1, \dots, 1/s_n$, $1 \le j \le n$) instead of m (resp. n) in (1), the only restrictions being given by (4).

Thus it seems natural to pick an m-tuple $\mathbf{r}$ and an n-tuple $\mathbf{s}$ satisfying (4) and ask for existence of a positive lower bound for values

$$\max\left(|L_1(\mathbf{x})|^{1/r_1}, \dots, |L_m(\mathbf{x})|^{1/r_m}\right) \cdot \max\left(|L_{m+1}(\mathbf{x})|^{1/s_1}, \dots, |L_{m+n}(\mathbf{x})|^{1/s_n}\right) \quad (5)$$

for $\mathbf{x}$ chosen from $\mathbb{Z}^{m+n} \smallsetminus \{0\}$ or a smaller subset of $\mathbb{Z}^{m+n}$.

We will say that a matrix $L \in G$ is $(\mathbf{r}, \mathbf{s})$-*loose* if the set of values of (5) for all $\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ is separated from zero (see Definition 6.1). In §6 we will prove that L is $(\mathbf{r}, \mathbf{s})$-loose iff the orbit $\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}\}$, with g_t as in (3), is bounded in Ω, and, moreover, express the "size" of this orbit via the best permissible lower bound for (5). Thus one can use the results of [KM] to conclude that $(\mathbf{r}, \mathbf{s})$-loose matrices form a thick subset of G.

However we have built our exposition starting from consideration of one-sided trajectories rather than orbits. In §2 we show how the expression (5) should be modified to allow similar treatment of trajectories $\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_\pm\}$. We define the notion of L being $(\mathbf{r}, \mathbf{s}, \pm)$-*loose* (see Definition 2.4) so that $A \in M_{m,n}(\mathbb{R})$ is badly approximable iff L_A is $(\mathbf{m}, \mathbf{n}, +)$-loose, and, more generally (see Theorem 2.5), L is $(\mathbf{r}, \mathbf{s}, \pm)$-loose iff the trajectory $\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_\pm\}$ is bounded in Ω.

This theorem, or rather its quantitative version, is proven in §3. Note that in order to get a quantitative correspondence between dynamical and Diophantine characteristics of L, one needs to employ the "size" function dependent on the choice (3) of the one-parameter subgroup. Specifically (cf. §2.1), for $\Lambda \in \Omega$ one defines

$$\delta_{\mathbf{r},\mathbf{s}}(\Lambda) \stackrel{\text{def}}{=} \inf_{\mathbf{x} \in \Lambda \smallsetminus \{0\}} \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} \left(|x_i|^{1/r_i}, |x_{m+j}|^{1/s_j}\right).$$

Mahler's Compactness Criterion implies that $\delta_{\mathbf{r},\mathbf{s}}(\Lambda)$ can be thought of as a (normalized) distance from Λ to infinity in Ω. On the other hand, the condition $\delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n}) \le \delta$ turns out (see Lemma 3.4) to be equivalent to the existence of a nonzero integral solution of certain system of inequalities. Thus the behavior of the function $\delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n})$ can be completely described in Diophantine language. In particular, one should take the greatest lower bound for $\delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n})$ as $t \in \mathbb{R}_\pm$ (resp. $t \in \mathbb{R}$) to be the "size" of the trajectory (resp. the orbit) in order to get values which can be easily expressed in Diophantine terms.

Besides the results mentioned above, the Diophantine interpretation of the function $\delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n})$ also yields:

- several equivalent definitions for L being $(\mathbf{r}, \mathbf{s}, +)$-loose, as well as a Diophantine description of the "asymptotical size" $\liminf_{t\to\infty} \delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n})$ of the trajectory $\{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$ (see §5);
- a connection between the rate of decay of $\inf_{0 \le t \le T} \delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n})$ as $T \to \infty$ and "the extent of L being not $(\mathbf{r}, \mathbf{s}, +)$-loose" (cf. the order of approximation of a real number or a system of linear forms) – see §8.3;

- a Diophantine description of the set $\big\{L \in G \mid \{g_t L\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$ is divergent$\big\}$, generalizing the notion of *singular systems of linear forms*, as well as Theorem 2.14 from Dani's paper [D1] – see §7.

It is worthwhile to note that Dani's proofs of Theorems 2.14 and 2.20 don't seem to be related. On the other hand, our approach allows one to make Dani's argument more transparent and obtain quantitative versions of both theorems as an immediate consequence of a more general fact. See §8 and [K, Chapter VII] for other general remarks as well as some open questions.

Acknowledgements. I am grateful to Professor G.A. Margulis for suggesting this problem and constant encouragement during my work, and to the referee for useful comments.

§1. Badly approximable systems of linear forms

1.1. One of the possible motivations for the definition of badly approximable numbers and its generalizations comes from Linear Forms Theorem due to Minkowski (cf. [C, Appendix B] or [S4, Chapter II]):

Theorem. *Let $L \in G$, $a_1, \dots, a_{m+n} > 0$, $\prod_{i=1}^{m+n} a_i = 1$. Then there exists a vector $\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ such that*

$$|L_i(\mathbf{x})| < a_i, \quad 1 \le i \le m\,, \tag{1.1a}$$

and

$$|L_i(\mathbf{x})| \le a_i, \quad m+1 \le i \le m+n\,. \tag{1.1b}$$

As in [C] or [S4], one can use this theorem to show that for any system $(A_1, \dots, A_m)$ of linear forms in n variables there exist integers p_i, $1 \le i \le m$, and q_j, $1 \le j \le n$, such that the differences $A_i(\mathbf{q}) - p_i$ are small, while the absolute values of q_j are not too big.

1.2. Theorem. *Let $A \in M_{m,n}(\mathbb{R})$. Then for any $t \in \mathbb{R}$ there exists $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ such that*

$$|A_i(\mathbf{q}) - p_i| < e^{-t/m}, \quad 1 \le i \le m\,, \tag{1.2a}$$

and

$$|q_j| \le e^{t/n}, \quad 1 \le j \le n\,. \tag{1.2b}$$

Proof. Apply Theorem 1.1 to the matrix $L = L_A \overset{\text{def}}{=} \begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix}$ and numbers $a_i = \begin{cases} e^{-t/m}, & i \le m \\ e^{t/n}, & i > m \end{cases}$. $\square$

Denote by $\|\cdot\|$ the norm in $\mathbb{R}^k$ given by $\|\mathbf{x}\| = \max_{1\le i\le k} |x_i|$. Then (1.2a) and (1.2b) can be rewritten as

$$\|A\mathbf{q} - \mathbf{p}\|^m < e^{-t} \tag{1.2a$'$}$$

and

$$\|\mathbf{q}\|^n \le e^t\,. \tag{1.2b$'$}$$

1.3. Corollary. *For any* $A \in M_{m,n}(\mathbb{R})$*, there exist infinitely many* $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ *such that* $\|A\mathbf{q} - \mathbf{p}\| < 1$ *and* $\|A\mathbf{q} - \mathbf{p}\|^m \|\mathbf{q}\|^n < 1$*; in other words,*

$$(\mathbf{p}, \mathbf{q}) \neq 0 \quad \textit{and} \quad \|A\mathbf{q} - \mathbf{p}\|^m \max(\|A\mathbf{q} - \mathbf{p}\|^m, \|\mathbf{q}\|^n) < 1 \,. \tag{1.3}$$

