ERGODIC THEORY ON HOMOGENEOUS SPACES AND METRIC NUMBER THEORY

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Article outline

This article gives a brief overview of recent developments in metric number theory, in particular, Diophantine approximation on manifolds, obtained by applying ideas and methods coming from dynamics on homogeneous spaces.

Glossary

Diophantine approximation
Diophantine approximation refers to approximation of real numbers by rational numbers, or more generally, finding integer points at which some (possibly vector-valued) functions attain values close to integers.

metric number theory
Metric number theory (or, specifically, metric Diophantine approximation) refers to the study of sets of real numbers or vectors with prescribed Diophantine approximation properties.

homogeneous spaces
A homogeneous space G/Γ of a group G by its subgroup Γ is the space of cosets $\{g\Gamma\}$. When G is a Lie group and Γ is a discrete subgroup, the space G/Γ is a smooth manifold and locally looks like G itself.

lattice; unimodular lattice
A lattice in a Lie group is a discrete subgroup of finite covolume; unimodular stands for covolume equal to 1.

ergodic theory
The study of statistical properties of orbits in abstract models of dynamical systems.

Hausdorff dimension
A nonnegative number attached to a metric space and extending the notion of topological dimension of "sufficiently regular" sets, such as smooth submanifolds of real Euclidean spaces.

1. Definition of the subject and its importance

The theory of Diophantine approximation, named after Diophantus of Alexandria, in its simplest set-up deals with the approximation of real numbers by rational numbers. Various higher-dimensional generalizations involve studying values of linear or polynomial maps at integer points. Often a certain "approximation property" is fixed, and one wants to characterize the set of numbers (vectors, matrices) which share this property, by means of certain measures (Lebesgue, or Hausdorff, or some other interesting measures). This is usually referred to as *metric* Diophantine approximation.

The starting point for the theory is an elementary fact that $\mathbb{Q}$, the set of rational numbers, is dense in $\mathbb{R}$, the reals. In other words, every real number can be approximated by rationals: for any $y \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $p/q \in \mathbb{Q}$ with

$$|y - p/q| < \varepsilon\,. \tag{1.1}$$

To answer questions like "how well can various real numbers be approximated by rational numbers? i.e., how small can ε in (1.1) be chosen for varying $p/q \in \mathbb{Q}$?", a natural approach has been to compare the accuracy of the approximation of y by p/q to the "complexity" of the latter, which can be measured by the size of its denominator q in its reduced form. This seemingly simlpe set-up has led to introducing many important Diophantine approximation properties of numbers/vectors/matrices, which show up in various fields of mathematics and physics, such as differential equations, KAM theory, transcendental number theory.

2. Introduction

As the first example of refining the statement about the density of $\mathbb{Q}$ in $\mathbb{R}$, consider a theorem by Kronecker stating that for any $y \in \mathbb{R}$ and any $c > 0$, there exist infinitely many $q \in \mathbb{Z}$ such that

$$|y - p/q| < c/|q|\,, \quad \text{i.e. } |qy - p| < c \tag{2.1}$$

for some $p \in \mathbb{Z}$. A comparison of (1.1) and (2.1) shows that it makes sense to multiply both sides of (1.1) by q, since in the right hand side of (2.1) one would still be able to get very small numbers. In other words, approximation of y by p/q translates into approximating integers by integer multiples of y.

Also, if y is irrational, (p, q) can be chosen to be relatively prime, i.e. one gets infinitely many different rational numbers p/q satisfying (2.1). However if $y \in \mathbb{Q}$ the latter is no longer true for small enough c. Thus it seems to be more convenient to talk about pairs (p, q) rather than $p/q \in \mathbb{Q}$, avoiding a necessity to consider the two cases separately.

At this point it is convenient to introduce the following central definition: if ψ is a function $\mathbb{N} \to \mathbb{R}_+$ and $y \in \mathbb{R}$, say that y is *ψ-approximable* (notation: $y \in \mathcal{W}(\psi)$) if there exist infinitely many $q \in \mathbb{N}$ such that

$$|qy - p| < \psi(q) \tag{2.2}$$

for some $p \in \mathbb{Z}$. Because of Kronecker's Theorem, it is natural to assume that $\psi(x) \to 0$ as $x \to \infty$. Often ψ will be assumed non-increasing, although many results do not require monotonicity of ψ.

One can similarly consider a higher-dimensional version of the above set-up. Note that $y \in \mathbb{R}$ in the above formulas plays the role of a linear map from $\mathbb{R}$ to another copy of $\mathbb{R}$, and one asks how close values of this map at integers are from integers. It is natural to generalize it by taking a linear operator Y from $\mathbb{R}^n$ to $\mathbb{R}^m$ for fixed $m, n \in \mathbb{N}$, that is, an $m \times n$-matrix (interpreted as a system of m linear forms Y_i on $\mathbb{R}^n$). We will denote by $M_{m,n}$ the space of $m \times n$ matrices with real coefficients. For ψ as above, one says that $Y \in M_{m,n}$ is *ψ-approximable* (notation: $Y \in \mathcal{W}_{m,n}(\psi)$) if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|Y\mathbf{q} + \mathbf{p}\| \le \psi(\|\mathbf{q}\|) \tag{2.3}$$

for some $\mathbf{p} \in \mathbb{Z}^m$. Here $\|\cdot\|$ is the supremum norm on $\mathbb{R}^k$ given by $\|\mathbf{y}\| = \max_{1\le i\le k} |y_i|$. (This definition is slightly different from the one used in [58], where powers of norms were considered.)

Traditionally, one of the main goals of metric Diophantine approximation has been to understand how big the sets $\mathcal{W}_{m,n}(\psi)$ are for fixed m, n and various functions ψ. Of course, (2.3) is not the only interesting condition that can be studied; various modifications of the approximation properties can also be considered. For example the Oppenheim Conjecture, now a theorem of Margulis [69] and a basis for many important recent developments [22, 34, 35], states that indefinite irrational quadratic forms can take arbitrary small values at integer points; Littlewood's conjecture, see (6.2) below, deals with a similar statement about products of linear forms. See the article "Ergodic Theory and Rigidity" by Nitica and surveys [70, 32] for details.

