NONDENSE ORBITS OF FLOWS ON HOMOGENEOUS SPACES

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Abstract. Let F be a nonquasiunipotent one-parameter (cyclic) subgroup of a unimodular Lie group G, Γ a discrete subgroup of G. We prove that for certain classes of subsets Z of the homogeneous space G/Γ, the set of points in G/Γ with F-orbits staying away from Z has full Hausdorff dimension. From this we derive applications to geodesic flows on manifolds of constant negative curvature.

Introduction

Given a dynamical system with phase space X and a fixed subset Z of X, consider points in X with orbits staying away from Z. Although orbits of almost all points may be dense, a number of recent results [Da1–2, Do, U, KM] indicate that for certain classes of dynamical systems and subsets Z, sets of those exceptional points are surprisingly big. More precisely, they are *thick*, i.e. have full Hausdorff dimension at any point of the space. These results motivated the following

Definition. For a set F of maps $X \to X$, say that a subset Z of X is *F-escapable* if the set $\{x \in X \mid \overline{Fx} \cap Z = \varnothing\}$ of points with F-orbits staying away from Z is thick. Similarly, $\{\infty\}$ is *F-escapable* if the set of points with bounded orbits is thick.

In all the cases considered below, F will be of the form $\{g_t\}$, where t runs through (a subset of) $\mathbb{R}$ or $\mathbb{Z}$; we will sometimes be referring to the pair (X, F) as to a *dynamical system*. Note that if the space X is not compact, it may happen that Fx diverges for a thick set of points x; in this case a lot of sets are escapable for trivial reasons. On the other hand, if the set of recurrent points is thick (e.g. if M carries a finite F-invariant measure), one can intuitively think of existence of large classes of escapable sets as of one of the features of chaotic behavior of the system. For example, escapability of certain sets relative to expanding endomorphisms or Anosov diffeomorphisms of Riemannian manifolds is the subject of recent papers by M. Urbanski [U] and D. Dolgopiat [Do]. This suggests the following problem: given a space X and a family of (partially) hyperbolic recurrent dynamical systems on X, describe the class of "dynamically small" (i.e. escapable relative to any member of this family) subsets of X.

Let now G be a Lie group, Γ a lattice in G, $F = \{g_t \mid t \in \mathbb{R}\}$ a one-parameter nonquasiunipotent (see §1.3.3) subgroup of G. In higher rank situation, the left

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action of F on G/Γ is only partially hyperbolic, hence one cannot count on existence of Markov partitions, which were essential in [U] and [Do]. The objective of this work is to compensate it by making use of the underlying rich algebraic structure.

It was proved by S.G. Dani for some special cases ([Da1, Da2], see also [AL]) and then conjectured by G.A. Margulis [Ma1, Conjecture (A)] that the abundance of bounded orbits is a general feature of nonquasiunipotent flows on homogeneous spaces of Lie groups. The latter conjecture was recently settled in the joint paper [KM] of G.A. Margulis and the author. Namely, we formulate necessary and sufficient conditions for $\{\infty\}$ to be escapable. The general problem was reduced to the case "G is a connected semisimple Lie group without compact factors and Γ is an irreducible lattice in G". Under these assumptions it is proved in [KM] that any F-invariant closed subset $Z \subset \Omega$ of Haar measure 0 is F-escapable. The argument in [KM] is based on mixing properties of the F-action on Ω. However, it turns out that some of the technique developed there allows one to state sufficient conditions for sets to be escapable relative to (possibly non-mixing) actions on homogeneous spaces of Lie groups.

In §1 we collect all the preliminaries needed, as well as introduce the concept of escapability. §2 is concerned with the advantages of partial hyperbolicity of actions on homogeneous spaces. Here we define the "horospherical decomposition" of the group G and stress the importance of the expanding leaves of the induced foliation of the space. We also give some background for the tesselation method first introduced in [KM].

The goal of the next section is to formulate a Hausdorff-dimension-free condition sufficient for the escapability of the set Z. Roughly speaking, one needs an asymptotical uniform bound on the number of small pieces of an expanding leaf which have nonempty intersection with "cylinders" of the form $g_{[0,t]}Z$ (see §3.3 for the exact statement). We prove (Propositions 3.3) that any set satisfying that condition is escapable, this being the only place where the definition of Hausdorff dimension (via an estimate due to C. McMullen and M. Urbanski, see Lemma 1.2.2) is crucially involved.

Let now Z be a smooth submanifold of Ω. We say that it is $\mathfrak{h}$-*transversal* (see §4.1.1) if for any $x \in Z$ the tangent space to the expanding leaf at x is not contained in T_xZ. In §4 we prove the following main result (see Corollary 4.3.2):

Theorem. *Let $F = \{g_t \mid t \in \mathbb{Z}\}$ be a cyclic nonquasiunipotent subgroup of G. Then any $\mathfrak{h}$-transversal compact C^1 submanifold of Ω is F^+-escapable (here $F^+ = \{g_t \mid t \in \mathbb{Z}^+\}$).*

See Corollary 4.4.2 for a continuous time analogue of this theorem, which solves a part of Conjecture (B) from [Ma1]. One can also apply these results to geodesic flows on manifolds of constant negative curvature. In particular, Corollary 4.4.4 asserts that if M is a Riemannian manifold of constant negative curvature and Y is a finite subset of M, then the elements (x, ξ) (here $x \in M$ and ξ is a unit tangent vector at x) such that the geodesic through x in the direction of ξ stays away from Y form a thick subset of the unit tangent bundle of M.

§1. PRELIMINARIES

1.1. Nondense orbits

1.1.1. For topological spaces X, Y, we will denote by X^Y the set of maps $Y \to X$

and by $\mathcal{H}(X)$ the group of self-homeomorphisms of X. Let F be a subset of X^X and Z a subset of X. Denote by $E(F,Z)$ the set of points of X with F-orbits escaping Z, that is

$$E(F,Z) \overset{\text{def}}{=} \{x \in X \mid \overline{F(x)} \cap Z = \varnothing\}\,.$$

We will say that Z is F*-escaped* by x if $x \in E(F,Z)$.

It is easy to see that the operation $E(\cdot,\cdot)$ is "order-reversing"; more precisely, the following is immediate from the definition:

- $E(F,\cup_{i\in I}Z_i) = \cap_{i\in I}E(F,Z_i)$ and $E(F,\cap_{i\in I}Z_i) = \cup_{i\in I}E(F,Z_i)$, in particular, $Z' \subset Z \Rightarrow E(F,Z) \subset E(F,Z')$;
- $E(\cup_{i\in I}F_i,Z) \subset \cap_{i\in I}E(F_i,Z)$ and $E(\cap_{i\in I}F_i,Z) \supset \cup_{i\in I}E(F_i,Z)$, with equalitiies if I is finite; in particular, $F' \subset F \Rightarrow E(F,Z) \subset E(F',Z)$.

The crucial role in this work will be played by the fact that sometimes one can deduce that $x \in E(F,Z)$ if one knows that another set Z', usually bigger than Z, is F'-escaped by x, where F' is another subset of X^X, perhaps smaller than F. This is formalized in the following

Lemma. *Let X be a locally compact topological space, $F' \subset X^X$, $Z \subset X$, and let F'' be a compact subset of a topological subgroup of $\mathcal{H}(X)$ acting continuously on X . Then $E\big((F'')^{-1}F',Z\big) = E\big(F',F''(Z)\big)$.*

Proof. Take $x \notin E\big((F'')^{-1}F',Z\big)$, pick $z \in \overline{(F'')^{-1}F'(x)} \cap Z$ and take a neighborhood of z with compact closure Q; then

$$z \in \overline{(F'')^{-1}\big(F'(x) \cap F''(Q)\big)} \subset \overline{(F'')^{-1}\Big(\overline{F'(x)} \cap F''(Q)\Big)}\,.$$

By compactness of F'', $F''(Q)$ is compact $\Rightarrow$ so is $\overline{F'(x)} \cap F''(Q)$ $\Rightarrow$ so is $(F'')^{-1}\Big(\overline{F'(x)} \cap F''(Q)\Big)$. Therefore $z \in (F'')^{-1}\Big(\overline{F'(x)} \cap F''(Q)\Big) \subset (F'')^{-1}\Big(\overline{F'(x)}\Big)$ $\Rightarrow \overline{F'(x)} \cap F''(z) \neq \varnothing \Rightarrow x \notin E\big(F',F''(Z)\big)$. The opposite direction is easy and does not require either compactness of F'' or local compactness of X. ☐

1.1.2. We now mention "functorial properties" of the operation $E(\cdot,\cdot)$ defined above. If X, Y are topological spaces and $\varphi \in X^Y$, we will say that $f \in Y^Y$ *factors through* φ if there exists ${}_\varphi f \in X^X$ such that $\varphi \circ f = {}_\varphi f \circ \varphi$ (${}_\varphi f$ is clearly unique if φ is surjective). If $F \subset Y^Y$ consists of maps factoring through φ, we will say that F *factors through* φ; in this case we let ${}_\varphi F \overset{\text{def}}{=} \{{}_\varphi f \mid f \in F\}$.

Lemma. *Let X, Y be topological spaces, $\varphi \in X^Y$, $Z \subset X$, and let $F \subset Y^Y$ consist of maps factoring through φ. Then*

(a) *φ is continuous $\Rightarrow \varphi^{-1}\big(E({}_\varphi F,Z)\big) \subset E\big(F,\varphi^{-1}(Z)\big)$;*

(b) *φ is closed $\Rightarrow \varphi^{-1}\big(E({}_\varphi F,Z)\big) \supset E\big(F,\varphi^{-1}(Z)\big)$.*

The proof is straightforward, see [K] for details.

1.2. Hausdorff dimension

1.2.1. All distances (diameters of sets) in various metric spaces will be denoted by "dist" ("diam"), and $B(x,r)$ (resp. $Y^{(r)}$) will stand for the r-neighborhood of a point x (resp. a set Y) To avoid confusion, we will sometimes put subscripts indicating the underlying metric space. If the metric space is a group and e is its identity element, we will simply write $B(r)$ instead of $B(e,r)$.

We will denote by $\dim(X)$ the Hausdorff dimension of a metric space X. See [F] for the definition and basic facts, such as

- $X \subset Y \quad \Rightarrow \quad \dim(X) \le \dim(Y)$;
- $X = \bigcup_{j\in\mathbb{N}} X_j \quad \Rightarrow \quad \dim(X) = \sup_{j\in\mathbb{N}} \dim(X_j)$;
- $\varphi \in X^Y$ is Lipschitz $\Rightarrow \quad \dim\big(\varphi(Y)\big) \le \dim(Y)$.

One can also estimate Hausdorff dimension of the direct product of two spaces in terms of the dimension of factors, or more generally, estimate Hausdorff dimension of a bundle in terms of the dimension of the base and fibers.

Lemma. *Let X_1 and X_2 be Riemannian manifolds, $A_1 \subset X_1$, $A_2 \subset X_2$, $B \subset X_1 \times X_2$. Denote by B_a the intersection of B and $\{a\} \times A_2$ (the slice of B at an element a of A_1) and assume that B_a is nonempty for all $a \in A_1$. Then*

(a) $\dim(B) \ge \dim(A_1) + \inf_{a\in A_1} \dim\,(B_a)$;

(b) $\dim(B) \le \dim(A_1) + \dim(X_2)$.

