

AN ‘ALMOST ALL VERSUS NO’ DICHOTOMY IN HOMOGENEOUS DYNAMICS AND DIOPHANTINE APPROXIMATION

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ABSTRACT. Let Y_0 be a not very well approximable $m \times n$ matrix, and let \mathcal{M} be a connected analytic submanifold in the space of $m \times n$ matrices containing Y_0 . Then almost all $Y \in \mathcal{M}$ are not very well approximable. This and other similar statements are cast in terms of properties of certain orbits on homogeneous spaces and deduced from quantitative nondivergence estimates for ‘quasi-polynomial’ flows on the space of lattices.

Dedicated to S.G. Dani on the occasion of his 60th birthday

1. INTRODUCTION

This work is motivated by a result from a recent paper [D2] by S.G. Dani. Let G be a connected Lie group and Γ a lattice in G . Suppose a is a semisimple element of G , and let

$$U = \{u \in G : a^{-n}ua^n \rightarrow e \text{ as } n \rightarrow \infty\}$$

be the *expanding horospherical subgroup* with respect to a . Now suppose that U is not contained in any proper closed normal subgroup of G , take an arbitrary sequence of natural numbers $n_k \rightarrow \infty$, and denote by \mathcal{A} the set $\{a^{n_k} : k \in \mathbb{N}\}$. Then it follows from results of N. Shah [Sh1] that for any $x \in G/\Gamma$, the set

$$\{u \in U : \mathcal{A}ux \text{ is dense in } G/\Gamma\} \quad (1.1)$$

has full (Haar) measure.

One of the themes in [D2] is a close investigation of sets of type (1.1). Namely, the following is a special case of [D2, Corollary 2.3]:

Theorem 1.1. *Let G , Γ , a , \mathcal{A} and U be as above, and let $\{u_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of U . Suppose that for some $t_0 \in \mathbb{R}$ and $x \in G/\Gamma$, $\mathcal{A}u_{t_0}x$ is dense in G/Γ . Then $\mathcal{A}u_t x$ is dense in G/Γ for almost all $t \in \mathbb{R}$.*

In other words, an interesting dichotomy takes place: either a one-parameter subgroup of U is contained in the complement to the set (1.1), or it intersects it in a set of full measure.

In this note we discuss other situations where analogous conclusions can be derived. That is, we consider certain properties of points in a big ‘ambient’ set (U in the above example) which happen to be generic (satisfied for almost all

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points in that set), and show that some ‘nice’ measures μ on this set satisfy a similar dichotomy: that is, those properties hold either for μ -almost all points, or for no points in the support of μ . In particular, such a phenomenon has been observed in metric theory of simultaneous Diophantine approximation, which will be the main context in the present paper. For positive integers m, n , we denote by $M_{m,n}$ the space of $m \times n$ matrices with real entries; this will be the ambient space of our interest. We will interpret elements $Y \in M_{m,n}$ as systems of m linear forms in n variables. Properties of Y of our interest will be cast in terms of existence or non-existence of not too large integer vectors $\mathbf{q} \in \mathbb{Z}^n$ such that $\text{dist}(Y\mathbf{q}, \mathbb{Z}^m)$ is small. Here are two examples.

Definition 1.2. *The Diophantine exponent $\omega(Y)$ of $Y \in M_{m,n}$ is the supremum of $v > 0$ for which*

$$\text{dist}(Y\mathbf{q}, \mathbb{Z}^m) < \|\mathbf{q}\|^{-v} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n. \quad (1.2)$$

Here $\|\cdot\|$ and ‘dist’ depend on the choice of norms, but the above definition does not. It is easy to see that $\omega(Y) = n/m$ for Lebesgue almost all $Y \in M_{m,n}$; those Y for which $\omega(Y)$ is strictly bigger than n/m are called *very well approximable* (VWA).

Definition 1.3. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing continuous function. Say that $Y \in M_{m,n}$ is φ -singular if for any $c > 0$ there is N_0 such that for all $N \geq N_0$ one can find $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ with*

$$\text{dist}(Y\mathbf{q}, \mathbb{Z}^m) < \frac{c\varphi(N)}{N^{m/n}} \text{ and } \|\mathbf{q}\| < cN. \quad (1.3)$$

As the previous one, this definition is norm-independent. One says that Y is *singular* if it is φ -singular with $\varphi \equiv 1$. Note that the property of being φ -singular depends only on the equivalence class of φ (its tail up to a multiplicative constant) and holds for Lebesgue almost no Y as long as φ is bounded, as shown by Khintchine. Also, in view of Khintchine’s Transference Principle, see [C, Chapter V], Y is singular or very well approximable if and only if so is its transpose.

Proving that almost all Y with respect to some natural measures other than Lebesgue do not have the above (and some other similar) properties has been an active direction of research. Its motivation comes from a conjecture of Mahler [M] (1932, settled by Sprindžuk in 1964, see [Sp1, Sp2]) that $(x \ x^2 \ \dots \ x^n) \in M_{n,1} \cong \mathbb{R}^n$ is not VWA for Lebesgue almost every $x \in \mathbb{R}$. In other words, in Sprindžuk’s terminology, the curve

$$\{(x \ x^2 \ \dots \ x^n) : x \in \mathbb{R}\} \quad (1.4)$$

is *extremal*. Later [KM1, Theorem A] the same conclusion was established for submanifolds of \mathbb{R}^n of the form $\{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in U\}$, where $U \subset \mathbb{R}^d$ is open and connected, $\mathbf{f} = (f_1, \dots, f_n) : U \rightarrow \mathbb{R}^n$ is real analytic, and

$$1, f_1, \dots, f_n \text{ are linearly independent over } \mathbb{R}. \quad (1.5)$$

This settled a conjecture made by Sprindžuk in 1980 [Sp3]. A version of this result with ‘VWA’ replaced by ‘singular’ can be found in [KW2] and in a stronger form in [KW3].