We remark that the standard tool for studying Diophantine approximation properties of real numbers ($m = n = 1$) is the continued fraction expansion, or, equivalently, the Gauss map $x \mapsto 1/x \mod 1$ of the unit interval, see [49]. However the emphasis of this survey lies in higher-dimensional theory, and the dynamical system described below can be thought of as a replacement for the continued fraction technique applicable in the one-dimensional case. Additional details about interactions between ergodic theory and number theory can be found in the article by Nitica mentioned above, in "Ergodic Theory: Recurrence" by Frantzikinakis and McCutcheon and "Ergodic Theory: Interactions with Combinatorics and Number Theory" by Ward, as well as in the survey papers [32, 33, 50, 58, 66, 70, 71].

Here is a brief outline of the rest of the article. In the next section we survey basic results, some classical, some obtained relatively recently, in metric Diophantine approximation. §4 is devoted to a description of the connection between Diophantine approximation and dynamics, specifically flows on the space of lattices. In §5 and §6 we specialize to the set-up of Diophantine approximation on manifolds, or, more generally, approximation properties of vectors with respect to measures satisfying some natural conditions, and show how applications of homogeneous dynamics contributed to important recent developments in the field. §7 mentions several open questions and directions for further investigation.

3. Basic facts

General references for this section: [17, 80].

The simplest choice for functions ψ happens to be the following: let us denote $\psi_{c,v}(x) = cx^{-v}$. It was shown by Dirichlet in 1842 that with the choice $c = 1$ and $v = n/m$, all $Y \in M_{m,n}$ are ψ-approximable. Moreover, Dirichlet's Theorem states that for any $Y \in M_{m,n}$ and for any $t > 0$ there exist $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ satisfying the following system of inequalities:

$$\|Y\mathbf{q} - \mathbf{p}\| < e^{-t/m} \quad \text{and} \quad \|\mathbf{q}\| \le e^{t/n} . \tag{3.1}$$

From this it easily follows that $\mathcal{W}_{m,n}(\psi_{1,n/m}) = M_{m,n}$. In fact, it is this paper of Dirichlet which gave rise to his box principle. Later another proof of the same result was given by Minkowski. The constant $c = 1$ is not optimal: the smallest value of c for which $\mathcal{W}_{1,1}(\psi_{c,1}) = \mathbb{R}$ is $1/\sqrt{5}$, and the optimal constants are not known in higher dimensions, although some estimates can be given [80].

Systems of linear forms which do not belong to $\mathcal{W}_{m,n}(\psi_{c,n/m})$ for some positive c are called *badly approximable*; that is, we set

$$\mathrm{BA}_{m,n} \stackrel{\mathrm{def}}{=} M_{m,n} \smallsetminus \cup_{c>0} \mathcal{W}_{m,n}(\psi_{c,n/m}) .$$

Their existence in arbitrary dimensions was shown by Perron. Note that a real number y ($m = n = 1$) is badly approximable if and only if its continued fraction coefficients are uniformly bounded. It was proved by Jarnik [46] in the case $m = n = 1$ and by Schmidt in the general case [78] that badly approximable matrices form a set of full Hausdorff dimension: that is, $\dim(\mathrm{BA}_{m,n}) = mn$.

On the other hand, it can be shown that each of the sets $\mathcal{W}_{m,n}(\psi_{c,n/m})$ for any $c > 0$ has full Lebesgue measure, and hence the complement $\mathrm{BA}_{m,n}$ to their intersection has measure zero. This is a special case of a theorem due to Khintchine [48] in the case $n = 1$ and to Groshev [42] in full generality, which gives the precise condition on the function ψ under which the set of ψ-approximable matrices has full measure. Namely, if ψ is non-increasing (this assumption can be removed in higher dimensions but not for $n = 1$, see [29]), then λ-almost no (resp. λ-almost every) $Y \in M_{m,n}$ is ψ-approximable, provided the sum

$$\sum_{k=1}^{\infty} k^{n-1}\psi(k)^m \tag{3.2}$$

converges (resp. diverges). (Here and hereafter λ stands for Lebesgue measure.) This statement is usually referred to as the Khintchine-Groshev Theorem. The convergence case of this theorem follows in a straightforward manner from the Borel-Cantelli Lemma, but the divergence case is harder. It was reproved and sharpened in 1960 by Schmidt [76], who showed that if the sum (3.2) diverges, then for almost all Y the number of solutions to (2.3) with $\|\mathbf{q}\| \le N$ is asymptotic to the partial sum of the series (3.2) (up to a constant), and also gave an estimate for the error term.

A special case of the convergence part of the theorem shows that $\mathcal{W}_{m,n}(\psi_{1,v})$ has measure zero whenever $v > n/m$. Y is said to be *very well approximable* if it belongs to $\mathcal{W}_{m,n}(\psi_{1,v})$ for some $v > n/m$. That is,

$$\mathrm{VWA}_{m,n} \stackrel{\mathrm{def}}{=} \cup_{v>n/m} \mathcal{W}_{m,n}(\psi_{1,v})\,.$$

More specifically, let us define the *Diophantine exponent* $\omega(Y)$ of Y (sometimes called "the exact order" of Y) to be the supremum of $v > 0$ for which $Y \in \mathcal{W}_{m,n}(\psi_{1,v})$. Then $\omega(Y)$ is always not less than n/m, and is equal to n/m for Lebesgue-a.e. Y; in fact, $\mathrm{VWA}_{m,n} = \{Y \in M_{m,n} : \omega(Y) > n/m\}$.

The Hausdorff dimension of the null sets $\mathcal{W}_{m,n}(\psi_{1,v})$ was computed independently by Besicovitch [14] and Jarnik [45] in the one-dimensional case and by Dodson [26] in general: when $v > n/m$, one has

$$\dim\big(\mathcal{W}_{m,n}(\psi_{1,v})\big) = (n-1)m + \frac{m+n}{v+1}\,. \tag{3.3}$$

See [27] for a nice exposition of ideas involved in the proof of both the aforementioned formula and the Khintchine-Groshev Theorem.

Note that it follows from (3.3) that the null set $\mathrm{VWA}_{m,n}$ has full Hausdorff dimension. Matrices contained in the intersection

$$\bigcap_v \mathcal{W}_{m,n}(\psi_{1,v}) = \{Y \in M_{m,n} : \omega(Y) = \infty\}$$

are called *Liouville* and form a set of Hausdorff dimension $(n-1)m$, that is, to the dimension of Y for which $Y\mathbf{q} \in \mathbb{Z}$ for some $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ (the latter belong to $\mathcal{W}_{m,n}(\psi)$ for any positive ψ).