Proof. Using countable coverings by normal neighborhoods, one can reduce the statement to X_1 and X_2 being open subsets in Euclidean spaces. Then the lower estimate (a) (which was referred to as Marstrand Slicing Theorem in [KM]) follows from a theorem of Marstrand ([Mrs] or [F, Theorem 5.8]). As for the upper estimate, one has $B \subset A_1 \times X_2$, and from a theorem of Besicovitch and Moran [BM] it follows that $\dim(A_1 \times X_2) \le \dim(A_1) + \dim(X_2)$. Actually, both theorems cited above are stated and proved for one-dimensional X_1 and X_2, but the same proofs work for arbitrary dimensions. $\square$

1.2.2. We now describe a construction (cf. [F, PW]) of a class of sets for which there is a natural lower estimate for Hausdorff dimension. Let X be a Riemannian manifold, ν a Borel measure on X, A_0 a compact subset of X. Say that a countable collection $\mathcal{A}$ of compact subsets of A_0 of positive measure ν is *tree-like* relative to ν if $\mathcal{A}$ is the union of finite nonempty subcollections $\mathcal{A}_j$, $j \in \mathbb{Z}^+$, such that $\mathcal{A}_0 = \{A_0\}$ and the following two conditions are satisfied:

$$\forall\, j \in \mathbb{N} \quad \forall\, A, B \in \mathcal{A}_j \quad \text{either } A = B \quad \text{or} \quad \nu(A \cap B) = 0\,; \tag{TL1}$$

$$\forall\, j \in \mathbb{N} \quad \forall\, B \in \mathcal{A}_j \quad \exists\, A \in \mathcal{A}_{j-1} \quad \text{such that} \quad B \subset A\,. \tag{TL2}$$

Say also that $\mathcal{A}$ is *strongly tree-like* if it is tree-like and in addition

$$d_j(\mathcal{A}) \overset{\text{def}}{=} \sup_{A\in\mathcal{A}_j} \operatorname{diam}(A) \to 0 \quad \text{as} \quad j \to \infty\,. \tag{STL}$$

Let $\mathcal{A}$ be a tree-like collection of sets. For each $j \in \mathbb{Z}^+$, let $\mathbf{A}_j = \bigcup_{A\in\mathcal{A}_j} A$. These are nonempty compact sets, and from (TL2) it follows that $\mathbf{A}_j \subset \mathbf{A}_{j-1}$ for any $j \in \mathbb{N}$. Therefore one can define the (nonempty) *limit set* of $\mathcal{A}$ to be

$$\mathbf{A}_\infty = \bigcap_{j\in\mathbb{Z}^+} \mathbf{A}_j\,.$$

Further, for any subset B of A_0 with $\nu(B) > 0$ and any $j \in \mathbb{N}$, define the *jth stage density* $\Delta_j(B,\mathcal{A})$ of B in $\mathcal{A}$ by

$$\Delta_j(B,\mathcal{A}) = \frac{\nu(\mathbf{A}_j \cap B)}{\nu(B)}\,,$$

and the *jth stage density* $\Delta_j(\mathcal{A})$ of $\mathcal{A}$ by $\Delta_j(\mathcal{A}) = \inf_{B\in\mathcal{A}_{j-1}} \Delta_j(B,\mathcal{A})$.

The following estimate, based on an application of Frostman's Lemma, is essentially proved in [Mc] and [U]:

Lemma. *Assume that there exists* $k > 0$ *such that*

$$\liminf_{r\to 0} \frac{\log\big(\nu(B(x,r))\big)}{\log(r)} \ge k \tag{1.1}$$

for any $x \in A_0$*. Then for any strongly tree-like (relative to* ν*) collection* $\mathcal{A}$ *of subsets of* A_0

$$\dim(\mathbf{A}_\infty) \ge k - \limsup_{j\to\infty} \frac{\sum_{i=1}^{j} \log\big(\Delta_i(\mathcal{A})\big)}{\log(d_j(\mathcal{A}))}\,.$$

1.2.3. Let X be a metric space. A subset A of X will be called *thick* (in X) if for any nonempty open subset W of X, $\dim(W\cap A) = \dim(W)$ (i.e. A has full Hausdorff dimension at any point of X). Clearly thick subsets are dense in X and, if $\dim(X) > 0$, have cardinality of continuum. However, thick subsets of Riemannian manifolds X that we are going to consider may be small from the point of view of both topology (i.e. they may be of first category in X) and measure theory (they may have zero volume).

Let X and Y be Riemannian manifolds. Say that a continuous map $\varphi : Y \to X$ is a *bi-Lipschitz covering* if for any $x \in X$ there exists a neighborhood U of x, a Riemannian manifold M and a bi-Lipschitz homeomorphism $U\times M \to \varphi^{-1}(U)$ sending φ to the natural projection $U\times M \to U$. Clearly such a map φ is automatically open and surjective. We will say that the covering is *compact* if M can be taken to be compact. Using the properties of Hausdorff dimension mentioned above, one can easily prove

Lemma. *Let* X *and* Y *be Riemannian manifolds,* $\varphi\in X^Y$ *a bi-Lipschitz covering and* A *a subset of* X*. Then* A *is thick iff* $\varphi^{-1}(A)$ *is thick.*

1.2.4. Let X be a locally compact metric space and $F\subset X^X$. Say that a subset Z of X is *escapable relative to* F (or, briefly, F*-escapable*) if the set $E(F,Z)$ is thick in X.

Clearly any subset of an F-escapable set is F-escapable, and for any subset F' of F, any F-escapable set is F'-escapable (but a union of two escapable sets need not be escapable). Furthermore, one can use the facts listed in the two previous sections to derive the following

Theorem. (a) *Let* X *be a locally compact metric space,* F' *a subset of* X^X *and* F'' *a compact subset of a topological subgroup of* $\mathcal{H}(X)$ *acting continuously on* X*. Then* $Z\subset X$ *is* $(F'')^{-1}F'$*-escapable iff* $F''(Z)$ *is* F'*-escapable.*

(b) *Let* X *and* Y *be Riemannian manifolds,* $\varphi\in X^Y$ *a compact bi-Lipschitz covering and* F *a subset of* Y^Y *which factors through* φ*. Then* $Z\subset X$ *is* ${}_\varphi F$*-escapable iff* $\varphi^{-1}(Z)$ *is* F*-escapable.*

Proof. Part (a) follows from Lemma 1.1.1, while part (b) is obtained by combining Lemma 1.1.2 with Lemma 1.2.3 (note that a compact covering is necessarily closed). ☐

1.3. Lie groups and homogeneous spaces

1.3.1. Let G be a unimodular real Lie group. We will denote by $\mathfrak{g}$ the Lie algebra of G and fix a Euclidean structure on $\mathfrak{g}$ inducing a right-invariant Riemannian metric on G. The fact that $\mathfrak{g}$ can serve as a good approximation for G at a neighborhood of identity will be recorded as a

Lemma. *There exists positive* $\sigma_0 < 1$ *such that*

(a) *the exponential map* $\exp : \mathfrak{g} \to G$ *is injective on* $B_{\mathfrak{g}}(4\sigma_0)$*; moreover,*

(b) *for any* $u, v \in B_{\mathfrak{g}}(4\sigma_0)$, $\frac{1}{2}\|u - v\| \le \operatorname{dist}\bigl(\exp(u), \exp(v)\bigr) \le 2\|u - v\|$ *(i.e. up to a factor of* 2, $\exp$ *is an isometry* $4\sigma_0$*-close to* $0 \in \mathfrak{g}$*).*

1.3.2. Let Γ be a discrete subgroup of G; we will always denote by Ω the homogeneous space G/Γ, and equip it with the Riemannian metric coming from G. For any $x \in \Omega$ we let π_x stand for the quotient map $G \to \Omega$, $g \to gx$, which will be an isometry restricted to a (depending on x) neighborhood of $e \in G$.

Note that the tangent space to Ω at any $x \in \Omega$ can be identified with $\mathfrak{g} = T_eG$ via $(d\pi_x)_e$; we will make use of this identification and denote $T_x\Omega$ by $\mathfrak{g}_x$. Similarly, for any subspace $\mathfrak{l} \subset \mathfrak{g}$, $\mathfrak{l}_x$ will stand for the corresponding subspace of $\mathfrak{g}_x$, i.e. $\mathfrak{l}_x = (d\pi_x)_e(\mathfrak{l})$. In other words, $\mathfrak{l}$ will be viewed as a G-invariant distribution on Ω. We also denote by $\exp_x : \mathfrak{g}_x \to \Omega$ the composition $\pi_x \circ \exp$.

The following lemma lists several elementary local geometric properties of Ω.

Lemma. *For any bounded* $Q \subset \Omega$ *there exists positive* $\sigma_1 = \sigma_1(Q) < \sigma_0$*, such that for all* $x \in Q$

(a) π_x *is injective (hence an isometry) on* $B(4\sigma_1)$*;*

(b) *for any* $y \in B(x, 2\sigma_1)$ *and* $g \in B(2\sigma_1)$, $\operatorname{dist}(gx, gy) \le 2{\cdot}\operatorname{dist}(x, y)$*.*

From (a) and Lemma 1.3.1 it follows immediately that for $x \in Q$, the map $\exp_x$ is injective and bi-Lipschitz on $B_{\mathfrak{g}}(2\sigma_1)$, i.e.

$$\frac{1}{2}\|u - v\| \le \operatorname{dist}\bigl(\exp(u), \exp(v)\bigr) \le 2\|u - v\| \quad \forall\, u, v \in B_{\mathfrak{g}_x}(2\sigma_1)\,. \tag{1.2}$$

We will denote by $\log_x : \Omega \to \mathfrak{g}_x$ the inverse of $\exp_x$ defined (and bi-Lipschitz) at least in $B(x, 2\sigma_1)$, $x \in Q$.

1.3.3. An element $g \in G$ will be called *quasiunipotent* if $|\lambda| = 1$ for all eigenvalues λ of $\operatorname{Ad} g$; a subset F of G will be called *quasiunipotent* if all elements of F are *quasiunipotent*, and *nonquasiunipotent* otherwise.

Example. Let $G = SO_{k+1,1}(\mathbb{R})$, Γ a discrete torsion-free subgroup of G, $\Omega = G/\Gamma$ and $K = \begin{pmatrix} SO_{k+1}(\mathbb{R}) & 0 \\ 0 & 1 \end{pmatrix}$ a maximal compact subgroup of G. Choose a Riemannian metric on G which is left K-invariant (as well as right invariant). Then it is well-known that the double coset space $M \overset{\text{def}}{=} K\backslash\Omega$ is a Riemannian manifold of constant negative curvature, and Ω can be identified with the bundle of orthonormal frames associated to M. Moreover, any complete connected Riemannian manifold of constant negative curvature can be realized in the above way. Consider also $C = \begin{pmatrix} SO_k(\mathbb{R}) & 0 \\ 0 & I_2 \end{pmatrix}$; then the double coset space $C\backslash\Omega$ can be identified with the unit tangent bundle of M, that is, $S(M) = \{(x, \xi) \mid x \in M,\ \xi \in T_xM,\ \|\xi\| = 1\}$.

Denote by φ the canonical quotient map $\Omega \to S(M)$, and consider a nonquasiunipotent one-parameter subgroup $F = \{g_t \mid t \in \mathbb{R}\}$ of G commuting with C, where

$$g_t = \begin{pmatrix} I_k & 0 & 0 \\ 0 & \operatorname{ch} t & \operatorname{sh} t \\ 0 & \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} . \tag{1.3}$$

Then the action of F on Ω factors through φ, and it is well-known (cf. [Mau]) that the induced action is exactly the geodesic flow on $S(M)$. In other words, let $\{\gamma_{x,\xi}(t)\}$ be the geodesic on M through x in the direction of ξ parametrized by arclength. With some abuse of notation, we let γ_t be the time-t map of the geodesic flow, i.e. $\gamma_t(x,\xi) = \big(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t)\big)$; then $\gamma_t = {}_\varphi g_t$.