On the other hand, it is easy to construct examples of non-extremal analytic submanifolds of \mathbb{R}^n , see §3.2 for more detail. More precisely, a necessary and sufficient condition for the extremality of an affine subspace $\mathcal{L} \subset \mathbb{R}^n$ is given in [K1]. This condition is explicitly written in terms of coefficients of parameterizing maps for \mathcal{L} , and, incidentally, it is shown that \mathcal{L} is not extremal if and only if all its points are VWA. Furthermore, the same dichotomy holds for any connected analytic submanifold \mathcal{M} of \mathbb{R}^n : either almost every¹ point of \mathcal{M} is not VWA, or all points of \mathcal{M} are. In [K1] this has been done by finding an explicit necessary and sufficient condition involving the smallest affine subspace \mathcal{L} containing \mathcal{M} and via so-called ‘inheritance theorems’ generalizing the work in [KM1]. See also [K2, Zh] for extensions of these results with VWA replaced by ‘having Diophantine exponent bigger than v ’ for an arbitrary v .

Note that the proofs in [KM1] and subsequent papers are based on homogeneous dynamics, that is, on quantitative nondivergence estimates for flows on the spaces of lattices, and on a possibility to phrase Diophantine exponents and other characteristics in terms of the behavior of certain orbits.

In the present paper we give a simple argument showing how for arbitrary m, n the dichotomy described above can be directly (without writing explicit necessary and sufficient conditions) derived from quantitative nondivergence. Here is a special case of our main result:

Theorem 1.4. *Let $\mathcal{M} \subset M_{m,n}$ be a connected analytic submanifold.*

- (a) *Let $v \geq n/m$ and suppose that $\omega(Y_0) \leq v$ for some $Y_0 \in \mathcal{M}$; then $\omega(Y) \leq v$ for almost every $Y \in \mathcal{M}$.*
- (b) *Let $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ be as in Definition 1.3, and suppose that $\exists Y_0 \in \mathcal{M}$ which is not φ -singular; then Y is not φ -singular for almost every $Y \in \mathcal{M}$.*

In other words, the aforementioned Diophantine properties² hold either for almost all or for no $Y \in \mathcal{M}$. In particular, \mathcal{M} is extremal if and only if it contains at least one not very well approximable point.

Clearly (by Fubini’s Theorem) the properties discussed in the above theorem hold for almost every translate of an arbitrary \mathcal{M} . It is also clear that if \mathcal{M} belongs to a proper rational affine subspace, that is, if $Y\mathbf{q} \in \mathbb{Z}^m$ for some $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and all $Y \in \mathcal{M}$, then all points of \mathcal{M} have infinite Diophantine exponents and are φ -singular for arbitrary positive φ . However there exist less trivial examples of those exceptional subspaces; these will be discussed in §3.2.

As was the case in the papers [K1, K2, Zh], Theorem 1.4 is deduced from statements involving orbits on the space of lattices. Namely, let us put

$$k = m + n, \quad G = \mathrm{SL}_k(\mathbb{R}), \quad \Gamma = \mathrm{SL}_k(\mathbb{Z}) \quad \text{and} \quad \Omega = G/\Gamma. \quad (1.6)$$

The space Ω can be viewed as the space of unimodular lattices in \mathbb{R}^k by means of the correspondence $g\Gamma \mapsto g\mathbb{Z}^k$. Denote by \mathcal{A} the set of $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$

¹This will always mean ‘with respect to the smooth measure class on \mathcal{M} ’.

²Recall that Y is called *Diophantine* if $\omega(Y) < \infty$. We remark that in [CY] a weaker result has been obtained by elementary methods (not using estimates on homogeneous spaces) in the case $m = 1$: if a connected analytic submanifold \mathcal{M} of \mathbb{R}^n contains a Diophantine vector, then almost all vectors in \mathcal{M} are Diophantine.

such that

$$t_1, \dots, t_k > 0 \quad \text{and} \quad \sum_{i=1}^m t_i = \sum_{j=1}^n t_{m+j}.$$

For $\mathbf{t} \in \mathcal{A}$ write

$$g_{\mathbf{t}} \stackrel{\text{def}}{=} \text{diag}(e^{t_1}, \dots, e^{t_m}, e^{-t_{m+1}}, \dots, e^{-t_k}) \in G. \quad (1.7)$$

We will consider subsets of G of the form

$$g_{\mathcal{T}} \stackrel{\text{def}}{=} \{g_{\mathbf{t}} : \mathbf{t} \in \mathcal{T}\}, \quad \text{where } \mathcal{T} \subset \mathcal{A} \text{ is unbounded,}$$

and will study their action on Ω . Also for $Y \in M_{m,n}$ define

$$u_Y \stackrel{\text{def}}{=} \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix},$$

where I_ℓ stands for the $\ell \times \ell$ identity matrix. Then it is clear that the group $\{u_Y : Y \in M_{m,n}\}$ is the expanding horospherical subgroup of G corresponding to $g_{\mathbf{t}}$ where \mathbf{t} belongs to the ‘central ray’ in \mathcal{A} , that is, to

$$\mathcal{R} \stackrel{\text{def}}{=} \left\{ \left(\frac{t}{m}, \dots, \frac{t}{m}, \frac{t}{n}, \dots, \frac{t}{n} \right) : t > 0 \right\}. \quad (1.8)$$

Our goal is to show a dichotomy similar to (and in fact, generalizing) the one from Theorem 1.4 for certain properties of $g_{\mathcal{T}}$ -orbits on Ω . Fix a norm $\|\cdot\|$ on \mathbb{R}^k and define a function $\delta : \Omega \rightarrow \mathbb{R}_+$ by

$$\delta(\Lambda) \stackrel{\text{def}}{=} \inf_{\mathbf{v} \in \Lambda \setminus \{0\}} \|\mathbf{v}\| \quad \text{for } \Lambda \in \Omega.$$

Ω is a noncompact space, and the function δ defined above can be used to describe its geometry at infinity. Namely, Mahler’s Compactness Criterion (see [Ra] or [BM]) says that a subset of Ω is relatively compact if and only if δ is bounded away from zero on this subset. Further, it follows from the reduction theory for $\text{SL}_k(\mathbb{Z})$ that the ratio of $1 + \log(1/\delta(\cdot))$ and $1 + \text{dist}(\cdot, \mathbb{Z}^k)$ is bounded between two positive constants for any right invariant Riemannian metric ‘dist’ on Ω . In other words, a lattice $\Lambda \in \Omega$ for which $\delta(\Lambda)$ is small is approximately $-\log \delta(\Lambda)$ away from the base point \mathbb{Z}^k . This justifies the following