Moreover, one can choose a sequence of solutions $(\mathbf{p}_k, \mathbf{q}_k)$*,* $k \in \mathbb{N}$*, of* (1.3) *such that* $\mathbf{q}_k \to \infty$ *as* $k \to \infty$*.*

Proof. Indeed, $\|A\mathbf{q} - \mathbf{p}\|^m \|\mathbf{q}\|^n < 1$ is a product of inequalities (1.2a′) and (1.2b′), while $\|A\mathbf{q} - \mathbf{p}\| < 1$ follows from (1.2a′) if one takes $t \geq 0$. If $A\mathbf{q} = \mathbf{p}$ for some $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$, one can take $\mathbf{p}_k = k\mathbf{p}$ and $\mathbf{q}_k = k\mathbf{q}$, $k \in \mathbb{N}$. Otherwise, for fixed $\mathbf{q}$, (1.2a′) can only hold for finitely many $\mathbf{p}$ and small enough t. Hence as $t \to +\infty$, one obtains infinitely many solutions with different values of $\mathbf{q}$. □

Note that for $(\mathbf{p}, \mathbf{q}) \neq 0$ with $\|A\mathbf{q} - \mathbf{p}\|^m \|\mathbf{q}\|^n < 1$, the condition $\|A\mathbf{q} - \mathbf{p}\| < 1$ is equivalent to $\mathbf{q} \neq 0$. Thus $\|\mathbf{q}\|^n > \|A\mathbf{q} - \mathbf{p}\|^m$, i.e. $\|\mathbf{q}\|^n = \max(\|A\mathbf{q} - \mathbf{p}\|^m, \|\mathbf{q}\|^n)$. Therefore (1.3) can be written in the form

$$\mathbf{q} \neq 0 \quad \text{and} \quad \|A\mathbf{q} - \mathbf{p}\|^m \|\mathbf{q}\|^n < 1 \,, \tag{1.3′}$$

with the left hand side of the second inequality in (1.3) equal to the left hand side of the second inequality in (1.3′).

1.4. Definition. A system of linear forms given by $A \in M_{m,n}(\mathbb{R})$ is called *badly approximable* if, roughly speaking, one can not replace 1 in the right hand side of (1.3) or (1.3′) by arbitrarily small constant. More precisely, let

$$\begin{aligned} c(A) &\stackrel{\text{def}}{=} \inf_{\substack{\mathbf{p} \in \mathbb{Z}^m \\ \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}}} \|A\mathbf{q} - \mathbf{p}\|^m \|\mathbf{q}\|^n \\ &= \inf_{(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \smallsetminus \{0\}} \|A\mathbf{q} - \mathbf{p}\|^m \max(\|A\mathbf{q} - \mathbf{p}\|^m, \|\mathbf{q}\|^n) \end{aligned} \tag{1.4}$$

(note that $c(A)$ is always less than 1 by Corollary 1.3). Then a system of linear forms given by A is badly approximable if $c(A) > 0$ and *well approximable* otherwise.

The case $m = n = 1$ corresponds to badly/well approximable numbers. W. Schmidt proved in [S3] that there exist continuum many badly approximable systems of linear forms in any dimensions; more precisely, the set of $A \in M_{m,n}(\mathbb{R})$ with $c(A) > 0$ is thick.

1.5. We now highlight the relation between this notion and dynamics of flows in the space Ω of unimodular lattices in $\mathbb{R}^{m+n}$. The following is a restatement of Theorem 2.20 from [D1]:

Theorem. *Let* $\{g_t\}$ *be the one-parameter subgroup of* G *defined by* $g_t = \mathrm{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n})$*. Then a system of linear forms given by* $A \in M_{m,n}(\mathbb{R})$ *is badly approximable iff the trajectory* $\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$*, with* L_A *as in the proof of Theorem 1.2, is bounded in* Ω*.*

Observe that Definition 1.4 is motivated by a very special case of Theorem 1.1. Indeed, we have chosen the matrix L and the numbers a_i of a special form. In the next section we repeat this procedure for L and a_i chosen with much greater degree of freedom, and then show that one can still relate the objects obtained this way to the boundedness of certain trajectories in the space Ω. Thus one can use results of [KM] on bounded orbits to establish an existence theorem generalizing W. Schmidt's results on badly approximable systems of linear forms.

§2. Bounded trajectories and $(\mathbf{r},\mathbf{s},\pm)$-loose matrices

2.1. To simplify the subsequent exposition, let us introduce the following notation: for a k-tuple $\mathbf{w} = (w_1, \dots, w_k)$, $k \in \mathbb{N}$, with positive components, define the $\mathbf{w}$-*quasinorm* $\|\cdot\|_{\mathbf{w}}$ on $\mathbb{R}^k$ by $\|\mathbf{x}\|_{\mathbf{w}} \stackrel{\text{def}}{=} \max_{1\le i\le k} |x_i|^{1/w_i}$. Of course it is not a norm unless all coordinates of $\mathbf{w}$ are equal to 1. A list of trivial properties of $\|\cdot\|_{\mathbf{w}}$ is given below as a

Lemma. (a) *For* $\lambda \in \mathbb{R}$, $\min_{1\le i\le k} |\lambda|^{1/w_i} \|\mathbf{x}\|_{\mathbf{w}} \le \|\lambda\mathbf{x}\|_{\mathbf{w}} \le \max_{1\le i\le k} |\lambda|^{1/w_i} \|\mathbf{x}\|_{\mathbf{w}}$;
(b) $\min_{1\le i\le k} \|\mathbf{x}\|^{1/w_i} \le \|\mathbf{x}\|_{\mathbf{w}} \le \max_{1\le i\le k} \|\mathbf{x}\|^{1/w_i}$; *in particular,*
(c) *if* $\mathbf{k}$ *stands for the* k*-tuple* $(\frac{1}{k}, \dots, \frac{1}{k})$, *then* $\|\cdot\|_{\mathbf{k}} = \|\cdot\|^k$.

In particular, this lemma means that the sets $\{\mathbf{x} \in \mathbb{R}^k \mid \|\mathbf{x}\|_{\mathbf{w}} < \delta\}$, $\delta > 0$, form a basis of open neighborhoods of 0 in $\mathbb{R}^k$.

Let now $\mathbf{w}$ be an $(m+n)$-tuple with positive components. Define a function $\delta_{\mathbf{w}} : \Omega \to \mathbb{R}_+$ by

$$\delta_{\mathbf{w}}(\Lambda) \stackrel{\text{def}}{=} \inf_{\mathbf{x}\in\Lambda\smallsetminus\{0\}} \|\mathbf{x}\|_{\mathbf{w}}\,,$$

i.e. the $\mathbf{w}$-quasinorm of a nonzero vector in a lattice Λ with minimal $\mathbf{w}$-quasinorm. For any $\Lambda \in \Omega$, $\delta_{\mathbf{w}}(\Lambda)$ is positive since Λ is discrete, and is not greater than 1 by Theorem 1.1 (note that $\Lambda = L\mathbb{Z}^{m+n}$ for some $L \in G$). Mahler's Compactness Criterion [R, Corollary 10.9] implies that the sets $\{\Lambda \in \Omega \mid \delta_{\mathbf{w}}(\Lambda) \ge \delta\}$, $\delta > 0$, are compact and exhaust the space Ω.