§2. The expanding horospherical subgroup and its tesselations

2.1. From now on, we will be considering actions of nonquasiunipotent one-parameter subgroups F of a unimodular Lie group G on $\Omega = G/\Gamma$, where Γ is a discrete subgroup of G. Specifically, we will talk about $F = \{g_t \mid t \in \mathcal{T}\}$, where $\mathcal{T}$ will stand for either $\mathbb{R}$ or $\mathbb{Z}$. We also introduce the notation $\mathcal{T}^+ \overset{\text{def}}{=} \{t \in \mathcal{T} \mid t \geq 0\}$ and $\mathcal{T}^- \overset{\text{def}}{=} \{t \in \mathcal{T} \mid t \leq 0\}$, and let $F^\pm = \{g_t \mid t \in \mathcal{T}^\pm\}$.

To study F-escapable sets, it is helpful to consider a special "horospherical" decomposition of G relative to F^+. Following [KM, §1.3], one can define subgroups H, H^0, H^- of G such that H^- is a horospherical subgroup with respect to g_1, while H is horospherical with respect to g_{-1}, and the Lie algebras $\mathfrak{h}$, $\mathfrak{h}^0$ and $\mathfrak{h}^-$ of H, H^0 and H^- span the Lie algebra $\mathfrak{g}$ of G. The subgroup F is nonquasiunipotent iff both $\mathfrak{h}$ and $\mathfrak{h}^-$ are nonzero. We will refer to H as to the *expanding horospherical subgroup* corresponding to F^+ (see §2.2 for motivation). By symmetry, H^- is of course expanding horospherical with respect to F^-. Note that both H and H^- are connected simply connected nilpotent Lie groups.

Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^0 \oplus \mathfrak{h}^-$, the product of H^-, H and H^0 (in any order) is open in G and contains the identity. Moreover, the direct product of H^-, H^0 and H also has $\mathfrak{g}$ as its Lie algebra, so one can apply Lemma 1.3.1 to it to get that the multiplication map from a neighborhood of identity in that direct product to G is one-to-one and distorts distances by at most the factor of 2.

We will denote by k the dimension of H and fix a Haar measure ν on H.

Example. If G and $F = \{g_t \mid t \in \mathbb{R}\}$ are as in Example 1.3.3, both H and H^- are isomorphic to $\mathbb{R}^k$. Trivial computation shows that the Lie algebra of H is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \mathbf{x} & -\mathbf{x} \\ \mathbf{x}^T & 0 & 0 \\ \mathbf{x}^T & 0 & 0 \end{pmatrix} \middle| \, \mathbf{x} \in \mathbb{R}^k \right\} . \tag{2.1}$$

2.2. The subalgebras $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{h}^-$ are invariant under $\operatorname{Ad} F$, which implies that the subgroups H, H^0, H^- are normalized by F. Moreover, it is easy to show that for all $t > 0$ the inner automorphism $\Phi_t : G \to G$, $g \to g_t g g_{-t}$, defines an *expanding* automorphism of H. More precisely, denote by S_t the restriction of $d\Phi_t$ on $\mathfrak{h}$. From the definition of $\mathfrak{h}$ it follows that the absolute values $\lambda_1^t, \dots, \lambda_k^t$ of the eigenvalues of S_t (ignoring multiplicities) are greater than 1. We will order them so that $\lambda_1 \leq \dots \leq \lambda_k$, and denote by $J = \lambda_1 \cdot \ldots \cdot \lambda_k$ the Jacobian of Φ_1; then

$$\nu\big(\Phi_t(V)\big) = J^t \nu(V) \tag{2.2}$$

for any $t \in \mathcal{T}$ and any measurable subset V of H.

Clearly if V is a small enough neighborhood of identity in H, then the contracting properties of the map Φ_t^{-1}, $t > 0$, on V are more or less determined by the bounds $\lambda_1^{-t}, \dots, \lambda_k^{-t}$ on the spectrum of S_t^{-1}. More precisely, the following is true:

Lemma. *There exists a constant $a_0 > 0$ such that for all $t \in \mathcal{T}^+$ and all $g, h \in B_H(\sigma_0)$, $g \neq h$, one has*

$$\check{a}(t) \overset{\text{def}}{=} \tfrac{1}{a_0} t^{-k} \lambda_k^{-t} \leq \frac{\operatorname{dist}\big(\Phi_t^{-1}(g), \Phi_t^{-1}(h)\big)}{\operatorname{dist}(g,h)} \leq \hat{a}(t) \overset{\text{def}}{=} a_0 t^k \lambda_1^{-t} . \tag{2.3}$$

Proof. Pass to the action of S_t on $\mathfrak{h}$ using Lemma 1.3.1, and then apply the estimates on iterations of linear transformations as in [Ma2, Lemma II.1.1]. $\square$

Note that the functions $\check{a}(t)$ and $\hat{a}(t)$ defined in (2.3) are decreasing for large t and tend to zero as $t \to \infty$. We put

$$\hat{t} \overset{\text{def}}{=} \inf\{\tau \in \mathcal{T}^+ \mid \hat{a}(t) \leq \frac{1}{16\sqrt{k}} \text{ and } \hat{a}'(t) \leq 0 \text{ for all } t \geq \tau\}$$

and will be in many cases taking t to be not less than $\hat{t}$.

2.3. We will consider the foliation of Ω by orbits of H, and will be estimating the Hausdorff dimension of subsets of Ω by looking at their intersections with leaves of this foliation. To make this idea more transparent, we introduce the following definition: say that a subset Z of Ω is *horospherically escapable relative to F^+* (or, briefly, *horospherically F^+-escapable*) if there exists an open dense subset Ω^+ of Ω such that for any $y \in \Omega^+$ and for any neighborhood V of identity in H

$$\dim\big(\{h \in V \mid hy \in E(F^+, Z)\}\big) = \dim(H) . \tag{2.4+}$$

In other words, if for any $y \in \Omega^+$ the set $H \cap \pi_y^{-1}\big(E(F^+, Z)\big)$ has full Hausdorff dimension at $e \in H$.

The importance of this notion is shown by the following

Theorem. (a) *$Z \subset \Omega$ is horospherically F^+-escapable $\Rightarrow$ it is F^+-escapable.*

(b) *Z is compact and horospherically escapable relative to both F^+ and F^- $\Rightarrow$ it is F-escapable.*

We remark that horospherical F^--escapability means existence of an open dense subset Ω^- of Ω such that for any $y \in \Omega^-$ and for any neighborhood V^- of identity in H^-

$$\dim\big(\{h^- \in V^- \mid h^- y \in E(F^-, Z)\}\big) = \dim(H^-) . \tag{2.4–}$$

Proof. Given a nonempty open $W \subset \Omega$, choose a point $x \in W \cap \Omega^+$ and put $Q = \overline{B_\Omega(x, 1)}$. Then take $U \subset B_G\big(\sigma_1(Q)\big)$ of the form VV^-V^0, where V, V^- and V^0 are small neighborhoods of identity in H, H^- and H^0 respectively, such that Ux is inside $W \cap \Omega^+$. Since $\pi_x|_U$ is an isometry, it suffices to prove that

$$\dim\left(\pi_x^{-1}\big(E(F^+, Z) \cap Ux\big)\right) = \dim\left(\{g \in U \mid gx \in E(F^+, Z)\}\right) = \dim(G) .$$

Furthermore, since U is bi-Lipschitz homeomorphic to $V \times V^0 \times V^-$, it is enough to show that

$$\dim\big(\{(h, h^0, h^-) \in V \times V^0 \times V^- \mid hh^0h^-x \in E(F^+, Z)\}\big) = \dim(G)\,. \tag{2.5}$$

Observe that for any $h^- \in V^-$ and $h^0 \in V^0$, the points $y = h^0h^-x$ are in Ω^+, hence satisfy (2.4+). Therefore, by Lemma 1.2.2(a), the right hand side of (2.5) is not less than (and therefore equal to) $k + \dim(V^0 \times V^-) = \dim(G)$, which proves (a).

For (b), we are given two dense open sets Ω^+ and Ω^- and an arbitrary nonempty open set W. Pick a point $x \in W \cap \Omega^+ \cap \Omega^-$ and again choose $U \subset B_G\big(\sigma_1(Q)\big)$, with $Q = \overline{B_\Omega(x, 1)}$ as above. However, U will now be of the form V^-VV^0, with V, V^- and V^0 being small neighborhoods of identity in H, H^- and H^0 respectively, such that Ux is inside $W \cap \Omega^+ \cap \Omega^-$. For brevity, we will skip the bi-Lipschitz argument and work with U as if it were the direct product of V, V^- and V^0.

Denote by B the set $\{g \in U \mid gx \in E(F^+, Z)\}$. Since $V^0x \subset \Omega^+$, the points $y = h^0x$ satisfy (2.4+) for any $h^0 \in V^0$; in other words, the slice $B \cap Vh^0$ has Hausdorff dimension k for any $h^0 \in V^0$. Thus, by Lemma 1.2.2,

$$\dim(B \cap VV^0) = \dim(V^0) + k\,. \tag{2.6}$$

We now claim that for all $g \in B$ there exists a neighborhood $V^-(g)$ of identity in H^- such that $V^-(g)g \subset B$. Indeed, for $g \in B$ denote by $\varepsilon(g)$ the distance between disjoint closed sets Z and $\overline{F^+gx}$; it is always positive because Z is compact. Since the map Φ_t, $t > 0$, is contracting on H^-, one can find an open neighborhood $V^-(g)$ of identity in H^- such that $V^-(g)gh \subset U$ and $\operatorname{diam}\big(\Phi_t(V^-(g)\big) \le \varepsilon(g)/2$ for any $t \in \mathcal{T}^+$. Then $g_tV^-(g)gx = \Phi_t\big(V^-(g)\big)g_tgx$ is for any $t \in \mathcal{T}^+$ disjoint from $Z^{(\varepsilon(g)/2)}$, hence the claim.