Note also that the aforementioned properties behave nicely with respect to transposition; this is described by the so-called Khintchine's Transference Principle [17, Ch. V]. For example, $Y \in \mathrm{BA}_{m,n}$ if and only if $Y^T \in \mathrm{BA}_{n,m}$, and $Y \in \mathrm{VWA}_{m,n}$ if and only if $Y^T \in \mathrm{VWA}_{n,m}$. In particular, many problems related to approximation properties of vectors ($n = 1$) and linear forms ($m = 1$) reduce to one another.

We refer the readers to [43] and [8] for very detailed and comprehensive recent accounts of various further aspects of the theory.

4. Connection with dynamics on the space of lattices

General references for this section: [4, 86].

Interactions between Diophantine approximation and the theory of dynamical systems has a long history. Already in Kronecker's Theorem one can see a connection.

Indeed, the statement of the theorem can be rephrased as follows: the points on the orbit of 0 under the rotation of the circle $\mathbb{R}/\mathbb{Z}$ by y approach the initial point 0 arbitrarily closely. This is a special case of the Poincare Recurrence Theorem in measurable dynamics. And, likewise, all the aforementioned properties of $Y \in M_{m,n}$ can be restated in terms of recurrence properties of the $\mathbb{Z}^n$-action on the m-dimensional torus $\mathbb{R}^m/\mathbb{Z}^m$ given by $\mathbf{x} \mapsto Y\mathbf{x} \mod \mathbb{Z}^m$. In other words, fixing Y gives rise to a dynamical system in which approximation properties of Y show up.

However the theme of this section is a different dynamical system, whose phase space is (essentially) the space of parameters Y, and which can be used to read the properties of Y from the behavior of the associated trajectory.

It has been known for a long time (see [81] for a historical account) that Diophantine properties of real numbers can be coded by the behavior of geodesics on the quotient of the hyperbolic plane by $\mathrm{SL}_2(\mathbb{Z})$. In fact, the latter flow can be viewed as the suspension flow of the Gauss map mentioned at the end of §2. There have been many attempts to construct a higher-dimensional analogue of the Gauss map so that it captures all the features of simultaneous approximation, see [47, 63, 65] and references therein. On the other hand, it seems to be more natural and efficient to generalize the suspension flow itself, and this is where one needs higher rank homogeneous dynamics.

As was mentioned above, in the basic set-up of simultaneous Diophantine approximation one takes a system of m linear forms $Y_1, \dots, Y_m$ on $\mathbb{R}^n$ and looks at the values of $|Y_i(\mathbf{q}) + p_i|$, $p_i \in \mathbb{Z}$, when $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n$ is far from 0. The trick is to put together

$$Y_1(\mathbf{q}) + p_1, \dots, Y_m(\mathbf{q}) + p_m \quad \text{and} \quad q_1, \dots, q_n,$$

and consider the collection of vectors

$$\left\{ \begin{pmatrix} Y\mathbf{q} + \mathbf{p} \\ \mathbf{q} \end{pmatrix} \middle| \, \mathbf{p} \in \mathbb{Z}^m, \, \mathbf{q} \in \mathbb{Z}^n \right\} = L_Y \mathbb{Z}^k$$

where $k = m + n$ and

$$L_Y \stackrel{\text{def}}{=} \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix}, Y \in M_{m,n} \, . \tag{4.1}$$

This collection is a unimodular lattice in $\mathbb{R}^k$, that is, a discrete subgroup of $\mathbb{R}^k$ with covolume 1. Our goal is to keep track of vectors in such a lattice having small projections onto the first m components of $\mathbb{R}^k$ and big projections onto the last n components. This is where dynamics comes into the picture. Denote by g_t the one-parameter subgroup of $\mathrm{SL}_k(\mathbb{R})$ given by

$$g_t = \operatorname{diag}(\underbrace{e^{t/m}, \dots, e^{t/m}}_{m \text{ times}}, \underbrace{e^{-t/n}, \dots, e^{-t/n}}_{n \text{ times}}) \, . \tag{4.2}$$

The vectors in the lattice $L_Y \mathbb{Z}^k$ are moved by the action of g_t, $t > 0$, and a special role is played by the moment t when the “small” and “big” projections equalize.

That is, one is led to consider a new dynamical system. Its phase space is the space of unimodular lattices in $\mathbb{R}^k$, which can be naturally identified with the homogeneous space

$$\Omega_k \stackrel{\text{def}}{=} G/\Gamma, \text{ where } G = \mathrm{SL}_k(\mathbb{R}) \text{ and } \Gamma = \mathrm{SL}_k(\mathbb{Z}) \, , \tag{4.3}$$

and the action is given by left multiplication by elements of the subgroup (4.2) of G, or perhaps other subgroups $H \subset G$. Study of such systems has a rich history; for example, they are known to be ergodic and mixing whenever H is unbounded [74]. What is important in this particular case is that the space Ω_k happens to be noncompact, and its structure at infinity is described via Mahler's Compactness Criterion, see [4, Chapter V]: a sequence of lattices $g_i\mathbb{Z}^k$ goes to infinity in $\Omega_k \iff$ there exists a sequence $\{\mathbf{v}_i \in \mathbb{Z}^k \smallsetminus \{0\}\}$ such that $g_i(\mathbf{v}_i) \to 0$ as $i \to \infty$. Equivalently, for $\varepsilon > 0$ consider a subset K_ε of Ω_k consisting of lattices with no nonzero vectors of norm less than ε; then all the sets K_ε are compact, and every compact subset of Ω_k is contained in one of them. Moreover, one can choose a metric on Ω_k such that $\text{dist}(\Lambda, \mathbb{Z}^k)$ is, up to a uniform multiplicative constant, equal to $-\log \min_{\mathbf{v}\in\Lambda\smallsetminus\{0\}} \|\mathbf{v}\|$ (see [25]); then the length of the smallest nonzero vector in a lattice Λ will determine how far away is this lattice in the "cusp" of Ω_k.

Using Mahler's Criterion, it is not hard to show that $Y \in \text{BA}_{m,n}$ if and only if the trajectory

$$\{g_t L_Y \mathbb{Z}^k : t \in \mathbb{R}_+\} \tag{4.4}$$

is bounded in Ω_k. This was proved by Dani [20] in 1985, and later generalized in [57] to produce a criterion for Y to be ψ-approximable for any non-increasing function ψ. An important special case is a criterion for a system of linear forms to be very well approximable: $Y \in \text{VWA}_{m,n}$ if and only if the trajectory (4.4) has linear growth, that is, there exists a positive γ such that $\text{dist}(g_t L_Y \mathbb{Z}^k, \mathbb{Z}^k) > \gamma t$ for an unbounded set of $t > 0$.