2.2. In what follows, we will fix an m-tuple $\mathbf{r}$ and an n-tuple $\mathbf{s}$ with positive components such that $\sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j$, and endow the space $\mathbb{R}^{m+n}$ with the $(\mathbf{r},\mathbf{s})$-quasinorm. To simplify the notation we will write $\|\cdot\|_{\mathbf{r},\mathbf{s}}$ instead of $\|\cdot\|_{(\mathbf{r},\mathbf{s})}$ and $\delta_{\mathbf{r},\mathbf{s}}(\cdot)$ instead of $\delta_{(\mathbf{r},\mathbf{s})}(\cdot)$. Note that for $\mathbf{p} \in \mathbb{R}^m$ and $\mathbf{q} \in \mathbb{R}^n$, $\|(\mathbf{p},\mathbf{q})\|_{\mathbf{r},\mathbf{s}}$ is by definition equal to $\max(\|\mathbf{p}\|_{\mathbf{r}}, \|\mathbf{q}\|_{\mathbf{s}})$.

Theorem. *For any* $L \in G$ *and* $t \in \mathbb{R}$ *there exists* $\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ *such that*

$$\|L^{(m)}\mathbf{x}\|_{\mathbf{r}} < e^{-t} \tag{2.1a}$$

and

$$\|L_{(n)}\mathbf{x}\|_{\mathbf{s}} \le e^{t}\,. \tag{2.1b}$$

Proof. Using the definition of the quasinorms $\|\cdot\|_{\mathbf{r}}$ and $\|\cdot\|_{\mathbf{s}}$, (2.1ab) can be rewritten as

$$|L_i(\mathbf{x})| < e^{-r_i t}, \quad 1 \le i \le m\,, \tag{2.1a$'$}$$

and

$$|L_{m+j}(\mathbf{x})| \le e^{s_j t}, \quad 1 \le j \le n\,, \tag{2.1b$'$}$$

and one simply has to take the numbers a_i in Theorem 1.1 to be right hand sides of (2.1a$'$) and (2.1b$'$). $\square$

One can easily get Theorem 1.2 as a special case of Theorem 2.2: take $L = L_A$ as in the proof of Theorem 1.2, $\mathbf{r} = \mathbf{m} \stackrel{\text{def}}{=} (\frac{1}{m}, \dots, \frac{1}{m})$ and $\mathbf{s} = \mathbf{n} \stackrel{\text{def}}{=} (\frac{1}{n}, \dots, \frac{1}{n})$. It is also clear that one can keep on generalizing the statements of the preceding section.

2.3. Corollary. *There exist infinitely many* $\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ *satisfying*

$$\|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} < 1 \tag{2.2}$$

and

$$\|L^{(m)}\mathbf{x}\|_{\mathbf{r}} < 1\,, \tag{2.2+}$$

i.e.

$$\mathbf{x} \neq 0 \quad \textit{and} \quad \|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \max(\|L^{(m)}\mathbf{x}\|_{\mathbf{r}}, \|L_{(n)}\mathbf{x}\|_{\mathbf{s}}) < 1\,; \tag{2.3+}$$

moreover, one can choose a sequence of solutions $\mathbf{x}_k$*,* $k \in \mathbb{N}$*, of* (2.3+) *such that* $L_{(n)}\mathbf{x}_k \to \infty$ *as* $k \to \infty$*. Similarly, there exist infinitely many* $\mathbf{x} \in \mathbb{Z}^{m+n}$ *satisfying*

$$\mathbf{x} \neq 0 \quad \textit{and} \quad \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} \max(\|L^{(m)}\mathbf{x}\|_{\mathbf{r}}, \|L_{(n)}\mathbf{x}\|_{\mathbf{s}}) < 1\,, \tag{2.3−}$$

with $L^{(m)}\mathbf{x}$ *being arbitrarily large.*

Proof. The proof of Corollary 1.3 applies almost verbatim. Indeed, (2.2) is a product of (2.1a) and (2.1b), while (2.2+) follows from (2.1a) if one takes $t \geq 0$. If $L^{(m)}\mathbf{x} = 0$ for some $\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$, one can take $\mathbf{x}_k = k\mathbf{x}$. Otherwise, for fixed $\mathbf{x}$, (2.1a) can only hold for small enough t; hence, as $t \to +\infty$, one gets infinitely many solutions $\mathbf{x}_k$ of (2.3+) with $L^{(m)}\mathbf{x}_k \to 0$ as $k \to \infty$. Thus $L_{(n)}\mathbf{x}_k \to \infty$ by discreteness of $L\mathbb{Z}^{m+n}$. The second part follows by the same argument with $t \to -\infty$, or by applying the first part to the matrix $\begin{pmatrix} L_{(n)} \\ L^{(m)} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} L$. $\square$

Note that in this generality, (2.2+) does not follow from (2.2) and $L_{(n)}\mathbf{x} \neq 0$, since it is possible that $0 < \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} < 1$. In fact, the two equal expressions for the constant $c(A)$ lead to, in general, different generalizations for the notion of badly approximable systems of linear forms (see Example 5.1). We will see that in order to define a notion most closely related to the boundedness of certain trajectories in Ω, one has to base upon the second expression in (1.4). Other possibilities are reviewed in §5.

2.4. Definition. Say that a matrix $L \in G$ is $(\mathbf{r}, \mathbf{s}, \pm)$-*loose* if, roughly speaking, one can not replace 1 in the right hand side of (2.3±) by arbitrarily small constant. More precisely, define the quantities

$$C_{\mathbf{r},\mathbf{s},+}(L) \stackrel{\text{def}}{=} \inf_{\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \max(\|L^{(m)}\mathbf{x}\|_{\mathbf{r}}, \|L_{(n)}\mathbf{x}\|_{\mathbf{s}})$$

and

$$C_{\mathbf{r},\mathbf{s},-}(L) \stackrel{\text{def}}{=} \inf_{\mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}} \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} \max(\|L^{(m)}\mathbf{x}\|_{\mathbf{r}}, \|L_{(n)}\mathbf{x}\|_{\mathbf{s}})$$

(both are always less than 1 by Corollary 2.3 and are not greater than $\delta_{\mathbf{r},\mathbf{s}}(L\mathbb{Z}^{m+n})^2$ by definition of $\delta_{\mathbf{r},\mathbf{s}}(\cdot)$). Then L will be called $(\mathbf{r}, \mathbf{s}, \pm)$-*loose* if $C_{\mathbf{r},\mathbf{s},\pm}(L) > 0$, and $(\mathbf{r}, \mathbf{s}, \pm)$-*tight* otherwise.

Clearly a system of linear forms given by $A \in M_{m,n}(\mathbb{R})$ is badly approximable iff $L = L_A$ is $(\mathbf{m}, \mathbf{n}, +)$-loose; moreover, $c(A) = C_{\mathbf{m},\mathbf{n},+}(L_A)$ for any $A \in M_{m,n}(\mathbb{R})$. What is even more significant, the ideas lying behind Dani's proof of Theorem 1.5 can be carried over to this more general situation, which results in the following dynamical interpretation of Definition 2.4:

2.5. Theorem. *Let $\{g_t\}$ be the one-parameter subgroup of G defined by $g_t = \mathrm{diag}(e^{r_1 t}, \dots, e^{r_m t}, e^{-s_1 t}, \dots, e^{-s_n t})$. Then $L \in G$ is $(\mathbf{r}, \mathbf{s}, \pm)$-loose iff the trajectory $\{g_t L \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_\pm\}$ is bounded in Ω.*

We will prove this theorem, or rather its quantitative version, in §3. Combined with the results from [KM], Theorem 2.5 gives the existence result, generalizing Schmidt's treatment of badly approximable systems of linear forms:

2.6. Theorem. *The sets $\mathcal{L}_{\mathbf{r},\mathbf{s},\pm}$ of $(\mathbf{r}, \mathbf{s}, \pm)$-loose matrices are thick in G and have zero Haar measure.*

In particular, both $\mathcal{L}_{\mathbf{r},\mathbf{s},+}$ and $\mathcal{L}_{\mathbf{r},\mathbf{s},-}$ are dense in G and have the cardinality of continuum. The proof of this theorem will be the subject of §4.