Definition 1.5. *For an unbounded subset \mathcal{T} of \mathcal{A} and $\Lambda \in \Omega$, define the growth exponent $\gamma_{\mathcal{T}}(\Lambda)$ of Λ with respect to \mathcal{T} by*

$$\gamma_{\mathcal{T}}(\Lambda) \stackrel{\text{def}}{=} \limsup_{\mathbf{t} \rightarrow \infty, \mathbf{t} \in \mathcal{T}} \frac{-\log(\delta(g_{\mathbf{t}}\Lambda))}{\|\mathbf{t}\|}.$$

In other words (in view of the remark preceding the above definition), $\gamma_{\mathcal{T}}(\Lambda) > \beta$ is equivalent to the existence of $\beta' > \beta$ such that $\text{dist}(g_{\mathbf{t}}\Lambda, \mathbb{Z}^k) \geq \beta' \|\mathbf{t}\|$ for an unbounded set of $\mathbf{t} \in \mathcal{T}$. Even though the definition involve various norms, it clearly does not depend on the choices of norms. Also the growth exponent does not change if \mathcal{T} is replaced with another set of bounded Hausdorff distance from \mathcal{T} , so in what follows we can and will choose \mathcal{T} to be countable and with distance between its different elements uniformly bounded from below. Note that it can be derived from the Borel-Cantelli lemma that

for any unbounded \mathcal{T} , the growth exponent³ of Λ with respect to \mathcal{T} is equal to zero for Haar-almost all $\Lambda \in \Omega$.

Here is another property related to asymptotic behavior of trajectories:

Definition 1.6. *Given $\mathcal{T} \subset \mathcal{A}$ and a bounded function $\psi : \mathcal{T} \rightarrow \mathbb{R}_+$, say that the trajectory $g_{\mathcal{T}}\Lambda$ diverges faster than ψ if*

$$\limsup_{\mathbf{t} \rightarrow \infty, \mathbf{t} \in \mathcal{T}} \frac{\delta(g_{\mathbf{t}}\Lambda)}{\psi(\mathbf{t})} = 0.$$

In other words, if for every $c > 0$ one has $\delta(g_{\mathbf{t}}\Lambda) < c\psi(\mathbf{t})$ for all $\mathbf{t} \in \mathcal{T}$ with large enough (depending on c) norm.

Again this definition is insensitive to choices of norms, and also depends only on the behavior of ψ at infinity up to a multiplicative constant. An example: if $\psi \equiv 1$, the above condition, in view of Mahler’s Compactness Criterion, says that the trajectory $g_{\mathcal{T}}\Lambda$ diverges (that is, eventually leaves any compact subset of Ω) as $\mathbf{t} \rightarrow \infty$ in \mathcal{T} . Clearly, because of mixing of the G -action on Ω , for any ψ and \mathcal{T} as above, $g_{\mathcal{T}}\Lambda$ diverges faster than φ for Haar-almost no $\Lambda \in \Omega$.

In this paper we show:

Theorem 1.7. *Suppose we are given $\Lambda \in \Omega$, an unbounded $\mathcal{T} \subset \mathcal{A}$, and a connected analytic submanifold \mathcal{M} of $M_{m,n}$. Then:*

- (a) *Let $\beta \geq 0$ and $Y_0 \in \mathcal{M}$ be such that $\gamma_{\mathcal{T}}(u_{Y_0}\Lambda) \leq \beta$; then $\gamma_{\mathcal{T}}(u_Y\Lambda) \leq \beta$ for almost all $Y \in \mathcal{M}$;*
- (b) *Let $\psi : \mathcal{T} \rightarrow \mathbb{R}_+$ be bounded, and suppose that $\exists Y_0 \in \mathcal{M}$ such that the trajectory $g_{\mathcal{T}}u_{Y_0}\Lambda$ does not diverge faster than ψ ; then $g_{\mathcal{T}}u_Y\Lambda$ does not diverge faster than ψ for almost every $Y \in \mathcal{M}$.*

A connection between the corresponding parts of Theorems 1.4 and 1.7 is well known. Namely, it is observed by Dani [D1] that Y is singular if and only if the trajectory $g_{\mathcal{R}}u_Y\mathbb{Z}^k$ diverges, where \mathcal{R} is as in (1.8), so that $g_{\mathcal{R}}$ is a one-parameter semigroup. Also it follows from [KM2, Theorem 8.5] that Y is VWA iff the growth exponent of $u_Y\mathbb{Z}^k$ with respect to \mathcal{R} is positive, and moreover, the latter growth exponent determines $\omega(Y)$. The aforementioned Diophantine implications of Theorem 1.7 correspond to the case $\mathcal{T} = \mathcal{R}$. However, choosing other unbounded subsets of \mathcal{A} also gives rise to interesting results, for example related to so-called multiplicative approximation ($\mathcal{T} = \mathcal{A}$) or approximation with weights (\mathcal{T} is a ray in \mathcal{A} different from \mathcal{R}). We will comment on this in §3.4.

The structure of this paper is as follows. In the next section we prove Theorem 1.7 using quantitative nondivergence estimates. Then in §3 we will go through the correspondence between Diophantine approximation and dynamics, and derive Theorem 1.4 from Theorem 1.7. We also present other Diophantine applications, including a solution to a matrix analogue of Mahler’s Conjecture (Corollary 3.1) suggested to the author by G.A. Margulis. In the

³See also [KM2] for finer growth properties of almost all orbits on homogeneous spaces of Lie groups.

last section we bring up some conjectures and open questions, and also remark that the methods employed in this paper are applicable to objects somewhat more general than analytic submanifolds of $M_{m,n}$.