This correspondence allows one to link various Diophantine and dynamical phenomena. For example, from the results of [55] on abundance of bounded orbits on homogeneous spaces one can deduce the aforementioned theorem of Schmidt [78]: the set $\text{BA}_{m,n}$ has full Hausdorff dimension. And a dynamical Borel-Cantelli Lemma established in [57] can be used for an alternative proof of the Khintchine-Groshev Theorem; see also [87] for an earlier geometric approach. Note that both proofs are based on the following two properties of the g_t-action: mixing, which forces points to return to compact subsets and makes preimages of cusp neighborhoods quasi-independent, and hyperbolicity, which implies that the behavior of points on unstable leaves is generic. The latter is important since the orbits of the group $\{L_Y\mathbb{Z}^k : Y \in M_{m,n}\}$ are precisely the unstable leaves with respect to the g_t-action.

We note that other types of Diophantine problems, such as conjectures of Oppenheim and Littlewood mentioned in the previous section, can be reduced to statements involving Ω_k by means of the same principle: Mahler's Criterion is used to relate small values of some function at integer points to excursions to infinity in Ω_k of orbit of the stabilizer of this function.

However most important and useful recent applications of homogeneous dynamics to metric Diophantine approximation are related to the circle of ideas roughly called "Diophantine approximation with dependent quantities" (terminology borrowed from [84]), to be surveyed in the next two sections.

5. Diophantine approximation with dependent quantities: the set-up

General references for this section: [12, 84].

Here we restrict ourselves to Diophantine properties of vectors in $\mathbb{R}^n$. In particular, we will look more closely at the set of very well approximable vectors, which we will simply denote by VWA, dropping the subscripts. In many cases it does not matter whether one works with row or column vectors, in view of the duality remark made at the end of §3.

We begin with a non-example of an application of dynamics to Diophantine approximation: a celebrated and difficult theorem which currently, to the best of the author's knowledge, has no dynamical proof. Suppose that $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ is such that each y_i is algebraic and $1, y_1, \dots, y_n$ are linearly independent over $\mathbb{Q}$. It was established by Roth for $n = 1$ [75] and then generalized to arbitrary n by Schmidt [79], that $\mathbf{y}$ as above necessarily belongs to the complement of VWA. In other words, vectors with very special algebraic properties happen to follow the behavior of a generic vector in $\mathbb{R}^n$.

We would like to view the above example as a special case of a general class of problems. Namely, suppose we are given a Radon measure μ on $\mathbb{R}^n$. Let us say that μ is *extremal* [85] if μ-a.e. $\mathbf{y} \in \mathbb{R}^n$ is not very well approximable. Equivalently, define the *Diophantine exponent* $\omega(\mu)$ of μ to be the μ-essential supremum of the function $\omega(\cdot)$; in other words,

$$\omega(\mu) \stackrel{\text{def}}{=} \sup\left\{ v \mid \mu\big(\mathcal{W}(\psi_{1,v})\big) > 0 \right\}.$$

Clearly it only depends on the measure class of μ. If μ is naturally associated with a subset $\mathcal{M}$ of $\mathbb{R}^n$ supporting μ (for example, if $\mathcal{M}$ is a smooth submanifold of $\mathbb{R}^n$ and μ is the measure class of the Riemannian volume on $\mathcal{M}$, or, equivalently, the pushforward $\mathbf{f}_*\lambda$ of λ by a smooth map $\mathbf{f}$ parametrizing $\mathcal{M}$), one defines the Diophantine exponent $\omega(\mathcal{M})$ of $\mathcal{M}$ to be equal to that of μ, and says that $\mathcal{M}$ is extremal if $\mathbf{f}(\mathbf{x})$ is not very well approximable for λ-a.e. $\mathbf{x}$.

Then $\omega(\mu) \geq n$ for any μ, and $\omega(\lambda) = \omega(\mathbb{R}^n)$ is equal to n. The latter justifies the use of the word "extremal": μ is *extremal* if $\omega(\mu)$ is equal to n, i.e. attains the smallest possible value. The aforementioned results of Roth and Schmidt then can be interpreted as the extremality of atomic measures supported on algebraic vectors without rational dependence relations.

Historically, the first measure (other than λ) to be considered in the set-up described above was the pushforward of λ by the map

$$\mathbf{f}(x) = (x, x^2, \dots, x^n)\,. \tag{5.1}$$

The extremality of $\mathbf{f}_*\lambda$ for $\mathbf{f}$ as above was conjectured in 1932 by K. Mahler [67] and proved in 1964 by Sprindžuk [82, 83]. It was important for Mahler's study of transcendental numbers: this result, roughly speaking, says that almost all transcendental numbers are "not very algebraic". At about the same time Schmidt [77] proved the extremality of $\mathbf{f}_*\lambda$ when $\mathbf{f} : I \to \mathbb{R}^2$, $I \subset \mathbb{R}$, is C^3 and satisfies

$$\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1''(x) & f_2''(x) \end{vmatrix} \neq 0 \quad \text{for } \lambda\text{-a.e. } x \in I\,;$$

in other words, the curve parametrized by $\mathbf{f}$ has nonzero curvature at almost all points. Since then, a lot of attention has been devoted to showing that measures $\mathbf{f}_*\lambda$ are extremal for other smooth maps $\mathbf{f}$.

To describe a broader class of examples, recall the following definition. Let $\mathbf{x} \in \mathbb{R}^d$ and let $\mathbf{f} = (f_1, \ldots, f_n)$ be a C^k map from a neighborhood of $\mathbf{x}$ to $\mathbb{R}^n$. Say that $\mathbf{f}$ is *nondegenerate at* $\mathbf{x}$ if $\mathbb{R}^n$ is spanned by partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ up to some order. Say that $\mathbf{f}$ is *nondegenerate* if it is nondegenerate at λ-a.e. $\mathbf{x}$. It was conjectured by Sprindžuk [84] in 1980 that $\mathbf{f}_*\lambda$ for real analytic nondegenerate $\mathbf{f}$ are extremal. Many special cases were established since then (see [12] for a detailed exposition of the theory and many related results), but the general case stood open until the mid-1990s [56], when Sprindžuk's conjecture was proved using the dynamical approach (later Beresnevich [6] succeeded in establishing and extending this result without use of dynamics). The proof in [56] uses the correspondence outlined in the previous section plus a measure estimate for flows on the space of lattices which is described below.