§3. Proof of Theorem 2.5

3.1. In this and several subsequent sections we fix an m-tuple $\mathbf{r}$, an n-tuple $\mathbf{s}$ and the one-parameter subgroup g_t as in Theorem 2.5. For the sake of brevity, we will mostly consider the $(\mathbf{r}, \mathbf{s}, +)$-case only; however it should be borne in mind that all the statements below have their $(\mathbf{r}, \mathbf{s}, -)$-counterparts. The passage from $(\mathbf{r}, \mathbf{s}, +)$-case to $(\mathbf{r}, \mathbf{s}, -)$-case is given by $(\mathbf{r}, \mathbf{s}) \to (\mathbf{s}, \mathbf{r})$, $L \to \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} L = \begin{pmatrix} L_{(n)} \\ L^{(m)} \end{pmatrix}$ and $g_t \to \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} g_t \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}^{-1} = \mathrm{diag}(e^{-s_1 t}, \dots, e^{-s_n t}, e^{r_1 t}, \dots, e^{r_m t})$.

We start by deriving another expression for the constant $C_{\mathbf{r},\mathbf{s},+}(L)$.

Lemma. *For $\mathbf{x} \in \mathbb{R}^{m+n}$ and $\delta > 0$, the following are equivalent:*

(i) $\|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \max(\|L^{(m)}\mathbf{x}\|_{\mathbf{r}}, \|L_{(n)}\mathbf{x}\|_{\mathbf{s}}) \le \delta^2$*;*

(ii) *there exists $t \in \mathbb{R}_+$ such that*

$$\begin{cases} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \le \delta e^{-t} \\ \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} \le \delta e^{t} \end{cases} . \tag{3.1}$$

Proof. Indeed, (i) can be written as

$$\begin{cases} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \le \delta \\ \|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} \le \delta^2 \end{cases} . \tag{3.1$'$}$$

The second inequality in (3.1′) is a product of two inequalities in (3.1), while the first inequality in (3.1′) follows from the first one in (3.1); thus (ii) implies (i). Now assume (3.1′). If $L^{(m)}\mathbf{x} = 0$, (3.1) clearly holds for large enough t. Otherwise take $e^t = \delta / \|L^{(m)}\mathbf{x}\|_{\mathbf{r}}$ and check that (3.1) is satisfied. This shows that (ii) follows from (i), and the proof is completed. □

Introduce a function $C_{\mathbf{r},\mathbf{s}}(L, \cdot) : \mathbb{R} \to \mathbb{R}_+$ by

$$C_{\mathbf{r},\mathbf{s}}(L, t) \stackrel{\text{def}}{=} \inf \left\{ \delta^2 \mid \exists\, \mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\} \text{ s. t. } (3.1) \text{ holds} \right\} .$$

¿From the definition of $\delta_{\mathbf{r},\mathbf{s}}(\cdot)$ it follows that $C_{\mathbf{r},\mathbf{s}}(L, 0) = \delta_{\mathbf{r},\mathbf{s}}(L\mathbb{Z}^{m+n})^2$ and $C_{\mathbf{r},\mathbf{s}}(L, t) \ge \delta_{\mathbf{r},\mathbf{s}}(L\mathbb{Z}^{m+n})^2 e^{-2|t|}$ for all t.[1] Also, Theorem 2.2 says that $C_{\mathbf{r},\mathbf{s}}(L, t) \le 1$ for all t. ¿From Lemma 3.1 one can easily deduce

[1]It is easy to see that this lower bound on the decay of $C_{\mathbf{r},\mathbf{s}}(L, t)$ as $t \to \infty$ can be attained: $C_{\mathbf{r},\mathbf{s}}(L, t) \le \mathrm{const} \cdot e^{-2|t|}$ whenever $\mathbb{Z}^{m+n} \cap \mathrm{Ker}\, L^{(m)} \ne \{0\}$.

3.2. Corollary. *For any $L \in G$,*

$$C_{\mathbf{r},\mathbf{s},+}(L) = \inf_{t \in \mathbb{R}_+} C_{\mathbf{r},\mathbf{s}}(L,t) = \inf\left\{\delta^2 \mid \exists\, t \in \mathbb{R}_+ \text{ and } \mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\} \text{ s. t. (3.1) holds}\right\}.$$

Thus L is $(\mathbf{r},\mathbf{s},+)$-tight iff for any $\delta > 0$ there exist $t \in \mathbb{R}_+$ such that the system (3.1) has a nonzero solution.

3.3. Example. For $A \in M_{m,n}(\mathbb{R})$, take $L = L_A$, $\mathbf{x} = (\mathbf{p},\mathbf{q})$, $\mathbf{r} = \mathbf{m}$ and $\mathbf{s} = \mathbf{n}$. Then the system (3.1) can be written as

$$\begin{cases} \|A\mathbf{q} - \mathbf{p}\|^m \le \delta e^{-t} \\ \|\mathbf{q}\|^n \le \delta e^t \end{cases}. \tag{3.1A}$$

Define the function $c(A,\cdot) : \mathbb{R} \to \mathbb{R}_+$ by

$$c(A,t) \stackrel{\text{def}}{=} C_{\mathbf{m},\mathbf{n}}(L_A,t) = \inf\left\{\delta^2 \mid \exists\, (\mathbf{p},\mathbf{q}) \in \mathbb{Z}^{m+n} \smallsetminus \{0\} \text{ s. t. (3.1A) holds}\right\}.$$

It is easy to see that for $t > 0$

$$c(A,t) = \inf\left\{\delta^2 \mid \exists\, \mathbf{p} \in \mathbb{Z}^m \text{ and } \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \text{ s. t. (3.1A) holds}\right\}.$$

The constant $c(A)$ defined in §1.4 is equal to $\inf_{t\in\mathbb{R}_+} c(A,t)$, and A is well approximable iff for any $\delta > 0$ there exist $t \in \mathbb{R}_+$ such that the system (3.1A) has a solution $(\mathbf{p},\mathbf{q})$ with $\mathbf{q} \neq 0$.