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2. QUANTITATIVE NONDIVERGENCE AND THEOREM 1.7

Notation: if $B = B(\mathbf{x}, r)$ is a ball in \mathbb{R}^d and $c > 0$, cB will denote the ball $B(\mathbf{x}, cr)$. Lebesgue measure on \mathbb{R}^d will be denoted by λ . Given $C, \alpha > 0$ and $U \subset \mathbb{R}^d$, say that a function $f : U \rightarrow \mathbb{R}$ is (C, α) -good on U if for any ball $B \subset U$ and any $\varepsilon > 0$ one has

$$\lambda(\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon\}) \leq C \left(\frac{\varepsilon}{\sup_{\mathbf{x} \in B} |f(\mathbf{x})|} \right)^\alpha \lambda(B).$$

This property captures ‘quasi-polynomial’ behavior of a function f . See [KM1, KLW] for a discussion and many examples. The following proposition, which is essentially implied by [K1, Corollary 3.3], will be useful:

Proposition 2.1. *Let U be a connected open subset of \mathbb{R}^d , and let \mathcal{F} be a finite-dimensional space of analytic real-valued functions on U . Then for any $\mathbf{x} \in U$ there exist $C, \alpha > 0$ and a neighborhood $W \ni \mathbf{x}$ contained in U such that every $f \in \mathcal{F}$ is (C, α) -good on W .*

Proof. Without loss of generality we can assume that \mathcal{F} contains constant functions. Let $1, f_1, \dots, f_N$ be the basis of \mathcal{F} , and consider the map $\mathbf{f} = (f_1, \dots, f_N) : U \rightarrow \mathbb{R}^N$. Then for any open subset U' of U , $\mathbf{f}(U')$ is not contained in any proper affine subspace of \mathbb{R}^N (otherwise, in view of the analyticity of all the functions, the same would be true for $\mathbf{f}(U)$, hence the functions would not be linearly independent). Therefore, again due to analyticity, \mathbf{f} is nondegenerate at every point of U (see [KM1] for a definition), and the conclusion follows from [KM1, Proposition 3.4]. \square

Recall that given $m, n \in \mathbb{N}$ we fixed $k = m + n$ and defined Ω as in (1.6). In order to state the main measure estimate we need to introduce some more notation. Let

$$\mathcal{W} \stackrel{\text{def}}{=} \text{the set of proper nonzero rational subspaces of } \mathbb{R}^k.$$

From here until the end of this section we let $\|\cdot\|$ stand for the Euclidean norm on \mathbb{R}^k , which we extend from \mathbb{R}^k to its exterior algebra. For $V \in \mathcal{W}$ and $g \in G$, let

$$\ell_V(g) \stackrel{\text{def}}{=} \|g(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_j)\|,$$

where $\{\mathbf{v}_1, \dots, \mathbf{v}_j\}$ is a generating set for $\mathbb{Z}^k \cap V$; note that $\ell_V(g)$ does not depend on the choice of $\{\mathbf{v}_i\}$.

Let us record the following elementary observation:

Lemma 2.2. *There exists a constant E depending only on k with the following property: for any $V \in \mathcal{W}$ and $g \in G$ there exists a one-dimensional rational subspace $V' \subset V$ such that $\ell_{V'}(g) \leq E \ell_V(g)^{1/\dim(V)}$. Consequently, one has*

$$\delta(g\mathbb{Z}^k) \leq E \cdot \inf_{V \in \mathcal{W}} \ell_V(g)^{1/\dim(V)}.$$

Proof. Indeed, $\ell_V(g)$ by definition is the covolume of the lattice $g\mathbb{Z}^k \cap V$ in V , and Minkowski’s Lemma, see [S], implies that such a lattice has a nonzero vector of length at most $\text{const} \cdot \ell_V(g)^{1/\dim(V)}$ where the constant depends only on the dimension of V ; thus one can choose V' to be the line passing through this vector. \square

Here is the main measure estimate on which our argument is based:

Theorem 2.3 ([K2], Theorem 2.2). *Given $d, k \in \mathbb{N}$ and positive constants C, D, α , there exists $C_1 = C_1(d, k, C, \alpha) > 0$ with the following property. Suppose $\tilde{B} \subset \mathbb{R}^d$ is a ball, $0 < \rho \leq 1$, and h is a continuous map $\tilde{B} \rightarrow G$ such that for each $V \in \mathcal{W}$,*

(i) *the function $\ell_V \circ h$ is (C, α) -good on \tilde{B} ,*

and

(ii) *$\ell_V \circ h(\mathbf{x}) \geq \rho^{\dim(V)}$ for some $\mathbf{x} \in B = 3^{-(k-1)}\tilde{B}$.*

Then for any $0 < \varepsilon \leq \rho$,

$$\lambda(\{\mathbf{x} \in B : \delta(h(\mathbf{x})\mathbb{Z}^k) < \varepsilon\}) \leq C_1 \left(\frac{\varepsilon}{\rho}\right)^\alpha \lambda(B). \quad (2.1)$$

This theorem is similar to its earlier versions, see [KM1, KLW]; however one crucial difference is the term $\rho^{\dim(V)}$ in (ii), as opposed to just ρ independent on the dimension of V . It is this improvement that will enable us to prove sharp results.

Now recall that in the theorems stated in the introduction we are given a connected analytic submanifold \mathcal{M} of $M_{m,n}$. We are going to parameterize it by an analytic map $F : U \rightarrow M_{m,n}$, where U is a connected open subset of \mathbb{R}^d , $d = \dim(\mathcal{M})$. Theorem 2.3 will be applied to $h : U \rightarrow G$ given by

$$h(\mathbf{x}) = g_{\mathbf{t}} u_{F(\mathbf{x})} g, \quad (2.2)$$

where $\mathbf{t} \in \mathcal{A}$ and $g \in G$ are fixed, and our goal will be to check conditions (i) and (ii) of Theorem 2.3 and then use (2.1).

The next corollary (from Proposition 2.1) will help us handle condition (i):

Corollary 2.4. *Let U be a connected open subset of \mathbb{R}^d , and let $F : U \rightarrow M_{m,n}$ be an analytic map. Then for any $\mathbf{x}_0 \in U$ there exist $C, \alpha > 0$ and a neighborhood $W \ni \mathbf{x}_0$ contained in U such that for any $V \in \mathcal{W}$, $\mathbf{t} \in \mathcal{A}$ and $g \in G$, functions $\mathbf{x} \mapsto \ell_V(g_{\mathbf{t}} u_{F(\mathbf{x})} g)$ are (C, α) -good on W .*

Proof. For any $\mathbf{v}_1, \dots, \mathbf{v}_j \in \mathbb{R}^k$, $\mathbf{t} \in \mathcal{A}$ and $g \in G$, the coordinates of

$$g_{\mathbf{t}} u_{F(\mathbf{x})} g(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_j)$$

in some fixed basis of the j -th exterior power of \mathbb{R}^k are linear combinations of products of matrix elements of F , where the number of factors in the products

is uniformly bounded from above. Therefore all those coordinate functions are analytic and span a finite-dimensional space, and the claim follows from Proposition 2.1 and [KLW, Lemma 4.1]. \square

We are now ready for the

Proof of Theorem 1.7. Recall that we are given $\Lambda \in \Omega$ (which we will write in the form $g\mathbb{Z}^k$, where $g \in G$ is fixed), an unbounded $\mathcal{T} \subset \mathcal{A}$, and a connected analytic submanifold \mathcal{M} of $M_{m,n}$ which we will parameterize by $F : U \rightarrow M_{m,n}$ where $U \subset \mathbb{R}^d$ is open and connected.