In the subsequent work the method of [56] was adapted to a much broader class of measures. To define it we need to introduce some more notation and definitions. If $\mathbf{x} \in \mathbb{R}^d$ and $r > 0$, denote by $B(\mathbf{x}, r)$ the open ball of radius r centered at x. If $B = B(\mathbf{x}, r)$ and $c > 0$, cB will denote the ball $B(\mathbf{x}, cr)$. For $B \subset \mathbb{R}^d$ and a real-valued function f on B, let

$$\|f\|_B \stackrel{\text{def}}{=} \sup_{\mathbf{x}\in B} |f(\mathbf{x})|\,.$$

If ν is a measure on $\mathbb{R}^d$ such that $\nu(B) > 0$, define $\|f\|_{\nu,B} \stackrel{\text{def}}{=} \|f\|_{B\,\cap\,\operatorname{supp}\nu}$; this is the same as the $L^\infty(\nu)$-norm of $f|_B$ if f is continuous and B is open. If $D > 0$ and $U \subset \mathbb{R}^d$ is an open subset, let us say that ν is *D-Federer on U* if for any ball $B \subset U$ centered at $\operatorname{supp}\nu$ one has $\frac{\nu(3B)}{\nu(B)} < D$ whenever $3B \subset U$. This condition is often called "doubling" in the literature. See [54, 72] for examples and references. ν is called *Federer* if for ν-a.e. $\mathbf{x} \in \mathbb{R}^d$ there exist a neighborhood U of x and $D > 0$ such that ν is D-Federer on U.

Given $C, \alpha > 0$, open $U \subset \mathbb{R}^d$ and a measure ν on U, a function $f : U \to \mathbb{R}$ is called *(C, α)-good on U with respect to ν* if for any ball $B \subset U$ centered in $\operatorname{supp}\nu$ and any $\varepsilon > 0$ one has

$$\nu\big(\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon\}\big) \leq C \left(\frac{\varepsilon}{\|f\|_{\nu,B}}\right)^{\alpha} \nu(B)\,. \tag{5.2}$$

This condition was formally introduced in [56] for ν being Lebesgue measure, and in [54] for arbitrary ν. A basic example is given by polynomials, and the upshot of the above definition is the formalization of a property needed for the proof of several basic facts [68, 19, 21] about polynomial maps into the space of lattices.

In [54] a strengthening of this property was considered: f was called *absolutely (C,α)-good on U with respect to ν* if for B and ε as above one has

$$\nu\big(\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon\}\big) \le C\left(\frac{\varepsilon}{\|f\|_B}\right)^{\alpha} \nu(B)\,. \tag{5.3}$$

There is no difference between (5.2) and (5.3) when ν has full support, but it turns out to be useful for describing measures supported on proper (e.g. fractal) subsets of $\mathbb{R}^d$.

Now suppose that we are given a measure ν on $\mathbb{R}^d$, an open $U \subset \mathbb{R}^d$ with $\nu(U) > 0$ and a map $\mathbf{f} = (f_1,\dots,f_n) : \mathbb{R}^d \to \mathbb{R}^n$. Following [62], say that a pair $(\mathbf{f},\nu)$ is *(absolutely) good on U* if any linear combination of $1, f_1,\dots,f_n$ is (absolutely) (C,α)-good on U with respect to ν. If for ν-a.e. $\mathbf{x}$ there exists a neighborhood U of $\mathbf{x}$ and $C,\alpha > 0$ such that ν is (absolutely) (C,α)-good on U, we will say that the pair $(\mathbf{f},\nu)$ is *(absolutely) good*.

Another relevant notion is the nonplanarity of $(\mathbf{f},\nu)$. Namely, $(\mathbf{f},\nu)$ is said to be *nonplanar* if whenever B is a ball with $\nu(B) > 0$, the restrictions of $1, f_1,\dots,f_n$ to $B \cap \operatorname{supp}\nu$ are linearly independent over $\mathbb{R}$; in other words, $\mathbf{f}(B \cap \operatorname{supp}\nu)$ is not contained in any proper affine subspace of $\mathbb{R}^n$. Note that absolutely good implies both good and nonplanar, but the converse is in general not true.

Many examples of (absolutely) good and nonplanar pairs $(\mathbf{f},\nu)$ can be found in the literature. Already the case $n = d$ and $\mathbf{f} = \mathrm{Id}$ is very interesting. A measure μ on $\mathbb{R}^n$ is said to be *friendly* (resp., *absolutely friendly*) if and only if it is Federer and the pair (Id,μ) is good and nonplanar (resp., absolutely good). See [54, 88, 89] for many examples. An important class of measures is given by limit measures of irreducible system of self-similar or self-conformal contractions satisfying the Open Set Condition [44]; those are shown to be absolutely friendly in [54]. The prime example is the middle-third Cantor set on the real line. The term "friendly" was cooked up as a loose abbreviation for "Federer, nonplanar and decaying", and later proved to be particularly friendly in dealing with problems arising in metric number theory, see e.g. [36].

Also let us say that a pair $(\mathbf{f},\nu)$ is *nondegenerate* if $\mathbf{f}$ is nondegenerate at ν-a.e. $\mathbf{x}$. When ν is Lebesgue measure on $\mathbb{R}^d$, it is proved in [56, Proposition 3.4] that a nondegenerate $(\mathbf{f},\nu)$ is good and nonplanar. The same conclusion is derived in [54, Proposition 7.3] assuming that ν is absolutely friendly. Thus volume measures on smooth nondegenerate manifolds are friendly, but not absolutely friendly.