3.4. We are now just one step short of having a dynamical interpretation of the function $C_{\mathbf{r},\mathbf{s}}(L,\cdot)$, thus of the constant $C_{\mathbf{r},\mathbf{s},+}(L)$ as well. This step is provided by the following elementary

Lemma. *For $L \in G$, $\mathbf{x} \in \mathbb{R}^{m+n}$ and $t \in \mathbb{R}$,*

(a) $\|(g_tL)^{(m)}\mathbf{x}\|_{\mathbf{r}} = e^t\|L^{(m)}\mathbf{x}\|_{\mathbf{r}}$ *and* $\|(g_tL)_{(n)}\mathbf{x}\|_{\mathbf{s}} = e^{-t}\|L_{(n)}\mathbf{x}\|_{\mathbf{s}}$*;*

(b) *for $\delta > 0$, (3.1) is equivalent to $\|g_tL\mathbf{x}\|_{\mathbf{r},\mathbf{s}} \le \delta$.*

Proof. By definition of the $\mathbf{r}$-quasinorm, $\|(g_tL)^{(m)}\mathbf{x}\|_{\mathbf{r}}$ is equal to

$$\max_{1\le i\le m} |(g_tL)_i(\mathbf{x})|^{1/r_i} = \max_{1\le i\le m} |e^{r_it}L_i(\mathbf{x})|^{1/r_i} = \max_{1\le i\le m} e^t|L_i(\mathbf{x})|^{1/r_i} = e^t\|L^{(m)}\mathbf{x}\|_{\mathbf{r}},$$

same for $\|(g_tL)_{(n)}\mathbf{x}\|_{\mathbf{s}}$. Also,

$$\|g_tL\mathbf{x}\|_{\mathbf{r},\mathbf{s}} \le \delta \Leftrightarrow \begin{cases} \|(g_tL)^{(m)}\mathbf{x}\|_{\mathbf{r}} \le \delta \\ \|(g_tL)_{(n)}\mathbf{x}\|_{\mathbf{s}} \le \delta \end{cases} \underset{\text{(a)}}{\Leftrightarrow} \begin{cases} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}} \le \delta e^{-t} \\ \|L_{(n)}\mathbf{x}\|_{\mathbf{s}} \le \delta e^t \end{cases}. \quad \square$$

3.5. Corollary. *For any $L \in G$ and $t \in \mathbb{R}$, $C_{\mathbf{r},\mathbf{s}}(L,t) = \delta_{\mathbf{r},\mathbf{s}}(g_tL\mathbb{Z}^{m+n})^2$, in particular,*

$$C_{\mathbf{r},\mathbf{s},+}(L) = \left(\inf_{t\in\mathbb{R}_+} \delta_{\mathbf{r},\mathbf{s}}(g_tL\mathbb{Z}^{m+n})\right)^2.$$

Proof. Use part (b) of the above lemma and the definitions of $C_{\mathbf{r},\mathbf{s}}(L,\cdot)$ and $\delta_{\mathbf{r},\mathbf{s}}(\cdot)$. $\square$

Proof of Theorem 2.5. ¿From Mahler's Compactness Criterion it follows that the trajectory $\{g_t\Lambda \mid t \in \mathbb{R}_+\}$ is bounded in Ω iff $\inf_{t\in\mathbb{R}_+} \delta_{\mathbf{r},\mathbf{s}}(g_t\Lambda) > 0$. $\square$

Corollary 3.5 is clearly a quantitative version of Theorem 2.5: the "size" $\inf_{t\in\mathbb{R}_+} \delta_{\mathbf{r},\mathbf{s}}(g_tL\mathbb{Z}^{m+n})$ of the trajectory $\{g_tL\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$ turns out to be exactly the square root of the Diophantine constant $C_{\mathbf{r},\mathbf{s},+}(L)$. Moreover, the rate of decay of $C_{\mathbf{r},\mathbf{s}}(L,t_k)$ for a sequence $t_k \to +\infty$ corresponds to the rate with which the sequence $\{g_{t_k}L\mathbb{Z}^{m+n}\}$ escapes to infinity in Ω. (See §8.3 for another Diophantine interpretation of the decay of $\delta_{\mathbf{r},\mathbf{s}}(g_tL\mathbb{Z}^{m+n})$.)

§4. Proof of Theorem 2.6

4.1. We start by reviewing the results from [KM] on bounded orbits of flows on homogeneous spaces of Lie groups. Let G^0 be a connected Lie group, Γ a lattice in G^0, $\{g_t \mid t \in \mathbb{R}\}$ a one-parameter subgroup of G^0 which is not *quasiunipotent* (that is, $\operatorname{Ad} g_1$ has an eigenvalue with modulus different from 1). Denote by H^+ the subgroup of G^0 which is *horospherical* with respect to $\{g_t \mid t \in \mathbb{R}_-\}$, i.e. $H^+ \stackrel{\text{def}}{=} \{h \in G^0 \mid g_t h g_{-t} \to e \text{ as } t \to -\infty\}$. The following is a special case of [KM, Theorems 1.1 and 1.5]:

Theorem. *Let G^0 be a connected semisimple Lie group without compact factors, Γ an irreducible lattice in G^0, $\{g_t\}$ a one-parameter nonquasiunipotent subgroup of G^0. Then*

(a) *for any $x \in G^0/\Gamma$ and any neighborhood V of identity in H^+,*

$$\dim\bigl(\{h \in V \mid \{g_t h x \mid t \in \mathbb{R}_+\} \text{ is bounded in } G^0/\Gamma\}\bigr) = \dim(H^+)\,;$$

(b) *for any $x \in G^0/\Gamma$ and any neighborhood U of identity in G^0,*

$$\dim\bigl(\{h \in U \mid \{g_t h x \mid t \in \mathbb{R}\} \text{ is bounded in } G^0/\Gamma\}\bigr) = \dim(G^0)\,.$$

4.2. We now take $G^0 = SL_{m+n}(\mathbb{R})$, the connected component of the identity in G, and let $\Gamma = SL_{m+n}(\mathbb{Z})$, G^0/Γ being identified with Ω. As before, put $g_t = \operatorname{diag}(e^{r_1 t}, \dots, e^{r_m t}, e^{-s_1 t}, \dots, e^{-s_n t})$. Then H^+ is contained in a subgroup of G^0 conjugate to the group of upper triangular unipotent matrices (and is exactly equal to this group if $r_1 > \dots > r_m$ and $s_1 < \dots < s_n$).

Theorems 2.5 and 4.1 can be combined to give

Theorem. *For any subgroup G' of G containing H^+ and any right coset $G'M$, $M \in G$, of G', the intersection $\mathcal{L}_{\mathbf{r},\mathbf{s},+} \cap G'M$ is thick in $G'M$. Equivalently, for any neighborhood V of identity in G',*

$$\dim\bigl(\{L \in V \mid LM \text{ is } (\mathbf{r},\mathbf{s},+)\text{-loose}\}\bigr) = \dim(G')\,.$$

Proof. By standard "slicing" argument, such as Marstrand Slicing Theorem (cf. [F, Theorem 5.8] or [KM, Lemma 1.4]), it suffices to prove the claim for $G' = H^+$. Take $M \in G$ and a neighborhood V of identity in H^+; in view of Theorem 2.5, one needs to prove that

$$\dim\bigl(\{L \in V \mid \{g_t L M \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\} \text{ is bounded in } \Omega\}\bigr) = \dim(H^+)\,.$$

Write $M = M'M''$ with $\det(M') = 1$ and $M'' \in GL_{m+n}(\mathbb{Z})$. Then $M\mathbb{Z}^{m+n} = M'\mathbb{Z}^{m+n}$, so one can apply Theorem 4.1(a) with $x = M'\mathbb{Z}^{m+n}$ to finish the proof. $\square$

4.3. *Proof of Theorem 2.6.* The dimension part follows from Theorem 4.2 by taking $G' = G$; the measure part is a consequence of Theorem 2.5 and ergodicity [Mo] of g_t-action on Ω. $\square$