For part (a) we are given $\beta \geq 0$ such that the set

$$A_1 \stackrel{\text{def}}{=} \{\mathbf{x} \in U : \gamma_{\mathcal{T}}(u_{F(\mathbf{x})}\Lambda) \leq \beta\} \quad (2.3)$$

is nonempty. Define

$$A_2 \stackrel{\text{def}}{=} \{\mathbf{x} \in U : \lambda(B \setminus A_1) = 0 \text{ for some neighborhood } B \text{ of } \mathbf{x}\}. \quad (2.4)$$

We claim that

$$A_2 = \overline{A_1} \cap U. \quad (2.5)$$

Since A_2 is obviously open and U is connected, this implies that $A_2 = U$, and therefore A_1 has full measure, which is what we were supposed to show.

It is clear from (2.4) that $A_2 \subset \overline{A_1}$. To prove equality in (2.5), take $\mathbf{x}_0 \in \overline{A_1}$, and choose a ball $B \ni \mathbf{x}_0$ such that $\tilde{B} \stackrel{\text{def}}{=} 3^{k-1}B$ is contained in W as in Corollary 2.4. This way, condition (i) of Theorem 2.3 for h as in (2.2) (uniformly in $\mathbf{t} \in \mathcal{T}$) is taken care of. Then choose $\mathbf{x}' \in B \cap A_1$; (2.3) implies that for any $\beta' > \beta$ and all large enough $\mathbf{t} \in \mathcal{T}$, one has $\delta(g_{\mathbf{t}}u_{F(\mathbf{x}_0)}g\mathbb{Z}^k) \geq e^{-\beta'\|\mathbf{t}\|}$. Applying Lemma 2.2, we can conclude that $\ell_V \circ h(\mathbf{x}_0) \geq (e^{-\beta'\|\mathbf{t}\|}/E)^{\dim(V)}$ for any $V \in \mathcal{W}$ and all large enough $\mathbf{t} \in \mathcal{T}$. Thus condition (ii) of Theorem 2.3 is satisfied with $\rho = e^{-\beta'\|\mathbf{t}\|}/E$. Taking $\varepsilon = e^{-\beta''\|\mathbf{t}\|}$ where $\beta'' > \beta'$ is arbitrary, we apply (2.1) and conclude that for large enough $\mathbf{t} \in \mathcal{T}$,

$$\lambda(\{\mathbf{x} \in B : \delta(g_{\mathbf{t}}u_{F(\mathbf{x})}\Lambda) < e^{-\beta''\|\mathbf{t}\|}\}) \leq C_1 E^\alpha e^{-\alpha(\beta''-\beta')\|\mathbf{t}\|} \lambda(B). \quad (2.6)$$

The sum of the right hand sides of the above inequality over all $\mathbf{t} \in \mathcal{T}$ is finite (recall that \mathcal{T} is assumed to be ‘uniformly discrete’), hence almost all $\mathbf{x} \in B$ belong to at most finitely many sets as in the left hand side of (2.6). Since β'' can be arranged to be as close to β as one wishes, it follows that $\gamma_{\mathcal{T}}(u_{F(\mathbf{x})}\Lambda) \leq \beta$ for almost all $\mathbf{x} \in B$, that is $\mathbf{x}_0 \in A_2$.

Part (b) is proved along the same lines: define

$$A_1 \stackrel{\text{def}}{=} \{\mathbf{x} \in U : g_{\mathcal{T}}(u_{F(\mathbf{x})}\Lambda) \text{ does not diverge faster than } \psi\}$$

and then A_2 by (2.4); as before, the claim would follow from (2.5). Again, take $\mathbf{x}' \in \overline{A_1}$ and $B \ni \mathbf{x}'$ such that $\tilde{B} \stackrel{\text{def}}{=} 3^{k-1}B \subset W$ as in Corollary 2.4, so that h is as in (2.2) satisfies condition (i) of Theorem 2.3 for any \mathbf{t} . Then choose $\mathbf{x}_0 \in B \cap A_1$. The latter implies that there exists $c > 0$ and an unbounded subset \mathcal{T}' of \mathcal{T} such that

$$\delta(g_{\mathbf{t}}u_{F(\mathbf{x}_0)}\Lambda) \geq c\psi(\mathbf{t}) \quad \forall \mathbf{t} \in \mathcal{T}'.$$

From Lemma 2.2 it then follows that

$$\ell_V \circ h(\mathbf{x}_0) \geq (c\psi(\mathbf{t})/E)^{\dim(V)}$$

for any $V \in \mathcal{W}$ and any $\mathbf{t} \in \mathcal{T}'$. Applying (2.1), we conclude that for any $0 < \varepsilon < 1$ and $\mathbf{t} \in \mathcal{T}'$,

$$\lambda(\{\mathbf{x} \in B : \delta(g_{\mathbf{t}}u_{F(\mathbf{x})}\Lambda) < \varepsilon c\psi(\mathbf{t})\}) \leq C_1 E^\alpha \varepsilon^\alpha \lambda(B). \quad (2.7)$$

But by definition of ‘divergence faster than ψ ’ and since \mathcal{T}' is unbounded, for any positive ε there exists $\mathbf{t} \in \mathcal{T}'$ such that $B \setminus A_1$ is contained in the set in the left hand side of (2.7). Hence $B \setminus A_1$ has measure zero, which proves that $\mathbf{x}' \in A_2$. \square

3. DIOPHANTINE APPLICATIONS

3.1. Proof of Theorem 1.4. In order to connect Theorem 1.4 with Theorem 1.7, we take $\mathcal{T} = \mathcal{R}$ as in (1.8), and denote

$$g_t \stackrel{\text{def}}{=} \text{diag}(e^{t/m}, \dots, e^{t/m}, e^{-t/n}, \dots, e^{-t/n}).$$

According to [KM2, Theorem 8.5], (1.2) holds if and only if the inequality

$$\delta(g_t u_Y \mathbb{Z}^k) < e^{-\frac{mv-n}{n(mv+1)}t}$$

is satisfied for an unbounded set of $t \in \mathbb{R}_+$. Consequently, one has

$$\gamma_{\mathcal{R}}(u_Y \mathbb{Z}^k) = \frac{\frac{m}{n}\omega(Y) - 1}{m\omega(Y) + 1},$$

and therefore Theorem 1.4(a) follows immediately from Theorem 1.7(a).