Take h^0 and h such that $hh^0 \in B \cap VV^0$. The points $y = hh^0x$ are in Ω^-, so one can apply (2.4−) with $V^-(hh^0)$ in place of V^-, and then combine it with (2.6) via Lemma 1.2.2(a) to conclude that $\dim\big(\{g \in B \mid gx \in E(F^-, Z)\}\big) = \dim(G)$. At the same time, $\{g \in B \mid gx \in E(F^-, Z)\} = \{g \in U \mid gx \in E(F^+, Z) \cap E(F^-, Z)\}$ is equal to $\{g \in U \mid gx \in E(F, Z)\}$, which finishes the proof. □

2.4. To prove horospherical escapability of subsets of Ω we will be using the following criterion.

Proposition. *For $Z \subset \Omega$, the following are equivalent:*

(i) *Z is horospherically F^+-escapable;*

(ii) *there exists an open dense subset Ω^+ of Ω such that for any $x \in \Omega^+$ there is a sequence of neighborhoods $V(s)$ of identity in H ($s \in \mathbb{N}$) with*

$$\operatorname{diam}\big(V(s)\big) \to 0 \text{ as } s \to \infty \tag{2.7}$$

and

$$\dim\big(\{h \in \overline{V(s)} \mid hx \in E(F^+, Z)\}\big) \to k \text{ as } s \to \infty\,. \tag{2.8}$$

Proof. From (i) it follows that the left hand side of (2.8) is equal to k for any open $V(s) \subset H$. Conversely, assume (ii) and take any neighborhood V of identity in H; by (2.7), $\overline{V(s)} \subset V$ for large enough s. Hence

$$\dim\big(\{h \in \overline{V} \mid hx \in E(F^+, Z)\}\big) \ge \dim\big(\{h \in \overline{V(s)} \mid hx \in E(F^+, Z)\}\big) \to k \text{ as } s \to \infty\,,$$

and (2.4+) follows. □

2.5. We now recall and refine some of the results of [KM, §3]. Say that an open subset V of H is a *tesselation domain* for the right action of H on itself relative to a countable subset Λ of H if

(i) $\nu(\partial V) = 0$,

(ii) $\gamma_1(V) \cap \gamma_2(V) = \varnothing$ for different $\gamma_1, \gamma_2 \in \Lambda$, and

(iii) $X = \bigcup_{\gamma \in \Lambda} \gamma\left(\overline{V}\right)$.

The pair (V, Λ) will be called a *tesselation* of H. We will use a one-parameter family of tesselations of H defined as follows: if $\{X_1, \dots, X_k\}$ is a fixed orthonormal *strong Malcev basis* of $\mathfrak{h}$ (see [CG] or [KM, Section 3.3] for the definition), we let $I = \left\{ \sum_{j=1}^{k} x_j X_j \mid |x_j| < \varepsilon/2 \right\}$ be the unit cube in $\mathfrak{h}$, and then take $V_r = \exp\left(\frac{r}{\sqrt{k}} I\right)$. It was proved in [KM] that V_r is a tesselation domain of H; let Λ_r be a corresponding set of translations. The properties of the family $\{V_r \mid r \le \sigma_0\}$, with σ_0 from Lemma 1.3.1, are listed below.

Proposition. *There exist positive constants a_1 and a_2 such that for any positive $r \le \sigma_0$*

(a) $B(\frac{r}{4\sqrt{k}}) \subset V_r \subset B(r)$*;*

(b) *for any positive* $r' \le r$, $V_{r'} \subset V_r$*;*

(c) *for any positive* $b \le 1$

$$\nu\big((\partial V_r)^{(br)}\big) \le a_1 \nu(V_r) \cdot b\,; \tag{2.9}$$

(d) *for any subset A of H*

$$\#\{\gamma \in \Lambda_r \mid V_r\gamma \cap A \ne \varnothing\} \le \frac{\nu(A^{(2r)})}{\nu(V_r)}\,; \tag{2.10}$$

in particular, for A of diameter not greater than $lr \le \sigma_0$

$$\#\{\gamma \in \Lambda_r \mid V_r\gamma \cap A \ne \varnothing\} \le a_2(l+4)^k\,. \tag{2.11}$$

Roughly speaking, parts (a) and (d) mean that one can think of the sets V_r as of balls of radius r: each of V_r is contained in a ball of radius r, and each such ball can be covered by at most $a_2 6^k$ translates of V_r.

Note that given V_r, there are many choices of Λ_r giving a tesselation (V_r, Λ_r) of H. In what follows we will arbitrarily choose Λ_r for all positive $r \le \sigma_0$; nothing will ever depend on these choices.

2.6. Recall that H is the expanding horospherical subgroup corresponding to F^+, which means that it comes with a one-parameter family $\{\Phi_t \mid t \in \mathcal{T}^+\}$ of expanding automorphisms. Given $r \le \sigma_0$ and $t \in \mathcal{T}^+$, let us denote by $\Lambda_r(t)$ the set of translations $\gamma \in \Lambda_r$ such that the translates $V_r\gamma$ lie entirely inside the image of V_r by the map Φ_t, i.e.

$$\Lambda_r(t) \overset{\text{def}}{=} \{\gamma \in \Lambda_r \mid V_r\gamma \subset \Phi_t(V_r)\}\,.$$

Let us show now that when t is large enough, the measure of the union of the translates $V_r\gamma$, $\gamma \in \Lambda_r(t)$, is approximately equal to the measure of $\Phi_t(V_r)$; in other words, boundary effects are negligible.

Proposition. *For any $r \le \sigma_0$ and any $t \in \mathcal{T}^+$*

(a) $J^t \ge \#\Lambda_r(t) \ge J^t\big(1 - 2a_1\hat{a}(t)\big)$;

(b) *the union $\bigcup_{\gamma\in\Lambda_r(t)} \overline{V_r}\gamma$ contains the Φ_t-image of the ball $B\left(\big(\frac{1}{4\sqrt{k}} - 2\hat{a}(t)\big)r\right)$ (note that the latter ball contains $B\big(\frac{r}{8\sqrt{k}}\big)$ whenever $t \ge \hat{t}$).*

Proof. The upper estimate for $\#\Lambda_r(t)$ is immediate from $\cup_{\gamma\in\Lambda_r(t)} V_r\gamma \subset \Phi_t(V_r)$ and (V_r, Λ_r) being a tesselation of H. For the lower estimate, one has

$$\begin{aligned}\Lambda_r(t) &= \big\{\gamma \in \Lambda_r \mid V_r\gamma \cap \Phi_t(V_r) \ne \varnothing\big\} \smallsetminus \big\{\gamma \in \Lambda_r \mid V_r\gamma \cap \partial\big(\Phi_t(V_r)\big) \ne \varnothing\big\}\\ &= \big\{\gamma \in \Lambda_r \mid V_r\gamma \cap \Phi_t(V_r) \ne \varnothing\big\} \smallsetminus \big\{\gamma \in \Lambda_r \mid \Phi_t^{-1}(V_r\gamma) \cap \partial(V_r) \ne \varnothing\big\}.\end{aligned}$$

Since (V_r, Λ_r) is a tesselation of H, the number of elements in the first set is not less than $\nu\big(\Phi_t(V_r)\big)/\nu(V_r) = J^t$, while the cardinality of the second one is not greater than

$$\begin{aligned}\frac{\nu\left((\partial V_r)^{(\mathrm{diam}(\Phi_t^{-1}(V_r)))}\right)}{\nu\big(\Phi_t^{-1}(V_r)\big)} &\underset{\text{(by (2.2) and (2.3))}}{\le} J^t\frac{\nu\left((\partial V_r)^{(\hat{a}(t)\mathrm{diam}(V_r))}\right)}{\nu(V_r)}\\ \text{(by Proposition 2.5(a))} \quad \le \quad J^t\frac{\nu\left((\partial V_r)^{(2\hat{a}(t)r)}\right)}{\nu(V_r)} &\underset{\text{(by (2.9))}}{\le} J^t 2a_1\hat{a}(t),\end{aligned}$$

hence (a). Now observe that the above argument also gives

$$\bigcup_{\gamma\in\Lambda_r(t)} \Phi_t^{-1}\left(\overline{V_r}\gamma\right) \supset V_r \smallsetminus (\partial V_r)^{(2\hat{a}(t)r)} \supset B\big(\big(\tfrac{r}{4\sqrt{k}} - 2\hat{a}(t)\big)r\big),$$

which implies (b). $\square$

§3. A CONDITION SUFFICIENT FOR ESCAPABILITY

3.1. Suppose a subset U of Ω, $y \in \Omega$, $t \in \mathcal{T}^+$ and positive $r \le \sigma_0$ are given. We want to generalize the definition of the set $\Lambda_r(t)$ in the following two ways: let $\Lambda_r^+(y, U, t)$ (resp. $\Lambda_r^-(y, U, t)$) consist of all translations $\gamma \in \Lambda_r(t)$ such that $V_r\gamma y$ has nonempty (resp. empty) intersection with U. Note that $\Lambda_r(t) = \Lambda_r^+(y, \Omega, t) = \Lambda_r^-(y, \varnothing, t)$ for any $y \in \Omega$, and that $\Lambda_r(t)$ is a disjoint union of $\Lambda_r^+(y, U, t)$ and $\Lambda_r^-(y, U, t)$ for any y and U. Define also the quantity $\delta_r^\pm(y, U, t)$ to be the density (relative measure) in $\Phi_t(V_r)$ of the union of $V_r\gamma$, $\gamma \in \Lambda_r^\pm(y, U, t)$, in other words

$$\delta_r^\pm(y, U, t) \overset{\text{def}}{=} \frac{\nu\left(\bigcup_{\gamma\in\Lambda_r^\pm(y,U,t)} V_r\gamma\right)}{\nu\big(\Phi_t(V_r)\big)} = J^{-t}\#\Lambda_r^\pm(y, U, t),$$

and put $\bar{\delta}_r^-(U, t) \overset{\text{def}}{=} \inf_{y\in\Omega} \delta_r^-(y, U, t)$ and $\bar{\delta}_r^+(U, t) \overset{\text{def}}{=} \sup_{y\in\Omega} \delta_r^+(y, U, t)$. By Proposition 2.6(a),

$$1 \ge \bar{\delta}_r^+(U, t) + \bar{\delta}_r^-(U, t) \ge 1 - 2a_1\hat{a}(t) \tag{3.1}$$

for any $t \in \mathcal{T}^+$ and $U \subset \Omega$.

Before stating the main result of this section, let us introduce more notation. If $t_1 < t_2 \in \mathcal{T}$, we let

$$[t_1, t_2] \overset{\text{def}}{=} \begin{cases} \{t_1 + 1, \dots, t_2\} \text{ if } \mathcal{T} = \mathbb{Z}\\ \{t \mid t_1 \le t \le t_2\} \text{ if } \mathcal{T} = \mathbb{R}.\end{cases}$$

In other words, $[t_1, t_2]$ stands for the compact segment of $\mathcal{T}$ with endpoint t_2 and "magnitude" $t_2 - t_1$ (where magnitude means either length or number of elements). We also put $g_{[t_1,t_2]} \overset{\text{def}}{=} \{g_t \mid t \in [t_1, t_2]\}$.