§5. More on $(\mathbf{r},\mathbf{s},+)$-loose matrices

5.1. Example. One may want to ask whether the condition

$$C^{(0)}_{\mathbf{r},\mathbf{s},+}(L) \stackrel{\text{def}}{=} \inf_{\substack{\mathbf{x}\in\mathbb{Z}^{m+n} \\ L_{(n)}\mathbf{x}\neq 0}} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}}\|L_{(n)}\mathbf{x}\|_{\mathbf{s}} > 0\,, \tag{5.1}$$

motivated by one of the equivalent expressions in (1.4), is equivalent to being $(\mathbf{r},\mathbf{s},+)$-loose. The answer is negative, which can be seen even in the simplest case $m=n=1$. Denote by L_α the matrix $\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$, and take

$$L = \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} L_\alpha = \begin{pmatrix} a & -a\alpha \\ c & -c\alpha + 1/a \end{pmatrix},$$

with $ac \neq 0$, α badly approximable and $\frac{1}{ac} - \alpha$ irrational and well approximable. The lattice $L_\alpha\mathbb{Z}^2$, as we already know, has bounded g_t-trajectory, $t\in\mathbb{R}_+$, and so does $L\mathbb{Z}^2$ (cf. [D1, Proposition 2.12]). On the other hand, one can choose $p,q\in\mathbb{Z}$ such that $|(\frac{1}{ac}-\alpha)q + p|\cdot|q|$ is arbitrarily small. This makes $|L^{(1)}(p,q)|\cdot|L_{(1)}(p,q)| = |ac|\cdot|\alpha q - p|\cdot|(\frac{1}{ac}-\alpha)q+p|$ arbitrarily small and shows that $C^{(0)}_{1,1,+}(L) = 0$. Similarly, one can find a matrix $L\in M_{1,1}(\mathbb{R})$ for which $C_{1,1,+}(L)$ is strictly smaller than $C^{(0)}_{1,1,+}(L)$, see [K, Example 7.4.2].

5.2. However, a modification of (5.1) leads to another equivalent definition of L being $(\mathbf{r},\mathbf{s},+)$-loose. Namely, consider a (nondecreasing) function $C^{(\cdot)}_{\mathbf{r},\mathbf{s},+}(L) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$C^{(\sigma)}_{\mathbf{r},\mathbf{s},+}(L) \stackrel{\text{def}}{=} \inf_{\substack{\mathbf{x}\in\mathbb{Z}^{m+n} \\ \|L_{(n)}\mathbf{x}\|_{\mathbf{s}}>\sigma}} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}}\|L_{(n)}\mathbf{x}\|_{\mathbf{s}}\,.$$

We will give a dynamical interpretation for its value at infinity, which is equal to

$$C^{(\infty)}_{\mathbf{r},\mathbf{s},+}(L) = \inf\left\{\varepsilon \mid \exists \text{ a sequence } \mathbf{x}_k\in\mathbb{Z}^{m+n} \text{ s. t. } \begin{cases} L_{(n)}\mathbf{x}_k \to \infty \text{ as } k\to\infty \\ \|L^{(m)}\mathbf{x}_k\|_{\mathbf{r}}\|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}} \le \varepsilon \end{cases}\right\}$$

(note that $C^{(\infty)}_{\mathbf{r},\mathbf{s},+}(L) < 1$ for any $L\in G$ by Corollary 2.3).

Proposition. *For any $L\in G$,*
(a) $C^{(\infty)}_{\mathbf{r},\mathbf{s},+}(L) = \left(\liminf_{t\to+\infty}\delta_{\mathbf{r},\mathbf{s}}(g_t\mathbb{Z}^{m+n})\right)^2$;
(b) *The following are equivalent:* (i) L *is* $(\mathbf{r},\mathbf{s},+)$*-loose;*
(ii) $C^{(\infty)}_{\mathbf{r},\mathbf{s},+}(L) > 0$;
(iii) $C^{(\sigma)}_{\mathbf{r},\mathbf{s},+}(L) > 0$ *for some* $\sigma > 0$;
(iv) $C^{(\sigma)}_{\mathbf{r},\mathbf{s},+}(L) > 0$ *for all* $\sigma > 0$;
(v) $C^{(\sigma)}_{\mathbf{r},\mathbf{s},+}(L) > \sigma^2$ *for small enough* $\sigma > 0$.

Thus L is $(\mathbf{r},\mathbf{s},+)$-loose iff one can not replace 1 in the right hand side of (2.2) by arbitrarily small constant and still get infinitely many solutions $\mathbf{x}_k$ with $L_{(n)}\mathbf{x}_k \to \infty$ (equivalently, with $\|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}}$ bounded from below). Note that the equality in (a) can be thought of as another quantitative version of Theorem 2.5.

Proof. In view of Corollary 3.5, to prove (a) it suffices to show that

$$C^{(\infty)}_{\mathbf{r},\mathbf{s},+}(L) = \liminf_{t\to+\infty} C_{\mathbf{r},\mathbf{s}}(L,t)\,. \tag{5.2}$$

Unwinding the definitions, one can see that (5.2) amounts to the following statement: for any positive $\delta < 1$,

$$\exists \text{ sequences } t_k \to \infty \text{ and } \mathbf{x}_k \in \mathbb{Z}^{m+n} \smallsetminus \{0\} \text{ s. t. } \begin{cases} \|L^{(m)}\mathbf{x}_k\|_{\mathbf{r}} \le \delta e^{-t_k} \\ \|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}} \le \delta e^{t_k} \end{cases} \tag{5.3}$$

is equivalent to

$$\exists \text{ a sequence } \mathbf{x}_k \in \mathbb{Z}^{m+n} \text{ s. t. } \begin{cases} L_{(n)}\mathbf{x}_k \to \infty \\ \|L^{(m)}\mathbf{x}_k\|_{\mathbf{r}}\|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}} \le \delta^2 \end{cases}. \tag{5.4}$$

Assuming (5.4), one can take $e^{t_k} = \|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}}/\delta$ and check that $t_k \to +\infty$ and that the inequalities in (5.3) are satisfied. Conversely, assume (5.3) and use the argument of the proof of Corollary 2.3 to get infinitely many different solutions $\mathbf{x}_k$ of the second inequality in (5.4) (which is just the product of the two inequalities in (5.3)) such that $L_{(n)}\mathbf{x}_k \to \infty$ as $k \to \infty$.

The implications (v)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii) in (b) are trivial, while (ii)$\Rightarrow$(i) follows from (a) and Mahler's Compactness Criterion. Finally, assume that L is $(\mathbf{r},\mathbf{s},+)$-loose, and take any positive $\sigma < \inf_{t\in\mathbb{R}^+}\delta_{\mathbf{r},\mathbf{s}}(L\mathbb{Z}^{m+n})$ and any $\mathbf{x} \in \mathbb{Z}^{m+n}$ with $\|L_{(n)}\mathbf{x}\|_{\mathbf{s}} > \sigma$. Define $t > 0$ by $e^t = \|L_{(n)}\mathbf{x}\|_{\mathbf{s}}/\sigma$, then, by Lemma 3.4(a), $\|(g_tL)_{(n)}\mathbf{x}\|_{\mathbf{s}}$ is exactly equal to σ. This, by definition of $\delta_{\mathbf{r},\mathbf{s}}(\cdot)$, forces $\|(g_tL)^{(m)}\mathbf{x}\|_{\mathbf{r}}$ to be greater than σ. Therefore $\|(g_tL)^{(m)}\mathbf{x}\|_{\mathbf{r}}\|(g_tL)_{(n)}\mathbf{x}\|_{\mathbf{s}} = \|L^{(m)}\mathbf{x}\|_{\mathbf{r}}\|L_{(n)}\mathbf{x}\|_{\mathbf{s}} > \sigma^2$, which is all one needs to prove (v). $\square$

This theorem, in particular, shows that $\{L \in G \mid C^{(0)}_{\mathbf{r},\mathbf{s},+}(L) > 0\}$ is a subset of $\mathcal{L}_{\mathbf{r},\mathbf{s},+}$. The fact that this subset is also thick in G follows from Theorem 6.3 below.