It turns out that all the aforementioned examples of measures can be proved to be extremal by a generalization of the argument from [56]. Specifically, let ν be a Federer measure on $\mathbb{R}^d$, U an open subset of $\mathbb{R}^d$, and $\mathbf{f} : U \to \mathbb{R}^n$ a continuous map such that the pair $(\mathbf{f},\nu)$ is good and nonplanar; then $\mathbf{f}_*\nu$ is extremal. This can be derived from the Borel-Cantelli Lemma, the correspondence described in the previous section, and the following measure estimate: if ν, U and $\mathbf{f}$ are as above, then for ν-a.e. $\mathbf{x}_0 \in U$ there exists a ball $B \subset U$ centered at $\mathbf{x}_0$ and $\tilde{C},\alpha > 0$ such that for any $t \in \mathbb{R}_+$ and any $\varepsilon > 0$,

$$\nu\Big(\big\{\mathbf{x} \in B : g_t L_{\mathbf{f}(\mathbf{x})}\mathbb{Z}^{n+1} \notin K_\varepsilon\big\}\Big) < \tilde{C}\varepsilon^{\alpha}\,. \tag{5.4}$$

Here g_t is as in (4.2) with $m = 1$ (assuming that the row vector viewpoint is adopted). This is a quantitative way of saying that for fixed t, the "flow" $\mathbf{x} \mapsto g_t L_{\mathbf{f}(\mathbf{x})}\mathbb{Z}^{n+1}$, $B \to \Omega_{n+1}$, cannot diverge, and in fact must spend a big (uniformly in t) proportion of time inside compact sets K_ε.

The inequality (5.4) is derived from a general "quantitative non-divergence" estimate, which can be thought of a substantial generalization of theorems of Margulis and Dani [68, 19, 21] on non-divergence of unipotent flows on homogeneous spaces. One of its most general versions [54] deals with a measure ν on $\mathbb{R}^d$, a continuous map $h : \widetilde{B} \to G$, where $\widetilde{B}$ is a ball in $\mathbb{R}^d$ centered at $\operatorname{supp}\nu$ and G is as in (4.3). To describe the assumptions on h, one needs to employ the combinatorial structure of lattices in $\mathbb{R}^k$, and it will be convenient to use the following notation: if V is a nonzero rational subspace of $\mathbb{R}^k$ and $g \in G$, define $\ell_V(g)$ to be the covolume of $g(V \cap \mathbb{Z}^k)$ in gV. Then, given positive constants C, D, α, there exists $C_1 = C_1(d, k, C, \alpha, D) > 0$ with the following property. Suppose ν is D-Federer on $\widetilde{B}$, $0 < \rho \le 1$, and h is such that for each rational $V \subset \mathbb{R}^k$

(i) $\ell_V \circ h$ is (C, α)-good on $\widetilde{B}$ with respect to ν,

and

(ii) $\|\ell_V \circ h\|_{\nu,B} \ge \rho$, where $B = 3^{-(k-1)}\widetilde{B}$.

Then

(iii) for any positive $\varepsilon \le \rho$, one has

$$\nu\big(\{\mathbf{x} \in B : h(\mathbf{x})\mathbb{Z}^k \notin K_\varepsilon\}\big) \le C_1(\varepsilon/\rho)^\alpha \nu(B)\,. \tag{5.5}$$

Taking $h(\mathbf{x}) = g_t L_{\mathbf{f}(\mathbf{x})}$ and unwinding the definitions of good and nonplanar pairs, one can show that (i) and (ii) can be verified for some balls B centered at ν-almost every point, and derive (5.4) from (5.5).

6. Further results

The approach to metric Diophantine approximation using quantitative non-divergence, that is, the implication (i) + (ii) $\Rightarrow$ (iii), is not omnipotent. In particular, it is difficult to use when more precise results are needed, such as for example computing/estimating the Hausdorff dimension of the set of $\psi_{1,v}$-approximable vectors on a manifold. See [9, 10] for such results. On the other hand, the dynamical approach can often treat much more general objects that its classical counterpart, and also can be perturbed in a lot of directions, producing many generalizations and modifications of the main theorems from the preceding section.

One of the most important of them is the so-called *multiplicative* version of the set-up of §5. Namely, define functions $\Pi(\mathbf{x}) \stackrel{\text{def}}{=} \prod_i |x_i|$ and $\Pi_+(\mathbf{x}) \stackrel{\text{def}}{=} \prod_i \max(|x_i|, 1)$ Then, given a function $\psi : \mathbb{N} \to \mathbb{R}_+$, one says that $Y \in M_{m,n}$ is *multiplicatively ψ-approximable* (notation: $Y \in \mathcal{W}^\times_{m,n}(\psi)$) if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\Pi(Y\mathbf{q} + \mathbf{p})^{1/m} \le \psi\big(\Pi_+(\mathbf{q})^{1/n}\big) \tag{6.1}$$

for some $\mathbf{p} \in \mathbb{Z}^m$. Since $\Pi(\mathbf{x}) \le \Pi_+(\mathbf{x}) \le \|\mathbf{x}\|^k$ for $\mathbf{x} \in \mathbb{R}^k$, any ψ-approximable Y is multiplicatively ψ-approximable; but the converse is in general not true, see e.g. [37]. However if one, as before, considers the family $\{\psi_{1,v}\}$, the critical parameter for which the drop from full measure to measure zero occurs is again n/m. That is, if one defines the *multiplicative Diophantine exponent* $\omega^\times(Y)$ of Y by $\omega^\times(Y) \stackrel{\text{def}}{=} \sup\{v : Y \in \mathcal{W}^\times_{m,n}(\psi_{1,v})\}$, then clearly $\omega^\times(Y) \ge \omega(Y)$ for all Y, and yet $\omega^\times(Y) = n/m$ for λ-a.e. $Y \in M_{m,n}$.

Now specialize to $\mathbb{R}^n$ (by the same duality principle as before, it does not matter whether to think in terms of row or column vectors, but we will adopt the row vector set-up), and define the *multiplicative exponent* $\omega^\times(\mu)$ of a measure μ on $\mathbb{R}^n$ by $\omega^\times(\mu) \stackrel{\text{def}}{=} \sup\left\{ v \mid \mu\big(\mathcal{W}^\times(\psi_{1,v})\big) > 0\right\}$; then $\omega^\times(\lambda) = n$. Following Sprindžuk [85], say that μ is *strongly extremal* if $\omega^\times(\mu) = n$. It turns out that all the results mentioned in the previous section have their multiplicative analogues; that is, the measures described there happen to be strongly extremal. This was conjectured by A. Baker [1] for the curve (5.1), and then by Sprindžuk in 1980 [85] for analytic nondegenerate manifolds. (We remark that only very few results in this set-up can be obtained by the standard methods, see e.g. [10].) The proof of this stronger statement is based on using the multi-parameter action of