Theorem. *Let an open subset U of Ω, $t \in \mathcal{T}^+$ with $\hat{t} \le t$, and positive $r \le \sigma_0$ be given; denote by F_t the semigroup $\{g_{jt} \mid j \in \mathbb{N}\}$. Then for any $y \in \Omega$ there is a subset $\mathbf{A}_\infty(y)$ of $\overline{V_r}$ such that*

$$\dim\bigl(\mathbf{A}_\infty(y)\bigr) \ge k - \frac{\log(\bar{\delta}_r^-(U,t))}{\log\bigl(\hat{a}(t)\bigr)} \tag{3.2}$$

and

$$\mathbf{A}_\infty(y)y \subset E(F_t, U)\,. \tag{3.3}$$

Proof. We will construct a family $\bigl\{\mathcal{A}(y) \mid y \in \Omega\bigr\}$ of strongly tree-like (relative to the Haar measure ν on H) collections of subsets of $\overline{V_r}$ inductively as follows. First let $\mathcal{A}_0(y) = \bigl\{\overline{V_r}\bigr\}$ for all $y \in \Omega$, then define

$$\mathcal{A}_1(y) = \{\Phi_t^{-1}\left(\overline{V_r}\gamma\right) \mid \gamma \in \Lambda_r^-(g_t y, U, t)\}\,. \tag{3.4}$$

More generally, if $\mathcal{A}_i(y)$ is defined for all $y \in \Omega$ and $i < j$, we let

$$\mathcal{A}_j(y) = \{\Phi_t^{-1}(A\gamma) \mid A \in \mathcal{A}_{j-1}(\gamma g_t y),\ \gamma \in \Lambda_r^-(g_t y, U, t)\}\,. \tag{3.5}$$

The properties (TL1) and (TL2) follow readily from the construction and (V_r, Λ_r) being a tesselation of H; hence it makes sense to talk about the limit set $\mathbf{A}_\infty(y)$ of $\mathcal{A}(y)$. Also, from (3.5) and Lemma 2.2 it follows that for all $j \in \mathbb{N}$ and $y \in \Omega$, the constant $d_j\bigl(\mathcal{A}(y)\bigr)$ is not greater than $2r\bigl(\hat{a}(t)\bigr)^j$, and therefore (STL) is satisfied (recall that $\hat{a}(t) < 1$ for $t \ge \hat{t}$). Let us now show by induction that the jth density $\Delta_j\bigl(\mathcal{A}(y)\bigr)$ of $\mathcal{A}(y)$ is for all $y \in \Omega$ and $j \in \mathbb{N}$ bounded from below by δ. Indeed, by definition

$$\begin{aligned}
\Delta_1\left(\overline{V_r}, \mathcal{A}(y)\right) = \frac{\nu\bigl(\mathbf{A}_1(y)\bigr)}{\nu\left(\overline{V}_r\right)} &\underset{\text{(by (3.4))}}{=} \frac{\nu\bigl(\bigcup_{\gamma \in \Lambda_r(g_t y, U, t)} \Phi_t^{-1}(V_r\gamma)\bigr)}{\nu(V_r)} \\
\text{(by relative } \Phi_t\text{-invariance of } \nu) &= J^{-t}\#\Lambda_r^-(g_t y, U, t) \\
&= \delta_r^-(g_t y, U, t) > \bar{\delta}_r^-(U,t)\,.
\end{aligned}$$

On the other hand, if $j \ge 2$ and $B \in \mathcal{A}_{j-1}(y)$ is of the form $\Phi_t^{-1}(A\gamma)$ for $A \in \mathcal{A}_{j-2}\bigl(\gamma g_t y\bigr)$, the formula (3.5) gives

$$\begin{aligned}
\Delta_j\bigl(B, \mathcal{A}(y)\bigr) &= \frac{\nu\bigl(B \cap \mathbf{A}_j(y)\bigr)}{\nu(B)} = \frac{\nu\left(\Phi_t^{-1}\bigl(B \cap \mathbf{A}_j(y)\bigr)\right)}{\nu\bigl(\Phi_t^{-1}(B)\bigr)} \\
&= \frac{\nu\bigl(A\gamma \cap \mathbf{A}_{j-1}(\gamma g_t y)\gamma\bigr)}{\nu(A\gamma)} = \frac{\nu\bigl(A \cap \mathbf{A}_{j-1}(\gamma g_t y)\bigr)}{\nu(A)} = \Delta_{j-1}\bigl(A, \mathcal{A}(\gamma g_t y)\bigr)\,,
\end{aligned}$$

and induction applies. Finally, the measure ν clearly satisfies (1.1) with $k = \dim(H)$, and an application of Lemma 1.2.2 yields that for all $y \in \Omega$

$$\dim\bigl(\mathbf{A}_\infty(y)\bigr) \ge k - \limsup_{j \to \infty} \frac{j \log\left(\bar{\delta}_r^-(U,t)\right)}{\log\left(2r\bigl(\hat{a}(t)\bigr)^j\right)}\,,$$

which is exactly the right hand side of (3.2).

It now remains to show (3.3). First note that from (3.4) it immediately follows that for all $y \in \Omega$, the set $\Phi_t\bigl(\mathbf{A}_1(y)\bigr)g_t y = g_t\mathbf{A}_1(y)y$ is disjoint from U. The definition (3.5) and induction then give $g_{jt}\mathbf{A}_j(y)y \cap U = \varnothing$ for all $y \in \Omega$ and $j \in \mathbb{N}$. But U is open, hence the closure of $F_t z$ is disjoint from U for any $z \in \mathbf{A}_\infty(y)y$; in other words, $z \in E(F_t, U)$, and the proof is completed. $\square$

3.2. Corollary. *Let* U*,* t *and* r *be as in the above theorem, and let* $Z \subset \Omega$ *be such that* $g_{[0,t]}Z \subset U$*. Then for any* $x \in \Omega$

$$\dim\bigl(\{h \in \overline{V_r} \mid hx \in E(F^+, Z)\}\bigr) \geq k - \frac{\log\bigl(\bar{\delta}_r^-(U,t)\bigr)}{\log\bigl(\hat{a}(t)\bigr)} \tag{3.6}$$

Proof. Indeed, one has $E(F_t, U) \subset E(F_t, g_{[0,t]}Z) = E(F^+, Z)$ by Lemma 1.1.1; therefore the set described in the left hand side of (3.6) contains $\mathbf{A}_\infty(x)$, and the claim follows from (3.2). $\square$

3.3. Now we are ready to use Proposition 2.4 and write down a condition[1] sufficient for horospherical escapability of $Z \subset \Omega$.

Proposition. *For* $Z \subset \Omega$*, assume that for any* $x \in \Omega$ *there exist sequences* $r_s \to 0$*,* $t_s \in \mathcal{T}^+$*,* $\hat{t} \leq t_s$*, and a sequence of open sets* U_s *containing* $g_{[0,t_s]}Z$*,* $s \in \mathbb{N}$*, with*

$$\lim_{s\to\infty} \frac{-\log\bigl(\bar{\delta}_{r_s}^-(U_s,t_s)\bigr)}{t_s} = 0 \tag{3.7}$$

(i.e. $\bar{\delta}_{r_s}^-(U_s,t_s)$ *decays at most subexponentially with respect to* t_s*). Then* Z *is horospherically* F^+*-escapable.*

Proof. Indeed, the sequence $\dfrac{\log\bigl(\hat{a}(t_s)\bigr)}{t_s} = \dfrac{\log(a_0) + k\log(t_s) + t_s\log(\hat{\lambda})}{t_s}$ is bounded between two negative constants, hence (3.7) is equivalent to

$$\lim_{s\to\infty} \frac{\log\bigl(\bar{\delta}_{r_s}^-(U_s,t_s)\bigr)}{\log\bigl(\hat{a}(t_s)\bigr)} = 0\,. \tag{3.8}$$

Clearly the sequence $V(s) = V_{r_s}$ satisfies (2.7), and (2.8) follows from (3.6) and (3.8); an application of Proposition 2.4 finishes the proof. $\square$

In what follows, we will be taking the sets U_s to be η_s-neighborhoods of $g_{[0,t_s]}Z$ for some positive η_s. In other words, we will be checking the following condition:

$$\liminf_{r\to 0,\, t\geq\hat{t},\, \eta>0} \frac{-\log\Bigl(\bar{\delta}_r^-\bigl((g_{[0,t]}Z)^{(\eta)}, t\bigr)\Bigr)}{t} = 0\,, \tag{$*^-$}$$

which has just been proven to be sufficient for escapability of Z.

3.4. In many cases it is more convenient to estimate $\bar{\delta}_r^+(U,t)$ from above than $\bar{\delta}_r^-(U,t)$ from below[2]. One immediately has

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[1] See [K] for a weaker condition also sufficient for escapability of Z, as well as for a unified exposition of methods and results from [KM] and the present paper.

[2] One of the reasons is the subadditivity of $\bar{\delta}_r^+(U,t)$ in the first variable: $U \subset \cup_{i=1}^n U_i$ implies $\bar{\delta}_r^+(U,t) \leq \sum_{i=1}^n \bar{\delta}_r^+(U_i,t)$ for any t and r.

Corollary. $(*^-)$ *is equivalent to*

$$\liminf_{r\to 0,\, t\geq \hat t,\, \eta>0} \frac{-\log\big(1-\bar\delta^+_r\big((g_{[0,t]}Z)^{(\eta)},t\big)\big)}{t} = 0\,, \tag{$*^+$}$$

i.e. to the existence of sequences $r_s \to 0$, $\eta_s > 0$ *and* $t_s \in \mathcal{T}^+$, $\hat t \leq t_s$, *with*

$$\lim_{s\to\infty} \frac{-\log\big(1-\bar\delta^+_{r_s}\big((g_{[0,t_s]}Z)^{(\eta_s)},t_s\big)\big)}{t_s} = 0\,; \tag{3.9}$$

in particular, any set satisfying $(*^+)$ *is horospherically* F^+*-escapable.*

Proof. By (3.1), $-\log\big(1-\bar\delta^+_r(U,t)\big) \leq -\log\big(\bar\delta^-_r(U,t)\big)$, hence $(*^-)$ implies $(*^+)$. On the other hand, take sequences satisfying (3.9) and let $U_s = (g_{[0,t_s]}Z)^{(\eta_s)}$. If $\{\tau_s\}$ is bounded, one necessarily has $\hat\delta^+_{r_s}(U_s,t_s) = 0$ for large s, hence $\bar\delta^-_{r_s}(U_s,t_s)\big) = 1$ and (3.8) is satisfied. Otherwise $\hat a(t_s)$ decays at least exponentially with respect to t_s, therefore

$$\lim_{s\to\infty} \frac{-\log\big(1-\bar\delta^+_{r_s}(U_s,t_s)\big)}{\tau_s} = \lim_{s\to\infty} \frac{-\log\big(1-\bar\delta^+_{r_s}(U_s,t_s)-2a_1\hat a(t_s)\big)}{t_s}\,,$$

while the denominator in the right hand side is not less that $-\log\big(\bar\delta^-_{r_s}(U_s,t_s)\big)$ by (3.1). $\square$

3.5. The use of η-neighborhoods has the following technical advantage: in order to estimate the quantity $\delta^+_r(x,Y^{(\eta)},t)$ uniformly for all $x\in\Omega$, one can in many cases worry only about points x in the set Y. Let us record this observation as a

Lemma. *Let a bounded subset* Y *of* Ω, $t\geq\hat t$, $\eta < 6\sigma_1(Y)$ *and* $r < \sigma_1(Y)\check a(t)$ *be given. Then for any* $x\in\Omega$ *there exists* $y\in Y$ *such that*

$$\delta^+_{r/16\sqrt{k}}(x,Y^{(\eta/3)},t) \leq a_2(32\sqrt{k}+4)^k\delta^+_r(y,Y^{(\eta)},t)\,. \tag{3.10}$$

Proof. Take any $x\in\Omega$; if $\Phi_t(V_{r/16\sqrt{k}})x\cap Y^{(\eta/3)} = \varnothing$, (3.10) is trivially satisfied for any $y\in Y$. Otherwise, there exists $y\in Y$ and $x'\in\Phi_t(V_{r/16\sqrt{k}})x$ such that $\operatorname{dist}(y,x') < \eta/3$. Since $\operatorname{diam}(V_{r/16\sqrt{k}}) \leq \frac{r}{8\sqrt{k}}$, $\Phi_t(V_{r/16\sqrt{k}})x$ is contained in $\Phi_t\big(B_H(\frac{r}{8\sqrt{k}})\big)x'$.