§6. Bounded two-sided orbits and $(\mathbf{r},\mathbf{s})$-loose matrices

6.1. Definition. Say that a matrix $L \in G$ is $(\mathbf{r},\mathbf{s})$-*loose* if it is both $(\mathbf{r},\mathbf{s},+)$- and $(\mathbf{r},\mathbf{s},-)$-loose; in other words, if

$$C_{\mathbf{r},\mathbf{s}}(L) \stackrel{\text{def}}{=} \min\big(C_{\mathbf{r},\mathbf{s},+}(L), C_{\mathbf{r},\mathbf{s},-}(L)\big) > 0\,.$$

A trivial identity $ab = \min\big(a\max(a,b), b\max(a,b)\big)$ shows that in fact

$$C_{\mathbf{r},\mathbf{s}}(L) = \inf_{\mathbf{x}\in\mathbb{Z}^{m+n}\smallsetminus\{0\}} \|L^{(m)}\mathbf{x}\|_{\mathbf{r}}\|L_{(n)}\mathbf{x}\|_{\mathbf{s}}\,.$$

Thus L is $(\mathbf{r},\mathbf{s})$-loose if one can not replace 1 in the right hand side of (2.2) by arbitrarily small constant unless $\mathbf{x} = 0$.

With this definition, one can use Theorem 3.2 and its $(\mathbf{r},\mathbf{s},-)$ analogue to deduce

6.2. Theorem. *For any $L \in G$, $C_{\mathbf{r},\mathbf{s}}(L) = \big(\inf_{t\in\mathbb{R}}\delta_{\mathbf{r},\mathbf{s}}(g_t\mathbb{Z}^{m+n})\big)^2$; in particular, L is $(\mathbf{r},\mathbf{s})$-loose iff the orbit $\{g_tL\mathbb{Z}^{m+n} \mid t \in \mathbb{R}\}$ is bounded in Ω.*

6.3. Denote by $\mathcal{L}_{\mathbf{r},\mathbf{s}} \subset G$ the set of $(\mathbf{r},\mathbf{s})$-loose matrices. Since $\mathcal{L}_{\mathbf{r},\mathbf{s}} \subset \mathcal{L}_{\mathbf{r},\mathbf{s},\pm}$ (in fact, $\mathcal{L}_{\mathbf{r},\mathbf{s}} = \mathcal{L}_{\mathbf{r},\mathbf{s},+} \cap \mathcal{L}_{\mathbf{r},\mathbf{s},-}$), Theorem 4.3 implies that $\mathcal{L}_{\mathbf{r},\mathbf{s}}$ has zero Haar measure. The fact that this set is still big enough is less trivial and follows from the results of [KM].

Theorem. *$\mathcal{L}_{\mathbf{r},\mathbf{s}}$ is a thick subset of G.*

Proof. We keep the notation $G^0 = SL_{m+n}(\mathbb{R})$ as in §4. Take a neighborhood U of identity in G^0; in view of Theorem 6.2, one has to prove that for any $M \in G$,

$$\dim\big(\{L \in U \mid \{g_t LM\mathbb{Z}^{m+n} \mid t \in \mathbb{R}\} \text{ is bounded in } \Omega\}\big) = \dim(G)\,.$$

As in the proof of Theorem 4.2, this can be reduced to the case $M \in G^0$, and an application of Theorem 4.1(b) finishes the proof. □

6.4. Example. Look at the simplest possible case $m = n = 1$. It is easy to see (cf. [D1, Proposition 2.12]) that a matrix $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ is $(1,1,+)$-loose iff b/a is badly approximable, and is $(1,1,-)$-loose iff d/c is badly approximable. Thus the set $\mathcal{L}_{1,1} \cap SL_2(\mathbb{R})$ can be described as

$$\left\{ L = \begin{pmatrix} a & \alpha a \\ \frac{1}{a(\beta-\alpha)} & \frac{\beta}{a(\beta-\alpha)} \end{pmatrix} \,\middle|\, a \neq 0,\ \alpha \neq \beta \text{ badly approximable} \right\},$$

with $C_{1,1}(L) = \min\big(c(\alpha), c(\beta)\big)$.

§7. Divergent trajectories and $(\mathbf{r},\mathbf{s},+)$-singular matrices

7.1. Definition. Let $A \in M_{m,n}(\mathbb{R})$. Consider the function $c(A,\cdot)$ defined in Example 3.3, and introduce the constant

$$c^*(A) \stackrel{\text{def}}{=} \limsup_{t\to+\infty} c(A,t) = \inf\left\{ \delta^2 \,\middle|\, \begin{array}{l} \exists\, t_0 \text{ such that } \forall\, t \geq t_0\ \exists\, \mathbf{p} \in \mathbb{Z}^m \\ \text{and } \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \text{ with } (3.1A) \end{array} \right\};$$

it is not greater than 1 by Theorem 1.2.

A system of linear forms given by $A \in M_{m,n}(\mathbb{R})$ is said to be *singular* (cf. [C] or [D1]) if $c^*(A) = 0$; in other words, if $c(A,t) \to 0$ as $t \to +\infty$; in other words, if for any $\delta > 0$ there exists t_0 such that for any $t \geq t_0$ the system (3.1A) has a solution $(\mathbf{p},\mathbf{q})$ with $\mathbf{q} \neq 0$.[2] We remark that it is proven in [C] that the set of matrices $A \in M_{m,n}(\mathbb{R})$ such that the corresponding system of linear forms is singular has zero Lebesgue measure.

7.2. In [D1, Theorem 2.14], S.G. Dani related this notion with the dynamics of flows in the space of lattices. The following is a restatement of Dani's result:

Theorem. *A system of linear forms given by $A \in M_{m,n}(\mathbb{R})$ is singular iff $\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\} \subset \Omega$, with g_t as in Theorem 1.5 and L_A as in the proof of Theorem 1.2, is a divergent trajectory.*

By now it should be perfectly clear that the results and methods of §3 can be used to immediately generalize both Definition 7.1 and Theorem 7.2, as well as give a dynamical interpretation of the constant $c^*(A)$ defined above.

[2] The standard definition uses the system $\begin{cases} \|A\mathbf{q} - \mathbf{p}\| \leq \varepsilon b^{-n/m} \\ \|\mathbf{q}\| \leq b \end{cases}$ instead of (3.1A), which is the same as (3.1A) if one puts $\varepsilon = \delta^2$ and $b = (\delta e^t)^{1/n}$.

7.3. Definition. For $L \in G$ and $\mathbf{r}$, $\mathbf{s}$ as before, define the constant

$$C^*_{\mathbf{r},\mathbf{s},+}(L) \stackrel{\text{def}}{=} \limsup_{t\to\infty} C_{\mathbf{r},\mathbf{s}}(L,t) = \inf\left\{\delta^2 \;\middle|\; \begin{array}{c} \exists\, t_0 \text{ such that } \forall\, t \geq t_0 \\ \exists\, \mathbf{x} \in \mathbb{Z}^{m+n} \smallsetminus \{0\} \text{ with } (3.1) \end{array}\right\};$$

it is not greater than 1 by Theorem 2.2.