$$g_{\mathbf{t}} = \operatorname{diag}(e^{t_1+\cdots+t_n}, e^{-t_1}, \dots, e^{-t_n})\,, \quad \text{where} \quad \mathbf{t} = (t_1, \dots, t_n)$$

instead of g_t considered in the previous section. One can show that the choice $h(\mathbf{x}) = g_{\mathbf{t}} L_{\mathbf{f}(\mathbf{x})}$ allows one to verify (i) and (ii) uniformly in $\mathbf{t} \in \mathbb{R}^n_+$, and the proof is finished by applying a multi-parameter version of the correspondence described in §4. Namely, one can show that $\mathbf{y} \in \mathrm{VWA}^\times_{1,n}$ if and only if the trajectory $\{g_{\mathbf{t}} L_{\mathbf{y}} \mathbb{Z}^k : \mathbf{t} \in \mathbb{R}^n_+\}$ grows linearly, that is, for some $\gamma > 0$ one has $\operatorname{dist}(g_{\mathbf{t}} L_Y \mathbb{Z}^{n+1}, \mathbb{Z}^{n+1}) > \gamma\|\mathbf{t}\|$ for an unbounded set of $\mathbf{t} \in \mathbb{R}^n_+$. A similar correspondence was recently used in [30] to prove that the set of exceptions to Littlewood's Conjecture, which, using the terminology introduced above, can be called *badly multiplicatively approximable* vectors:

$$\mathrm{BA}^\times_{n,1} \stackrel{\text{def}}{=} \mathbb{R}^n \smallsetminus \bigcup_{c>0} \mathcal{W}^\times_{n,1}(\psi_{c,1/n}) = \left\{ \mathbf{y} : \inf_{q \in \mathbb{Z}\smallsetminus\{0\},\, \mathbf{p} \in \mathbb{Z}^n} |q| \cdot \Pi(q\mathbf{y} - \mathbf{p}) > 0 \right\}, \tag{6.2}$$

has Hausdorff dimension zero. This was done using a measure rigidity result for the action of the group of diagonal matrices on the space of lattices. See [18] for an implicit description of this correspondence and [32, 66, 70] for more detail.

The dynamical approach also turned out to be fruitful in studying Diophantine properties of pairs $(\mathbf{f}, \nu)$ for which the nonplanarity condition fails. Note that obvious examples of non-extremal measures are provided by proper affine subspaces of $\mathbb{R}^n$ whose coefficients are rational or are well enough approximable by rational numbers. On the other hand, it is clear from a Fubini argument that almost all translates of any given subspace are extremal. In [51] the method of [56] was pushed further to produce criteria for the extremality, as well as the strong extremality, of arbitrary affine subspaces $\mathcal{L}$ of $\mathbb{R}^n$. Further, it was shown that if $\mathcal{L}$ is extremal (resp. strongly extremal), then so is any smooth submanifold of $\mathcal{L}$ which is nondegenerate in $\mathcal{L}$ at a.e. point. (The latter property is a straightforward generalization of the definition of nondegeneracy in $\mathbb{R}^n$: a map $\mathbf{f}$ is *nondegenerate in $\mathcal{L}$ at $\mathbf{x}$* if the linear part of $\mathcal{L}$ is spanned by partial derivatives of $\mathbf{f}$ at $\mathbf{x}$.) In other words, extremality and strong extremality pass from affine subspaces to their nondegenerate submanifolds.

A more precise analysis makes it possible to study Diophantine exponents of measures with supports contained in arbitrary proper affine subspaces of $\mathbb{R}^n$. Namely, in [53] it is shown how to compute $\omega(\mathcal{L})$ for any $\mathcal{L}$, and furthermore proved that if ν is a Federer measure on $\mathbb{R}^d$, U an open subset of $\mathbb{R}^d$, and $\mathbf{f} : U \to \mathbb{R}^n$ a continuous map such that the pair $(\mathbf{f}, \nu)$ is good and nonplanar in $\mathcal{L}$, then $\omega(\mathbf{f}_*\nu) = \omega(\mathcal{L})$. Here we say, generalizing the definition from §5, that $(\mathbf{f}, \nu)$ is *nonplanar in $\mathcal{L}$* if for any ball B with $\nu(B) > 0$, the $\mathbf{f}$-image of $B \cap \operatorname{supp}\nu$ is not contained in any proper affine subspace of $\mathcal{L}$. (It is easy to see that for a smooth map $\mathbf{f} : U \to \mathcal{L}$, $(\mathbf{f}, \lambda)$ is good and nonplanar in $\mathcal{L}$ whenever $\mathbf{f}$ is nondegenerate in $\mathcal{L}$ at a.e. point.) It is worthwhile to point out that these new applications require a strengthening of the measure estimate described at the end of §5: it was shown in [53] that (i) and (ii) would still imply (iii) if ρ in (ii) is replaced by $\rho^{\dim V}$.

Another application concerns badly approximable vectors. Using the dynamical description of the set $\mathrm{BA} \subset \mathbb{R}^n$ due to Dani [20], it turns out to be possible to find badly approximable vectors inside supports of certain measures on $\mathbb{R}^n$. Namely, if a subset K of $\mathbb{R}^n$ supports an absolutely friendly measure, then $\mathrm{BA} \cap K$ has Hausdorff dimension not less than the Hausdorff dimension of this measure. In particular, it proves that limit measures of irreducible system of self-similar/self-conformal contractions satisfying the Open Set Condition, such as e.g. the middle-third Cantor set on the real line, contain subsets of full Hausdorff dimension consisting of badly approximable vectors. This was established in [60] and later independently in [64] using a different approach. See also [36] for a stronger result.

The proof in [60] uses quantitative nondivergence estimates and an iterative procedure, which requires the measure in question to be absolutely friendly and not just friendly. A similar question for even the simplest not-absolutely friendly measures is completely open. For example, it is not known whether there exist uncountably many badly approximable pairs of the form (x, x^2). An analogous problem for atomic measures supported on algebraic numbers, that is, a "badly approximable" version of Roth's Theorem, is currently beyond reach as well – there are no known badly approximable (or, for that matter, well approximable) algebraic numbers of degree bigger than two.