Consider

$$V' \overset{\text{def}}{=} \bigcup\nolimits_{\gamma\in\Lambda_r(t)} \overline{V_r}\gamma$$

and

$$V'' \overset{\text{def}}{=} \bigcup\nolimits_{\gamma\in\Lambda^+_r(y,Y^{(\eta)},t)} \overline{V_r}\gamma\,,$$

clearly $V''\subset V'\subset\Phi_t\left(\overline{V_r}\right)$. Note that Lemma 2.2 implies $\Phi_t\left(\overline{V_r}\right)\subset\overline{B(\frac{r}{\check a(t)})}\subset B\big(2\sigma_1(Y)\big)$. On the other hand, by Proposition 2.6(b), V' contains $\Phi_t\big(B_H(\frac{r}{8\sqrt{k}})\big)$; therefore

$$V'x' \supset \Phi_t\big(B_H(\tfrac{r}{8\sqrt{k}})\big)x' \supset \Phi_t(V_{r/16\sqrt{k}})x\,.$$

We now claim that $\Phi_t(V_{r/16\sqrt{k}})x\cap Y^{(\eta/3)}\subset V''x'$. Indeed, it suffices to show that $V'x'\cap Y^{(\eta/3)}\subset V''x'$. Take $h\in V'$ such that $\operatorname{dist}(hx',Y)<\eta/3$. From the choice of r and η it follows that $h \in B_H(2\sigma_1)$ and $x' \in B_\Omega(y, 2\sigma_1)$. Thus by Lemma 1.3.2(b)

$$\operatorname{dist}_\Omega(hx', hy) < 2\eta/3 \Rightarrow \operatorname{dist}_\Omega(hy, Y) < \eta \Rightarrow h \in V''\,.$$

Finally, V'' is the union of at most $\#\Lambda_r^+(y, Y^{(\eta)}, t)$ translates $\overline{V_r}\gamma$, each of diameter at most $2r$. By (2.11), each of them can be covered by at most $a_2(32\sqrt{k}+4)^k$ translates of the form $V_{r/16\sqrt{k}}\gamma$. This implies

$$\#\Lambda^+_{r/16\sqrt{k}}(x, Y^{(\eta/3)}, t) \le a_2(32\sqrt{k}+4)^k \#\Lambda_r^+(y, Y^{(\eta)}, t)\,.$$

as desired. $\square$

3.6. We now illustrate the use of the above lemma by proving escapability of a finite set in the discrete time case (the corresponding fact for $\mathcal{T} = \mathbb{R}$ is also true and will be proved in §4.4).

Corollary. *Finite subsets of Ω are F-escapable for any nonquasiunipotent cyclic subgroup F of G.*

Proof. Let Z be a finite subset of Ω. For any $s \in \mathbb{N}$, let $t = t_s = s + \hat{t}$. Our goal is to find $r_s \to 0$ and η_s such that (3.9) is satisfied. Denote $Y = g_{[0,t]}Z$; we will prove that for some (decreasing to zero as $t \to \infty$) $r = r(t)$ and $\eta = \eta(t)$, one has $\bar{\delta}_r^+(Y^{(\eta)}, t) \le \text{const}\cdot J^{-t}$, with "const" being independent of t. By virtue of Lemma 3.5, it suffices to find r and η such that for any $y \in Y$, $\delta_r^+(y, Y^{(\eta)}, t) \le \text{const}\cdot J^{-t}$. Since Y is finite, one can choose $\sigma > 0$ such that σ-neighborhood of any point of Y contains no other points of Y. Denote the closure of $Y^{(1)}$ by Q, and assume also that $\sigma < \sigma_1(Q)$.

Now put $r = \eta = \frac{1}{2}\check{a}(t)\sigma$; then (2.3) implies that for any $y \in Y$, $\Phi_t(V_r)y$ is contained in $B(y, \sigma/2)$. Thus $Y^{(\eta)} \cap \Phi_t(V_r)y = B(y, \eta) \cap \Phi_t(V_r)y$ has diameter at most $2\eta = 2r$. Then one can use (2.11) to get $\#\Lambda_r^+(y, Y^{(\eta)}, t) \le a_2 6^k$, or $\delta_r^+(y, Y^{(\eta)}, t) \le c_0 6^k J^{-t}$. This immediately implies $(*^+)$, and it remains to apply Corollary 3.4 and both parts of Theorem 2.3 to get the desired result. $\square$

One can notice that for finite sets Z the quantity $\bar{\delta}_r^+\big((g_{[0,t]}Z)^{(\eta)}, t\big)$ decreases (in t for suitable r and η) much faster than it is needed to satisfy the condition (3.9). Later in this chapter we will study a large class of sets for which this quantity decays exponentially. The simple argument we used to deduce the above corollary will serve as a model for the proof of the main result.

§4. COMPACT $\mathfrak{h}$-TRANSVERSAL SUBMANIFOLDS

4.1. Transversality conditions

4.1.1. Let $\mathfrak{l}$ and $\mathfrak{m}$ be two distributions (not necessarily G-invariant) defined on some subsets of Ω. We will say that $\mathfrak{l}$ is $\mathfrak{m}$-*transversal at a point* $x \in \Omega$ if $\mathfrak{l}_x, \mathfrak{m}_x \subset \mathfrak{g}_x$ are defined and $\mathfrak{m}_x$ is not contained in $\mathfrak{l}_x$ (in other words, if the intersection $\mathfrak{l}_x \cap \mathfrak{m}_x$ has positive codimension in $\mathfrak{m}_x$). We will say that $\mathfrak{l}$ is $\mathfrak{m}$-*transversal* if for any $x \in \Omega$ where $\mathfrak{l}_x$ is defined, $\mathfrak{m}_x$ is also defined and $\mathfrak{l}$ is $\mathfrak{m}$-transversal at x.

If Z is a C^1 submanifold of Ω, the tangent bundle TZ of Z can be thought of as a distribution defined on Z. We will say that Z is $\mathfrak{m}$-*transversal* (resp. $\mathfrak{m}$-*transversal at* $z \in Z$) if TZ is such. An important special case is when $\mathfrak{m} = \mathfrak{h}$, the distribution corresponding to the foliation of Ω by orbits of the expanding horospherical subgroup H of G: Z is $\mathfrak{h}$-transversal at $z \in Z$ if $\dim(\mathfrak{h}_z \cap T_z Z) < k$. Clearly any C^1 submanifold of dimension less than k is automatically $\mathfrak{h}$-transversal.

As a quantitative approach to $\mathfrak{h}$-transversality, we will consider a function $\theta_{\mathfrak{h}} : Z \to \mathbb{R}$,

$$\theta_{\mathfrak{h}}(z) \overset{\text{def}}{=} \sup_{v \in \mathfrak{h}_z,\, \|v\|=1} \operatorname{dist}_{\mathfrak{h}_z}(v, T_z Z)\,.$$

Clearly $\theta_{\mathfrak{h}}(z) \neq 0$ iff Z is $\mathfrak{h}$-transversal at z. It is also straighforward to verify that this function is continuous in $z \in Z$. Therefore the following holds:

Lemma. *A compact C^1 submanifold Z of Ω is $\mathfrak{h}$-transversal iff there exist positive $c_1 = c_1(Z)$ such that $\theta_{\mathfrak{h}}(z) \geq c_1$ for all $z \in Z$.*

4.1.2. In the continuous time case, another important example of a G-invariant distribution is the one defined by the Lie algebra $\mathfrak{f}$ of $F = \{g_t \mid t \in \mathbb{R}\}$. This gives a special case of the above definition: a C^1 submanifold Z of Ω is $\mathfrak{f}$-transversal iff the F-orbit of any point of Z is not tangent to Z. We will need the following simple

Lemma. *Let $F = \{g_t \mid t \in \mathbb{R}\}$ and let Z be a compact $\mathfrak{f}$-transversal C^1 submanifold of Ω. Then*

(a) *there exists positive $\varepsilon_1 = \varepsilon_1(Z)$ such that $g_{[-\varepsilon_1,\varepsilon_1]}Z$ is a C^1 manifold;*

(b) *if, in addition, $TZ \oplus \mathfrak{f}$ is $\mathfrak{h}$-transversal, then there exists positive $\varepsilon_2 = \varepsilon_2(Z) \leq \varepsilon_1(Z)$ such that $Z_{[0,\tau]}$ is $\mathfrak{h}$-transversal.*

Note that $TZ \oplus \mathfrak{f}$ is automatically $\mathfrak{h}$-transversal whenever Z is $\mathfrak{f}$-transversal and $\dim(Z) < k$.

Proof. Using a finite covering of Z by appropriate coordinate charts of Ω, one can without loss of generality assume that Z is of the form $\varphi\left(\overline{U}\right)$ for some bounded open $U \subset \mathbb{R}^d$, with φ being a C^1, nonsingular imbedding defined in an open set U' strictly containing $\overline{U}$. Define $\tilde{\varphi} : U' \times \mathbb{R} \to \Omega$ by putting $\tilde{\varphi}(u,t) = g_t\big(\varphi(u)\big)$. From $\mathfrak{f}$-transversality of Z it follows that $\tilde{\varphi}$ is nonsingular at $t = 0$ and $u \in \overline{U}$. Hence $\tilde{\varphi}$ is a nonsingular imbedding of $U'' \times [-\varepsilon_1, \varepsilon_1]$ into Ω for some $\varepsilon_1 > 0$ and an open set U'' strictly containing $\overline{U}$, and (a) is proved.

Clearly $(TZ \oplus \mathfrak{f})_z = T(g_{[-\varepsilon_1,\varepsilon_1]}Z)_z$ for $z \in Z$, so $\mathfrak{h}$-transversality of $TZ \oplus \mathfrak{f}$ implies that $g_{[-\varepsilon_1,\varepsilon_1]}Z$ is $\mathfrak{h}$-transversal at any point of Z, hence (by continuity of the function $\theta_{\mathfrak{h}}$ introduced in the previous subsection) at any its point which is close enough to Z. Therefore one can choose $\varepsilon_2 \leq \varepsilon_1$ such that $Z_{[0,\tau]}$ is $\mathfrak{h}$-transversal. $\square$

4.2. The main theorem

4.2.1. It turns out that $\mathfrak{h}$-transversality property of a compact C^1 submanifold Z of Ω implies certain asymptotic behavior of the quantities $\bar{\delta}_r^+\big(Z^{(\eta)}, t\big)$ introduced in §3. The crucial fact is the following

Theorem. *Let U and $U' \supset \overline{U}$ be two bounded open subsets of $\mathbb{R}^d$, $\varphi : U' \to \Omega$ a C^1 nonsingular imbedding. Assume that $Z = \varphi\left(\overline{U}\right)$ is $\mathfrak{h}$-transversal. Then for any $t \geq \hat{t}$ there exists $r_0 = r_0(Z,t) > 0$ such that for all positive $r \leq r_0$ there exists $\eta_0 = \eta_0(Z,r) > 0$ with*

$$\bar{\delta}_r^+(Z^{(\eta)}, t) \leq \bar{C} t^{2^k} \lambda_1^{-t}$$

for all positive $\eta \le \eta_0$, the constant $\bar{C} > 0$ being independent on Z, t, r and η. In other words,

$$\limsup_{r\to 0}\ \limsup_{\eta\to 0}\ \bar{\delta}_r^+(Z^{(\eta)},t) \le \bar{C}t^{2^k}\lambda_1^{-t}\,. \tag{4.1}$$

Proof. The main idea is roughly the same as that of the proof of Corollary 3.6. Using compactness and smoothness of Z, one takes a small σ such that the intersection of any σ-ball with Z lies in a very thin neighborhood of its tangent hyperplane. For fixed t, one picks r such that the set $\Phi_t(V_r)$ has diameter less than σ. Then from $\mathfrak{h}$-transversality of Z it follows that one can choose η such that for all $z \in Z$, the intersection of $\Phi_t(V_r)z$ with $Z^{(\eta)}$ has very small relative measure in $\Phi_t(V_r)z$, hence can be covered by very small number of translates of V_r.