Now say that L is $(\mathbf{r},\mathbf{s},+)$*-singular* if $C^*_{\mathbf{r},\mathbf{s},+}(L) = 0$; in other words, if $C_{\mathbf{r},\mathbf{s}}(L,t) \to 0$ as $t \to +\infty$; in other words, if for any $\delta > 0$ there exists t_0 such that for any $t \geq t_0$ the system (3.1) has a nonzero solution.

Clearly for $A \in M_{m,n}(\mathbb{R})$, $c^*(A) = C^*_{\mathbf{m},\mathbf{n},+}(L_A)$; therefore A is singular iff L_A is $(\mathbf{m},\mathbf{n},+)$-singular. To complete the picture, take g_t as in Theorem 2.5 and obtain

7.4. Theorem. *For any $L \in G$, $C^*_{\mathbf{r},\mathbf{s},+}(L) = \big(\limsup_{t\to+\infty} \delta_{\mathbf{r},\mathbf{s}}(g_t\mathbb{Z}^{m+n})\big)^2$; hence L is $(\mathbf{r},\mathbf{s},+)$-singular iff $\{g_tL\mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\} \subset \Omega$ is a divergent trajectory.*

Proof. From Mahler's Compactness Criterion it follows that

$$\{g_t\Lambda \mid t \in \mathbb{R}_+\} \text{ is divergent} \quad \Leftrightarrow \quad \limsup_{t\to+\infty} \delta_{\mathbf{r},\mathbf{s}}(g_t\Lambda) = 0\,,$$

and it remains to apply Corollary 3.5. $\square$

As a consequence of this theorem and ergodicity of g_t-action on Ω, one gets

7.5. Corollary. *The set of $(\mathbf{r},\mathbf{s},+)$-singular matrices $L \in G$ has zero Haar measure in G.*

§8. Concluding remarks and open questions

8.1. In the paper [S3], W. Schmidt proved that the set of matrices such that the corresponding system of linear forms is badly approximable is a *winning* (cf. [S2, D3]) subset of $M_{m,n}(\mathbb{R})$.

Question. Is it true that the sets $\mathcal{L}_{\mathbf{r},\mathbf{s},\pm}$ are winning subsets of G?

8.2. It seems rather natural to say that a system of linear forms given by $A \in M_{m,n}(\mathbb{R})$ is $(\mathbf{r},\mathbf{s})$*-badly approximable* if L_A is $(\mathbf{r},\mathbf{s},+)$-loose. In other words, if

$$C_{\mathbf{r},\mathbf{s},+}(L_A) = \inf_{\substack{\mathbf{p}\in\mathbb{Z}^m \\ \mathbf{q}\in\mathbb{Z}^n\smallsetminus\{0\}}} \|A\mathbf{q} - \mathbf{p}\|_{\mathbf{r}} \|\mathbf{q}\|_{\mathbf{s}} > 0\,, \tag{8.1}$$

the only difference between this definition and Definition 1.4 being the use of "quasi-norms" $\|\cdot\|_{\mathbf{r}}$ and $\|\cdot\|_{\mathbf{s}}$ with arbitrarily chosen $\mathbf{r}$ and $\mathbf{s}$, rather than $\mathbf{r} = \mathbf{m}$ and $\mathbf{s} = \mathbf{n}$. One can think of components of $\mathbf{r}$ and $\mathbf{s}$ as of weights measuring the importance of forms A_i and variables q_j. In particular, one can talk about an $\mathbf{r}$-badly approximable m-tuple (the case $n = 1$) or an $\mathbf{s}$-badly approximable linear form ($m = 1$).

Questions. For arbitrary choice of $\mathbf{r}$ and $\mathbf{s}$, (i) do there exist $(\mathbf{r},\mathbf{s})$-badly approximable systems of linear forms? (ii) is the set of $A \in M_{m,n}(\mathbb{R})$ satisfying (8.1) thick? (iii) is it a winning subset of $M_{m,n}(\mathbb{R})$?

8.3. One of the problems arising in Diophantine approximation is describing sets of real numbers (or systems of linear forms) which admit certain *order of approximation* (cf. [Kh2]). With this in mind, one can modify the main definition of §2 as follows. Let $\psi(x)$ be a positive function of $x \in [x_0, \infty)$. Say that $L \in G$ is $(\mathbf{r}, \mathbf{s}, +)$*-tighter than* ψ if there exists a sequence $\mathbf{x}_k \in \mathbb{Z}^{m+n}$ such that

$$\begin{cases} L_{(n)}\mathbf{x}_k \to \infty \text{ as } k \to \infty \\ \|L^{(m)}\mathbf{x}_k\|_{\mathbf{r}} \|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}} \le \psi(\|L_{(n)}\mathbf{x}_k\|_{\mathbf{s}}) \end{cases} . \tag{8.2}$$

By Corollary 2.3, any $L \in G$ is $(\mathbf{r}, \mathbf{s}, +)$-tighter than 1. Proposition 5.2 says that L is $(\mathbf{r}, \mathbf{s}, +)$-tight iff it is $(\mathbf{r}, \mathbf{s}, +)$-tighter than ε for any $\varepsilon > 0$; moreover, $C^{(\infty)}_{\mathbf{r},\mathbf{s},+}(L) = \inf\{\varepsilon \mid L \text{ is } (\mathbf{r}, \mathbf{s}, +)\text{-tighter than } \varepsilon\}$. Note also that L is $(\mathbf{r}, \mathbf{s}, +)$-tighter than ψ for any positive $\psi(x)$ iff $\mathbb{Z}^{m+n} \cap \operatorname{Ker} L^{(m)} \neq \{0\}$.

A modification of the proof of Proposition 5.2 (a) yields the following result:

Theorem. *Let $\psi(x)$ be a positive nonincreasing function of $x \in [x_0, \infty)$. Then there exists a positive nonincreasing function $f(t)$ of $t \in [t_0, \infty)$ such that $L \in G$ is $(\mathbf{r}, \mathbf{s}, +)$-tighter than ψ iff $\delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n}) \le f(t)$ for infinitely many arbitrarily large values of t.*

In other words, the rate of decay of $\inf_{0 \le t \le T} \delta_{\mathbf{r},\mathbf{s}}(g_t L\mathbb{Z}^{m+n})$ as $T \to \infty$ reflects certain Diophantine properties of L, generalizing the notion of the order of approximation of a real number or a system of linear forms. The details are left to the reader.

8.4. As in metrical theory of Diophantine approximation, a natural problem is to describe the class of functions ψ for which almost every $L \in G$ is $(\mathbf{r}, \mathbf{s}, +)$-tighter than ψ. Comparing with the situation in the approximation theory of real numbers [Kh1] or systems of linear forms [G, S1], one can conjecture that this class is defined by the condition $\int_{x_0}^{\infty} \frac{\psi(x)}{x}\, dx = \infty$ (in fact, the main result in [G] settles the case $\mathbf{r} = \mathbf{m}$ and $\mathbf{s} = \mathbf{n}$).

In view of Theorem 8.3, this conjecture means that for almost all $\Lambda \in \Omega$ the value of $\delta_{\mathbf{r},\mathbf{s}}(g_t\Lambda)$ is not greater than $f(t)$ for infinitely many arbitrarily large t if and only if a certain integral[3] diverges – a statement that can be thought of as a lattice analogue of Sullivan's logarithm law for geodesics (see [Su]). These and other related ideas are to be discussed in a forthcoming paper.

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