The connection between parts (b) of these theorems is analogous. Given φ as in Definition 1.3, define $N = N(t)$ by

$$e^{\frac{m+n}{mn}t} = N^{1+n/m} \varphi(N)^{-1} \quad (3.1)$$

(this is well defined in view of the continuity and monotonicity of φ), and then let

$$\psi(t) = e^{-t/n} N. \quad (3.2)$$

Then, for any $c > 0$,

$$e^{t/m} \frac{c\varphi(N)}{N^{m/n}} = e^{-t/n} cN = c\psi(t);$$

thus the solvability of (1.3) is equivalent to $\delta(g_t u_Y \mathbb{Z}^k) < c\psi(t)$. Hence Y is φ -singular if and only if $\gamma_{\mathcal{R}}(u_Y \mathbb{Z}^k)$ diverges faster than ψ (here we identify \mathcal{R} with \mathbb{R}_+ and view ψ as a function on \mathcal{R}), which readily proves Theorem 1.4(b). Note that given ψ one can define N by (3.1) and then ψ by (3.2), thus there is at most one function φ for which both (3.1) and (3.2) hold. For example, $\varphi \equiv \text{const}$ would give rise to $N(t) = e^{t/n}$ and thus $\psi(t) \equiv \text{const}$; and the faster is the decay of φ , the faster would be the decay of ψ .

3.2. Examples. Here we take $m = 1$, that is, consider \mathbb{R}^n as the space of row vectors (linear forms). Let an s -dimensional affine subspace \mathcal{L} of \mathbb{R}^n be parametrized by

$$\mathbf{x} \mapsto (\mathbf{x}, \mathbf{x}A' + \mathbf{a}_0), \quad (3.3)$$

where $A' \in M_{s, n-s}$ and $\mathbf{a}_0 \in \mathbb{R}^{n-s}$ (here both \mathbf{x} and \mathbf{a}_0 are row vectors). Denote by $\tilde{\mathbf{x}}$ the row vector $(1, \mathbf{x}) \in \mathbb{R}^{n+1}$, and put $A = \begin{pmatrix} \mathbf{a}_0 \\ A' \end{pmatrix} \in M_{s+1, n-s}$. It is easy to show, see [K2, Lemma 5.4], that all points of \mathcal{L} have Diophantine exponents at least as big as $\omega(A)$; in other words, a good rational approximation to A gives rise to a good approximation to all points of \mathcal{L} . Choosing subspaces \mathcal{L} for which $\omega(A)$ is arbitrary large one can produce examples of ‘irrational’ subspaces consisting of arbitrarily well approximable vectors. Similarly one can construct nontrivial examples of subspaces consisting of φ -singular vectors. For the sake of completeness let us work out those examples here, following the argument of [K2, Lemma 5.4]. Equation (3.3) can be rewritten as $\mathbf{x} \mapsto (\mathbf{x}, \tilde{\mathbf{x}}A)$. Suppose A is φ -singular (it is known from the work of Khintchine that nontrivial examples of such matrices exist for any φ). Then for any $c > 0$ there is N_0 such that for all $N \geq N_0$ one can find $\mathbf{p} = (p_0, p_1, \dots, p_s) \in \mathbb{Z}^{s+1}$ and $\mathbf{q} \in \mathbb{Z}^{n-s} \setminus \{0\}$ such that

$$\|A\mathbf{q} + \mathbf{p}\| < \frac{c\varphi(N)}{N^{m/n}} \text{ and } \|\mathbf{q}\| < cN. \quad (3.4)$$

Now take any $\mathbf{x} \in \mathbb{R}^s$, denote (p_1, \dots, p_s) by \mathbf{p}' and write

$$\left| p_0 + (\mathbf{x}, \tilde{\mathbf{x}}A) \begin{pmatrix} \mathbf{p}' \\ \mathbf{q} \end{pmatrix} \right| = |p_0 + \mathbf{x}\mathbf{p}' + \tilde{\mathbf{x}}A\mathbf{q}| = |\tilde{\mathbf{x}}(A\mathbf{q} + \mathbf{p})| \leq \|\tilde{\mathbf{x}}\| \|A\mathbf{q} + \mathbf{p}\|.$$

Therefore one has $\left| p_0 + (\mathbf{x}, \tilde{\mathbf{x}}A) \begin{pmatrix} \mathbf{p}' \\ \mathbf{q} \end{pmatrix} \right| \leq \frac{C_1 c \varphi(N)}{N^{m/n}}$, where C_1 depends only on \mathbf{x} . Also, it follows from (3.4) that $\|\mathbf{p}\|$ is bounded from above by $C_2 \|\mathbf{q}\|$, where C_2 depends only on A ; hence $\left\| \begin{pmatrix} \mathbf{p}' \\ \mathbf{q} \end{pmatrix} \right\| < C_2 c N$. Since c can be chosen to be arbitrary small, it follows that $(\mathbf{x}, \tilde{\mathbf{x}}A)$ is φ -singular for all \mathbf{x} .