We now give a detailed proof. Let Q be, say, 1-neighborhood of Z; consider $\sigma_1 = \sigma_1(Q)$ from Lemma 1.3.2. Then choose $\sigma_2 < \sigma_1$ such that the closure of $Z^{(\sigma_2)}$ is contained in $\varphi(U')$. For any $u \in \overline{U}$, denote by W_u the φ-preimage of $B\big(\varphi(u),\sigma_2\big)$, a neighborhood of u contained in U'. Denote by φ_u the composition $\log_{\varphi(u)}\circ\varphi : W_u \to \mathfrak{g}_{\varphi(u)}$, $u \to 0$.

The assumption that φ is a C^1 nonsingular imbedding implies that the maps φ_u, $u \in \overline{U}$, are C^1 and nonsingular imbeddings. Moreover, since $\overline{U}$ is compact, the norm of the first derivative of φ_u at u is for all $u \in \overline{U}$ bounded from below. In other words, there exists a constant c_2 such that for all $u \in \overline{U}$ and all $u' \in W_u$

$$\|\varphi_u(u') - \varphi_u(u)\| = \|\varphi_u(u')\| \ge c_2\|u' - u\|\,. \tag{4.2}$$

Furthermore, from the uniform continuity of $d\varphi$ on $\overline{U}$ it follows that for any $\alpha > 0$ there exists $\sigma_3(\alpha)$ such that for all $u \in \overline{U}$ and all $u' \in W_u$ with $\|u' - u\| < \sigma_3(\alpha)$ one has

$$\|d\varphi_u(u)(u' - u) - \varphi_u(u')\| \le \alpha\|u' - u\|\,. \tag{4.3}$$

Combining (4.2) and (4.3), we get the following: given any $\alpha > 0$ and $u \in \overline{U}$, if $v' = \varphi_u(u')$ is a point in $\varphi_u(W_u) = \log_{\varphi(u)}\big(B(\varphi(u),\sigma_2)\big)$ with $\|v\| \le c_2\sigma_3(\alpha)$, then there exists a point $v'' = d\varphi_u(u)(u' - u) \in T_{\varphi(u)}z$ such that

$$\|v'' - v'\| \le \frac{\alpha}{c_2}\|v'\|\,. \tag{4.4}$$

Fix $t \ge \hat{t}$; our goal is to prove (4.1) for some universal constant $\bar{C}$. Take any

$$\sigma \le \min\left(\sigma_2, \tfrac{1}{2}c_2\sigma_3\big(\frac{c_1c_2\check{a}(t)}{4}\big)\right)\,. \tag{4.5σ}$$

Then let

$$r = \tfrac{1}{4}\check{a}(t)\sigma\,, \tag{4.5r}$$

and take any

$$\eta \le c_1 r\,. \tag{4.5η}$$

Fix a point $z = \varphi(u)$, $u \in \overline{U}$, and denote by A the set $\{h \in \Phi_t(V_r) \mid hz \in Z^{(\eta)}\}$. Note that $\Lambda_r^+(z, Z^{(\eta)}, t)$ is exactly equal to $\{\gamma \in \Lambda_r(t) \mid V_r\gamma \cap A \ne \varnothing\}$.

Claim. *There exists a constant $\tilde{C}$ independent on Z and x such that*

$$\nu(A^{(2r)}) \le \tilde{C} t^{2^k} (\lambda_2 \cdot \ldots \cdot \lambda_k)^t \nu(V_r) . \tag{4.6}$$

To derive (4.1) from the claim, one can use (2.10) to get

$$\#\Lambda_r^+(z, Z^{(\eta)}, t) \le a_1 \tilde{C} t^{2^k} (\lambda_2 \cdot \ldots \cdot \lambda_k)^t$$

for all $z \in Z$, or, by Lemma 3.5,

$$\bar{\delta}^+_{r/16\sqrt{k}}(Z^{(\eta/3)}, t) \le a_1 a_2 (32\sqrt{k}+4)^k \tilde{C} t^{2^k} (\lambda_2 \cdot \ldots \cdot \lambda_k)^t J^{-t} = a_1 a_2 (32\sqrt{k}+4)^k \tilde{C} t^{2^k} \lambda_1^{-t} .$$

Since r (and η after the choice of r) can be made arbitrarily small, this completes the proof of the theorem modulo the claim which will be demonstrated below. $\square$

4.2.2. *Proof of Claim.* From Lemma 2.2 and the choice of η, r and σ it follows that $\big(\Phi_t(V_r)z\big)^{(\eta)} \subset B(z,\sigma) \subset B(z,\sigma_2)$. Thus it makes sense to consider $\log_z$-images, being sure (cf. (1.2)) that the metric is being distorted by at most a factor of 2. Define

$$\mathfrak{a} \overset{\text{def}}{=} \log_z(Az) = S_t(\tfrac{r}{\sqrt{k}} I) \cap \log_z(Z^{(\eta)}) \subset S_t(\tfrac{r}{\sqrt{k}} I) \cap (\log_z Z)^{(2\eta)} \subset \mathfrak{h}_z .$$

Since σ was chosen to be less than σ_0, one has

$$\nu(A^{(2r)_H}) \le 2^k \cdot \mathrm{vol}(\mathfrak{a}^{(4r)_\mathfrak{h}}) , \tag{4.7}$$

thus it suffices to estimate

$$\mathrm{vol}(\mathfrak{a}^{(4r)_\mathfrak{h}}) \le \mathrm{vol}\left(\big(S_t(\tfrac{r}{\sqrt{k}} I) \cap (\log_z Z)^{(2\eta)_\mathfrak{g}}\big)^{(4r)_\mathfrak{h}}\right) . \tag{4.8}$$

It is now time to use differentiability of φ, that is, the inequality (4.4). Take any $z' = \varphi(u') \in Z \cap B(z,\sigma)$, $u' \in W_u$, and denote $\log_z(z') = \varphi_u(u')$ by v'. Then by (4.5σ), $\|v'\| < 2\sigma < c_2 \sigma_3\big(\frac{c_1 c_2 \breve{a}(t)}{4}\big)$, thus by (4.4) and (4.5$r$)

$$\mathrm{dist}(v', T_z Z) \le \frac{1}{c_2} \frac{c_1 c_2 \breve{a}(t)}{4} \|v'\| < \frac{c_1 \breve{a}(t)}{2} \sigma \le 2 c_1 r .$$

This clearly implies

$$(\log_z Z)^{(2\eta)_\mathfrak{g}} \subset (T_z Z)^{(2\eta + 2c_1 r)_\mathfrak{g}} , \tag{4.9}$$

in other words, we have passed from the manifold to its tangent space. The last step is provided by the assumption of $\mathfrak{h}$-transversality of Z. From Lemma 4.1.1 one gets

$$c_1 \le \theta_\mathfrak{h}(z) = \sup_{v \in \mathfrak{h}_z ,\, \|v\|=1} \mathrm{dist}(v, T_z Z) = \sup_{v \in \mathfrak{h}_z \smallsetminus T_z Z} \frac{\mathrm{dist}(v, T_z Z)}{\mathrm{dist}(v, \mathfrak{h}_x \cap T_z Z)} ;$$

therefore

$$(T_z Z)^{(2\eta + 2c_1 r)_\mathfrak{g}} \subset (T_z Z \cap \mathfrak{h}_z)^{(2\eta/c_1 + 2r)_\mathfrak{h}} \underset{(\text{by } (4.5\eta))}{\subset} (T_z Z \cap \mathfrak{h}_z)^{(4r)_\mathfrak{h}} . \tag{4.10}$$

Combining (4.7)–(4.10), one obtains

$$\begin{aligned}\nu(A^{(2r)_H}) &\le 2^k \cdot \mathrm{vol}\left(\left(S_t(\tfrac{r}{\sqrt{k}}I) \cap (T_z Z \cap \mathfrak{h}_z)^{(4r)_\mathfrak{h}}\right)^{(4r)_\mathfrak{h}}\right) \\ &\le 2^k \cdot \mathrm{vol}\left(\left(S_t(\tfrac{r}{\sqrt{k}}I) \cap T_z Z\right)^{(8r)_\mathfrak{h}}\right) .\end{aligned} \tag{4.11}$$

Let $q = \dim(T_z Z \cap \mathfrak{h}_z) < k$. It is easy to observe that[3]

$$\mathrm{vol}\left(\left(S_t(\tfrac{r}{\sqrt{k}}I) \cap T_z Z\right)^{(8r)}\right) \le C_q r^{k-q} \cdot \mathrm{vol}_q\big(S_t(\tfrac{r}{\sqrt{k}}I) \cap T_z Z\big) , \tag{4.12}$$

where vol_q is the q-dimensional Lebesgue measure. But this gives one a chance to translate everything into $\frac{r}{\sqrt{k}}I$:

$$\mathrm{vol}_q\big(S_t(\tfrac{r}{\sqrt{k}}I) \cap T_z Z\big) \le \|S_t\|_{\wedge q} \cdot \mathrm{vol}_q\big(\tfrac{r}{\sqrt{k}}I \cap S_t^{-1}(T_z Z \cap \mathfrak{h}_z)\big) , \tag{4.13}$$

where by $\|S_t\|_{\wedge q}$ we mean the norm of S_t acting on the qth external power of $\mathfrak{h}$. Thus it remains to estimate this norm (again using [Ma2, Lemma II.1.1]):

$$\|S_t\|_{\wedge q} \le C'_q t^{\binom{k}{q}} (\lambda_{k-q+1} \cdot \ldots \cdot \lambda_k)^t \tag{4.14}$$

and the q-dimensional volume of a section of $\frac{r}{\sqrt{k}}I$ by a q-dimensional affine space:

$$\mathrm{vol}_q\big(\tfrac{r}{\sqrt{k}}I \cap S_t^{-1}(T_z Z \cap \mathfrak{h}_z)\big) \le C''_q r^q . \tag{4.15}$$

Putting together the estimates (4.11)–(4.15), one obtains that $\nu(A^{(2r)})$ is not greater than

$$\begin{aligned}&\nu(A^{(2r)}) \le 2^k C_q C'_q C''_q t^{\binom{k}{q}} (\lambda_{k-q+1} \cdot \ldots \cdot \lambda_k)^t r^k \\ &\le 2^k C_q C'_q C''_q t^{\binom{k}{q}} (\lambda_{k-q+1} \cdot \ldots \cdot \lambda_k)^t \sqrt{k}^k (\mathrm{vol}(\tfrac{r}{\sqrt{k}}I) \le \tilde{C} t^{2^k} (\lambda_2 \cdot \ldots \cdot \lambda_k)^t \nu(V_r) ,\end{aligned}$$

where $\tilde{C} = 4^k \sqrt{k}^k \max_{0 \le q < k} C_q C'_q C''_q$, thus completing the proof of (4.6). □

4.3. The discrete time case

4.3.1. Observe that Theorem 4.3.1 does not say that the set Z in question satisfies $(*^+)$. Indeed, to prove that condition one needs to have an analogue of (4.1) with Z being replaced by $g_{[0,t]}Z$. It turns out that in the case $\mathcal{T} = \mathbb{Z}$ this can be easily achieved.