It has been recently understood that the quantitative nondivergence method can be applied to the question of improvement to Dirichlet's Theorem (see the beginning of §3). Given a positive $\varepsilon < 1$, let us say that Dirichlet's Theorem *can be ε-improved* for $Y \in M_{m,n}$, writing $Y \in \mathrm{DI}_{m,n}(\varepsilon)$, if for every sufficiently large t the system

$$\|Y\mathbf{q} - \mathbf{p}\| < \varepsilon e^{-t/m} \quad \text{and} \quad \|\mathbf{q}\| < \varepsilon e^{t/n} \tag{6.3}$$

(that is, (3.1) with the right hand side terms multiplied by ε) has a nontrivial integer solution $(\mathbf{p}, \mathbf{q})$. It is a theorem of Davenport and Schmidt [24] that $\lambda\big(\mathrm{DI}_{m,n}(\varepsilon)\big) = 0$

for any $\varepsilon < 1$; in other words, Dirichlet's Theorem cannot be improved for Lebesgue-generic systems of linear forms. By a modification of the correspondence between dynamics and approximation, (6.3) is easily seen to be equivalent to $g_t L_Y \mathbb{Z}^k \in K_\varepsilon$, and since the complement to K_ε has nonempty interior for any $\varepsilon < 1$, the result of Davenport and Schmidt follows from the ergodicity of the g_t-action on Ω_k.

Similar questions with λ replaced by $\mathbf{f}_*\lambda$ for some specific smooth maps $\mathbf{f}$ were considered in [23, 2, 3, 15]. For example, [15, Theorem 7] provides an explicitly computable constant $\varepsilon_0 = \varepsilon_0(n)$ such that for $\mathbf{f}$ as in (5.1),

$$\mathbf{f}_*\lambda\big(\mathrm{DI}_{1,n}(\varepsilon)\big) = 0 \text{ for } \varepsilon < \varepsilon_0\,.$$

This had been previously done in [23] for $n = 2$ and in [2] for $n = 3$. In [62] this is extended to a much broader class of measures using estimates described in §5. In particular, almost every point of any nondegenerate smooth manifold is proved not to lie in $\mathrm{DI}(\varepsilon)$ for small enough ε depending only on the manifold. Earlier this was done in [61] for the set of *singular* vectors, defined as the intersection of $\mathrm{DI}(\varepsilon)$ over all positive ε; those correspond to divergent g_t-trajectories. As before, the advantage of the method is allowing a multiplicative generalization of the Dirichlet-improvement set-up; see [62] for more detail.

It is also worthwhile to mention that a generalization of the measure estimate discussed in §5 was used in [13] to estimate the measure of the set of points $\mathbf{x}$ in a ball $B \subset \mathbb{R}^d$ for which the system

$$\begin{cases} |\mathbf{f}(\mathbf{x})\cdot\mathbf{q} + p| < \varepsilon \\ |\mathbf{f}'(\mathbf{x})\cdot\mathbf{q}| < \delta \\ |q_i| < Q_i,\ i = 1,\dots,n, \end{cases}$$

where $\mathbf{f}$ is a smooth nondegenerate map $B \to \mathbb{R}^n$, has a nonzero integer solution. For that, $L_{\mathbf{f}(\mathbf{x})}$ as in (5.4) has to be replaced by the matrix

$$\begin{pmatrix} 1 & 0 & \mathbf{f}(\mathbf{x}) \\ 0 & 1 & \mathbf{f}'(\mathbf{x}) \\ 0 & 0 & I_n \end{pmatrix},$$

and therefore (i) and (ii) turn into more complicated conditions, which nevertheless can be checked when $\mathbf{f}$ is smooth and nondegenerate and ν is Lebesgue measure. This has resulted in proving the convergence case of Khintchine-Groshev Theorem for nondegenerate manifolds [13], in both standard and multiplicative versions. The aforementioned estimate was also used in [7] for the proof of the divergence case, and in [38, 40] for establishing the convergence case of Khintchine-Groshev theorem for affine hyperplanes and their nondegenerate submanifolds. This generalized results obtained by standard methods for the curve (5.1) by Bernik and Beresnevich [11, 6].

Finally, let us note that in many of the problems mentioned above, the ground field $\mathbb{R}$ can be replaced by $\mathbb{Q}_p$, and in fact several fields can be taken simultaneously, thus giving rise to the S-arithmetic setting where $S = \{p_1,\dots,p_s\}$ is a finite set of normalized valuations of $\mathbb{Q}$, which may or may not include the infinite valuation (cf.

[83, 90]). The space of lattices in $\mathbb{R}^{n+1}$ is replaced there by the space of lattices in $\mathbb{Q}_S^{n+1}$, where $\mathbb{Q}_S$ is the product of the fields $\mathbb{R}$ and $\mathbb{Q}_{p_1},\dots,\mathbb{Q}_{p_s}$. This is the subject of the paper [59], where S-arithmetic analogues of many results reviewed in §5 have been established. Similarly one can consider versions of the above theorems over local fields of positive characteristic [39]. See also [52] where Sprindžuk's solution [83] of the complex case of Mahler's Conjecture has been generalized (the latter involves studying small values of linear forms with coefficients in $\mathbb{C}$ at real integer points), and [31] which establishes a p-adic analogue of the result of [30] on the set of exceptions to Littlewood's Conjecture.

7. Future directions

Interactions between ergodic theory and number theory have been rapidly expanding during the last two decades, and the author has no doubts that new applications of dynamics to Diophantine approximation will emerge in the near future. Specializing to the topics discussed in the present paper, it is fair to say that the list of "further results" contained in the previous section is by no means complete, and many even further results are currently in preparation. This includes: extending proofs of extremality and strong extremality of certain measures to the set-up of systems of linear forms (namely, with $\min(m,n) > 1$; this was mentioned as work in progress in [56]); proving Khintchine-type theorems (both convergence and divergence parts) for p-adic and S-arithmetic nondegenerate manifolds, see [73] for results in this direction; extending [38, 40] to establish Khintchine-type theorems for submanifolds of arbitrary affine subspaces. The work of Druţu [28], who used ergodic theory on homogeneous spaces to compute the Hausdorff dimension of the intersection of $\mathcal{W}_{n,1}(\psi_{1,v})$, $v > 1$, with some rational quadratic hypersurfaces in $\mathbb{R}^n$ deserves a special mention; it is plausible that using this method one can treat more general situations. Several other interesting open directions are listed in [41, §9], in the final sections of papers [7, 54], in the book [16], and in surveys by Frantzikinakis-McCutcheon, Nitica and Ward in this volume.

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