3.3. A matrix analogue of Mahler’s Conjecture. We now describe an application of Theorem 1.4 suggested to the author by G.A. Margulis. Given $m, n \in \mathbb{N}$, consider the m^2 -dimensional submanifold of $M_{m, mn}$ given by

$$\{(X \ X^2 \ \dots \ X^n) : X \in M_{m, m}\},$$

which is a matrix analogue of (1.4). Then one can ask⁴ whether the above manifold is extremal. The answer turns out to be affirmative and follows from the dichotomy established in Theorem 1.4. In fact a more general statement can be proved:

Corollary 3.1. *Given $n \in \mathbb{N}$ and $v \geq n$, let $\mathbf{f} = (f_1, \dots, f_n)$ be an analytic map from a neighborhood of $x_0 \in \mathbb{R}$ to $M_{1, n}$, and suppose that $\omega(\mathbf{f}(x_0)) \leq v$. Take $m \in \mathbb{N}$, and let U be a neighborhood of $x_0 I_m \in M_{m, m}$ such that the map*

$$F : X \mapsto (f_1(X) \ \dots \ f_n(X)) \in M_{m, mn} \quad (3.5)$$

⁴This question was asked during the author’s talk at Moscow State University.

is defined for $X \in U$. Then $\omega(F(X)) \leq v$ for a.e. $X \in U$.

In particular, if \mathbf{f} satisfies (1.5), then $\omega(\mathbf{f}(x)) = n$ for almost all x in view of [KM1, Theorem A], hence the manifold $\{F(X) : X \in U\}$ is extremal.

Proof of Corollary 3.1. Note that $\omega(F(xI_m)) > v$ is equivalent to the existence of $w > v$ such that there are infinitely many $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n) \in (\mathbb{Z}^m)^n$ with

$$\text{dist}(f_1(xI_m)\mathbf{q}_1 + \dots + f_n(xI_m)\mathbf{q}_n, \mathbb{Z}^m) < \|\mathbf{q}\|^{-w}. \quad (3.6)$$

(Here it is convenient to define $\|\cdot\|$ and ‘dist’ via the supremum norm.) Write $\mathbf{q}_i = (q_{i,1}, \dots, q_{i,m})$ and $\mathbf{q}^{(j)} = (q_{1,j}, \dots, q_{n,j})$, and choose $j = 1, \dots, m$ such that $\|\mathbf{q}\| = \|\mathbf{q}^{(j)}\|$ for infinitely many \mathbf{q} satisfying (3.6). Then, since $f_i(xI_m) = f_i(x)I_m$ for every i , by looking at the j th component of vectors in the left hand side of (3.6) one concludes that $\text{dist}(f_1(x)q_{1,j} + \dots + f_n(x)q_{n,j}, \mathbb{Z}) < \|\mathbf{q}^{(j)}\|^{-w}$ for infinitely many $\mathbf{q}^{(j)} \in \mathbb{Z}^n$, which implies $\omega(\mathbf{f}(x)) > v$. Thus we have shown that $\omega(F(xI_m)) \leq \omega(\mathbf{f}(x))$ whenever $\mathbf{f}(x)$ is defined (the opposite inequality is also easy to show, although not needed for our purposes). The claim is therefore an immediate consequence of Theorem 1.4(a). \square

Similarly one can conclude, using [KW2] and Theorem 1.4(b), that under the assumption (1.5) $F(X)$ as in (3.5) is not singular for a.e. X .

3.4. Other applications. Here we describe two more corollaries from Theorem 1.7 which deal with Diophantine properties more general than those discussed in Theorem 1.4.

3.4.1. For $\mathbf{x} = (x_i) \in \mathbb{R}^\ell$ define

$$\Pi(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{\ell} |x_i| \quad \text{and} \quad \Pi_+(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{\ell} \max(|x_i|, 1).$$

Then say that $Y \in M_{m,n}$ is *very well multiplicatively approximable* (VWMA) if for some $\delta > 0$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\Pi(Y\mathbf{q} + \mathbf{p}) < \Pi_+(\mathbf{q})^{-(1+\delta)}$$

for some $\mathbf{p} \in \mathbb{Z}^m$. Since $\Pi(Y\mathbf{q} + \mathbf{p}) \leq \|Y\mathbf{q} + \mathbf{p}\|^m$ and $\Pi_+(\mathbf{q}) \leq \|\mathbf{q}\|^n$ for $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$, VWA implies VWMA. Still it can be easily shown that Lebesgue-a.e. Y is not VWMA⁵. To show that a submanifold \mathcal{M} of $M_{m,n}$ is *strongly extremal*, that is, its almost every point is not VWMA, is usually more difficult than to prove its extremality. For example, the multiplicative version of Mahler’s Conjecture, that is, the strong extremality of the curve (1.4) which was conjectured by Baker in the 1970s, has not been solved until the introduction of the methods of homogeneous dynamics to metric Diophantine approximation, and up to the present time there is no other proof than the one from [KM1]. Note that applications of dynamics to multiplicative Diophantine problems are based on the fact that Y is VWMA if and only if $\gamma_{\mathcal{A}}(u_Y \mathbb{Z}^{m+n}) = 0$; that is, the orbit of the lattice $u_Y \mathbb{Z}^k$ under the action of the whole semigroup $\{g_{\mathbf{t}} : \mathbf{t} \in \mathcal{A}\}$ has sublinear growth. This was shown in

⁵Also it is known [SW] that Y is VWMA iff so is the transpose of Y .

[KM1] and [KLW] in the cases $m = 1$ and $n = 1$ respectively. The proof for the general case can be found in [KMW], see also [KM2, Theorem 9.2] for a related statement. Therefore from Theorem 1.7 one derives

Corollary 3.2. *A connected analytic submanifold of $M_{m,n}$ is strongly extremal if and only if it contains at least one not VWMA point.*

The case $\min(m, n) = 1$ of the above statement is established in [K1].