Theorem. *Let $F = \{g_t \mid t \in \mathbb{Z}\}$ be a cyclic nonquasiunipotent subgroup of G. Then for any $\mathfrak{h}$-transversal compact C^1 submanifold Z of Ω*

$$\begin{gathered}\exists\, C(Z) > 0 \text{ such that } \forall\, t \ge \hat{t} \\ \limsup_{r \to 0} \limsup_{\eta \to 0} \bar{\delta}_r^+\big((g_{[0,t]}Z)^{(\eta)}, t\big) \le C(Z) t^{2^k+1} \lambda_1^{-t} ;\end{gathered} \tag{4.16}$$

---

[3] All the constants C_q, C'_q, C''_q below depend only on the dimension of $\mathfrak{h}$.

in particular, Z as above satisfies $(*^+)$.

Proof. Clearly Z can be covered by finite number of sets Z_i, $1 \le i \le N$, with Z_i being of the form $\varphi_i\left(\overline{U}_i\right)$ for some bounded open subset U_i of a Euclidean space and φ_i being a C^1 nonsingular imbedding defined in an open set strictly containing $\overline{U}_i$. Observe that the manifolds $g_l Z_i$, $1 \le l \le t$, are all of the above form. Moreover, since g preserves the H-orbit foliation of Ω, all these manifolds are $\mathfrak{h}$-transversal. Subadditivity of $\bar{\delta}_r^+$ gives

$$\limsup_{r\to 0}\ \limsup_{\eta\to 0}\ \bar{\delta}_r^+\big((g_{[0,t]}Z)^{(\eta)},t\big) \le \sum_{i=1}^{N}\sum_{l=1}^{t}\limsup_{r\to 0}\ \limsup_{\eta\to 0}\ \bar{\delta}_r^+\big((g_l Z_i)^{(\eta)},t\big)\,,$$

which is not greater than $Nt\bar{C}t^{2^k}\lambda_1^{-t}$ by Theorem 4.2.1. $\square$

4.3.2. Corollary. *For F and Z as in the above theorem, Z is horospherically F^+-escapable, hence F^+-escapable. Consequently, any compact C^1 submanifold of Ω of dimension less than* $\min\big(\dim(H),\dim(H^-)\big)$ *is F-escapable.*

Proof. Immediate from Corollary 3.4 and Theorem 2.3. $\square$

4.3.3. Remark. An important feature of the condition (4.16) is its additivity: a set satisfies (4.16) iff it can be covered by finitely many sets satisfying (4.16). Therefore the conclusions of Theorem 4.3.1 and Corollary 4.3.2 hold when Z can be covered by finitely many $\mathfrak{h}$-transversal compact C^1 submanifolds. In particular, self-intersections of at most finite multiplicity do not cause a problem.

The last observation can be put in a more general form: consider a condition

$$\limsup_{t\to+\infty}\ \limsup_{r\to 0}\ \limsup_{\eta\to 0}\ \bar{\delta}_r^+\big((g_{[0,t]}Z)^{(\eta)},t\big) = 0\,. \qquad (**^+)$$

Clearly (4.16) $\Rightarrow (**^+) \Rightarrow (*^+)$, and one can immediately see that a finite union of sets satisfying $(**^+)$ also satisfies $(**^+)$. This is an important difference between $(**^+)$ and other escapability conditions that were considered before.

4.3.4. We now consider the situation of Example 1.3.3 and apply Corollary 4.3.2 to geodesic flows on manifolds of constant negative curvature.

Corollary. *Let M be a $k+1$-dimensional complete connected Riemannian manifold of constant negative curvature, and let $Y \subset M$ be a compact C^1 submanifold of dimension less than k (or a finite union thereof, see Remark 4.3.3). Then for any positive τ the set $S(Y)$ is escapable relative to $\{\gamma_{i\tau} \mid i \in \mathbb{Z}\}$ (the geodesic shift on $S(M)$).*

Proof. We use the notation of Example 1.3.3. Define F_τ to be equal to $\{g_{i\tau} \mid i \in \mathbb{Z}\}$, with g_t as in (1.3). Then the expanding horospherical subgroup corresponding to F_τ^+ is $\exp(\mathfrak{h})$, with $\mathfrak{h}$ as in (2.1). Let Z stand for $\varphi^{-1}\big(S(Y)\big) = \pi^{-1}(Y)$, where π is the canonical quotient map of G/Γ onto M, and let d be the dimension of Y. Then for all $z \in Z$, $T_z Z$ is a direct sum of the space isomorphic to the Lie algebra $\mathfrak{k} = \begin{pmatrix} \mathfrak{so}_{k+1}(\mathbb{R}) & 0 \\ 0 & 0 \end{pmatrix}$ of K and a d-dimensional subspace of $\mathfrak{g}_z$. One can easily check that $\mathfrak{h}$ as in (2.1) has trivial intersection with $\mathfrak{k}$. Therefore the dimension of $\mathfrak{h}_z \cap T_z Z$ is at most $d < k$, so Z is $\mathfrak{h}$-transversal. Moreover, Z is clearly a compact C^1 submanifold of Ω, hence it is horospherically F_τ^+-escapable by Corollary 4.3.2. Similar argument shows that Z is $\mathfrak{h}^-$-transversal, thus horospherically F_τ^--escapable. Theorem 2.3(b) then yields that Z is F_τ-escapable. It remains to notice that $\{\gamma_{i\tau} \mid i \in \mathbb{Z}\} = {}_{\varphi}(F_\tau)$ and φ is a compact bi-Lipschitz covering, and apply Theorem 1.2.4(b) to get the desired result. $\square$

4.4. The continuous time case

4.4.1. In this section we take F to be of the form $\{g_t \mid t \in \mathbb{R}\}$. Instead of directly applying the methods of §3, we will use a trick of Lemma 4.1.2 and reduce the problem to the discrete time case.

Theorem. *Let $F = \{g_t \mid t \in \mathbb{R}\}$ be a one-parameter nonquasiunipotent subgroup of G, and let Z be an $\mathfrak{f}$-transversal compact C^1 submanifold of Ω such that $TZ \oplus \mathfrak{f}$ is $\mathfrak{h}$-transversal. Then Z satisfies* (4.16).

Proof. Take $t \ge \hat{t}$, and pick a positive $\varepsilon \le \varepsilon_2(Z)$ from Lemma 4.1.2(b) such that t is an integer multiple of ε; one can do this with $\varepsilon \ge \frac{1}{2}\varepsilon_2(Z)$. Denote by $F' = \{g'_s \mid s \in \mathbb{Z}^+\}$ the cyclic semigroup generated by g_ε, i.e. put $g'_s = g_{s\varepsilon}$. Also let Z' stand for the ($\mathfrak{h}$-transversal compact C^1) manifold $g_{[-\varepsilon_2(Z),0]}Z$. Clearly both F and F' induce the same horospherical decomposition of G. However, the constants introduced in §2.3 are parametrization-dependent; more precisely, if by λ'_1 and $\hat{t}'$ we denote the constants λ_1 and $\hat{t}$ defined for the group F', it is easy to see that $\lambda'_1 = \lambda_1^\varepsilon$ and $\hat{t}' = \min\{s \mid \varepsilon s \ge \hat{t}\}$. Moreover, if $\bar{\delta}'^+_r$ stands for the function $\bar{\delta}^+_r$ defined starting from the group F', one has $\bar{\delta}'^+_r(U, s) = \bar{\delta}^+_r(U, s\varepsilon)$ for any $U \subset \Omega$, $r > 0$ and $s \in \mathbb{N}$.

Put $s = t/\varepsilon$; then $s \ge \hat{t}'$, so by Theorem 4.3.1 applied to F' and Z', there exists positive $C(Z')$ such that

$$\limsup_{r\to 0}\ \limsup_{\eta\to 0}\ \bar{\delta}'^+_r\big((g'_{[0,s]}Z')^{(\eta)}, s\big) \le C(Z')s^{2^k+1}(\lambda'_1)^{-s}\,. \tag{4.17}$$

But $g'_{[0,s]}Z'$ clearly contains $g_{[0,t]}Z$, so (4.17) implies

$$\limsup_{r\to 0}\ \limsup_{\eta\to 0}\ \bar{\delta}^+_r\big((g_{[0,t]}Z)^{(\eta)}, t\big) \le C(Z')\varepsilon^{-(2^k+1)}t^{2^k+1}\lambda_1^{-t}\,,$$

which means that the constant $C(Z) = C(Z')\big(\frac{2}{\varepsilon_2(Z)}\big)^{2^k+1}$ will satisfy (4.16). $\square$

4.4.2. As in the previous section, we immediately get

Corollary. *For F and Z as in the above theorem, Z is horospherically F^+-escapable, hence F^+-escapable. Consequently, any finite union of $\mathfrak{f}$-transversal compact C^1 submanifolds of dimension less than* $\min\big(\dim(H), \dim(H^-)\big)$ *is F-escapable.*

In particular, any finite subset of Ω is always F-escapable. Note that this solves a part of Conjecture (B) from [Ma2] (the latter asserts that for any finite subset Z of Ω, $Z \cup \{\infty\}$ is escapable).

4.4.3. Remark. Another (less painful) way to prove the F^+-escapability of Z (without the stronger uniform condition (4.16)) is provided by a direct application of Theorem 1.2.4(a). Indeed, put $F'' = g_{[-\varepsilon_2(Z),0]}$; we saw that $F''(Z)$ is an $\mathfrak{h}$-transversal compact C^1 submanifold of Ω, so it is F'-escapable by Corollary 4.3.2, therefore Z is $(F'')^{-1}F'$-escapable, and $(F'')^{-1}F'$ clearly contains F^+.

4.4.4. The assumption of $\mathfrak{f}$-transversality of Z is actually very strong and does not seem to be natural. In particular, in the situation of Corollary 4.3.4 $S(Y)$ is not $\mathfrak{f}$-transversal whenever a submanifold Y of M has positive dimension. However we are able to prove

Corollary. *Let M be as in Corollary 4.3.4, and let $Y \subset M$ be a finite set. Then $S(Y)$ is escapable relative to $\{\gamma_t \mid t \in \mathbb{R}\}$ (the geodesic flow on $S(M)$).*

Proof. We use the notation of Corollary 4.3.4. For any $z \in Z \overset{\text{def}}{=} \pi^{-1}(Y)$, $T_z Z$ is isomorphic to $\mathfrak{k} = \begin{pmatrix} \mathfrak{so}_{k+1}(\mathbb{R}) & 0 \\ 0 & 0 \end{pmatrix}$. Hence

$$\mathfrak{f}_z = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \middle| \, a \in \mathbb{R} \right\}$$

is not contained in $T_z Z$, and the direct sum

$$\mathfrak{h}_z \oplus \mathbb{R} f_z = \left\{ \begin{pmatrix} 0 & \mathbf{x} & -\mathbf{x} \\ \mathbf{x}^T & 0 & a \\ \mathbf{x}^T & a & 0 \end{pmatrix} \middle| \, \mathbf{x} \in \mathbb{R}^k, a \in \mathbb{R} \right\}$$

has empty intersection with $T_z Z$. Therefore Z is $\mathfrak{f}$-transversal and $TZ \oplus \mathfrak{f}$ is $\mathfrak{h}$-transversal, so Theorem 4.4.1 applies; the rest of the proof of Corollary 4.3.4 goes without changes. ☐

We remark that from the results of [Do] it follows that any countable subset of $S(M)$ is escapable relative to the geodesic flow.

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