3.4.2. Let us generalize Definition 1.3 as follows: suppose $\varphi : \mathcal{A}_+ \rightarrow \mathbb{R}_+$ is a function which is continuous and nonincreasing in each variable; that is,

$$\varphi(t_1, \dots, t_i, \dots, t_k) \geq \varphi(t_1, \dots, t'_i, \dots, t_k) \quad \text{whenever } t_i \leq t'_i.$$

Also let \mathcal{T} be an unbounded subset of \mathcal{A} . Now say that $Y \in M_{m,n}$ is (φ, \mathcal{T}) -singular if for any $c > 0$ there is N_0 such that for all $\mathbf{t} \in \mathcal{T}$ with $\|\mathbf{t}\| \geq N_0$ one can find $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ with

$$\begin{cases} |Y_i \mathbf{q} - p_i| < c\varphi(\mathbf{t})e^{-t_i}, & i = 1, \dots, m \\ |q_j| < c\varphi(\mathbf{t})e^{t_{m+j}}, & j = 1, \dots, n \end{cases}$$

In other words, those systems Y of linear forms Y_1, \dots, Y_m admit a drastic improvement of the multiplicative (Minkowski's) form of Dirichlet's Theorem, see [KW3] or [Sh3]. It is not hard to show that the set of (φ, \mathcal{T}) -singular matrices has Lebesgue measure zero for any unbounded \mathcal{T} . Arguing as in the proof of Theorem 1.4(b), see §3.1, one can relate (φ, \mathcal{T}) -singularity of Y to the trajectory $g_{\mathbf{T}} u_Y \mathbb{Z}^k$ being divergent faster than ψ , where ψ and \mathcal{T}' are determined by φ and \mathcal{T} . Thus from Theorem 1.7 one can derive

Corollary 3.3. *Let φ and \mathcal{T} be as above, and suppose a connected analytic submanifold \mathcal{M} of $M_{m,n}$ contains Y_0 which is not (φ, \mathcal{T}) -singular; then Y is not (φ, \mathcal{T}) -singular for almost every $Y \in \mathcal{M}$.*

4. GENERALIZATIONS AND OPEN QUESTIONS

It seems natural to conjecture that other Diophantine or dynamical properties might exhibit a dichotomy of the same type as discussed in this paper. Here are some examples. For a function $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ one says that $Y \in M_{m,n}$ is φ -approximable if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that $\|Y\mathbf{q} + \mathbf{p}\| \leq \varphi(\|\mathbf{q}\|)$ for some $\mathbf{p} \in \mathbb{Z}^m$. (This definition is slightly different from the one used in [KM2], where powers of norms were considered.) The Khintchine-Groshev theorem gives the precise condition on the function φ under which the set of φ -approximable matrices has full measure. Namely, if φ is non-increasing (this assumption can be removed in higher dimensions but not for $n = 1$), then Lebesgue measure of the set of φ -approximable $Y \in M_{m,n}$ is zero if

$$\sum_{k=1}^{\infty} k^{n-1} \varphi(k)^m < \infty, \quad (4.1)$$

and full otherwise. Now suppose (4.1) holds and a connected analytic submanifold \mathcal{M} of $M_{m,n}$ contains a point which is not φ -approximable; is it true that almost all $Y \in \mathcal{M}$ are not φ -approximable, or at least not $\tilde{\varphi}$ -approximable,

where $\tilde{\varphi} = C\varphi$ with $C > 0$ depending on Y ? Our methods are not powerful enough to answer this question. Note that [KM2] provides a dynamical interpretation of φ -approximability along the lines of Definition 1.5. Namely, the choice of φ as above uniquely defines a continuous function $r : [t_0, \infty) \mapsto \mathbb{R}_+$ such that $Y \in M_{m,n}$ is φ -approximable if and only if there exist arbitrarily large positive t such that

$$\delta(g_t u_Y \mathbb{Z}^k) < r(t).$$

Likewise, one can modify the definition of φ -singularity by fixing the constant c ; as in the previous example, it is not clear if the ‘almost all versus no’ dichotomy would still hold. Here is an important special case. Given positive $\varepsilon < 1$, one says that Dirichlet’s Theorem *can be ε -improved* for Y , writing $Y \in \text{DI}_\varepsilon$, if for every sufficiently large t one can find $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ with

$$\|Y\mathbf{q} - \mathbf{p}\| < \varepsilon e^{-t/m} \quad \text{and} \quad \|\mathbf{q}\| < \varepsilon e^{t/n}.$$

Clearly Y is singular iff it belongs to $\cup_{\varepsilon>0} \text{DI}_\varepsilon$. It was proved by Davenport and Schmidt [DS] that the sets DI_ε have Lebesgue measure zero. In fact, the latter statement follows from the ergodicity of the G -action on G/Γ : arguing as in §3.1, it is not hard to see that $Y \in \text{DI}_\varepsilon$ iff the $g_{\mathcal{R}}$ -orbit of $u_Y \mathbb{Z}^k$ misses a certain nonempty open subset of G/Γ . This motivates questions extending both Theorem 1.1 and (in some direction) Theorem 1.7(b). Namely, let G , Γ , a , \mathcal{A} and $\{u_t : t \in \mathbb{R}\}$ be as in Theorem 1.1, and suppose that for some $t_0 \in \mathbb{R}$ and $x \in G/\Gamma$, the trajectory $\mathcal{A}u_{t_0}x$ has a limit point in an open subset W of G/Γ . Is it true that the intersection of $\overline{\mathcal{A}u_t x}$ with W is nonempty for almost all $t \in \mathbb{R}$? Or else let G and Γ be as in (1.6), take an open subset W of G/Γ and $\mathcal{T} \subset \mathcal{A}_+$, and suppose that a connected analytic submanifold \mathcal{M} of $M_{m,n}$ contains a point Y_0 such that $g_{\mathbf{t}} u_{Y_0} \mathbb{Z}^k \in W$ for an unbounded set of $\mathbf{t} \in \mathcal{T}$; then is the same true for almost every $Y \in \mathcal{M}$? An affirmative answer to the latter question would imply that for any positive $\varepsilon < 1$ and any \mathcal{M} as above, the set $\mathcal{M} \setminus \text{DI}_\varepsilon$ is either empty or of full measure. Note that it follows from the methods of proof of [Sh2] that almost all $Y \in \mathcal{M}$ are not in DI_ε for any $\varepsilon < 1$ whenever \mathcal{M} contains a point Y_0 such that the $g_{\mathcal{R}}$ -orbit of $u_{Y_0} \mathbb{Z}^k$ is dense in G/Γ .

Finally we would like to mention that the assumption of analyticity of manifolds \mathcal{M} in the main results of the paper cannot be replaced by differentiability. Indeed, it is not hard to smoothly glue an extremal C^∞ curve in \mathbb{R}^n to a rational line. On the other hand, one of important advantages of the use of the quantitative nondivergence method has been a possibility to treat measures on $M_{m,n}$ other than volume measures on analytic submanifolds. The reader is referred to [KLW, K2, KW3] and a recent paper [KMW] for a description of more general classes of measures allowing a similar ‘almost all vs. no’ dichotomy.

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