BOUNDED TRAJECTORIES IN HOMOGENEOUS SPACES OF SEMISIMPLE LIE GROUPS

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1. Introduction

This is a report on a joint work with G.A. Margulis. The problem to be discussed is apparently motivated by number theory, more precisely, by problems in Diophantine approximation. Let us start by recalling that the set $\mathcal{B}$ of badly approximable real numbers has full Hausdorff dimension at any point, in other words, for any nonempty open subset W of $\mathbb{R}$, $\dim(W \cap \mathcal{B}) = 1$ (cf. [J,S]). On the other hand, there is a well known connection between Diophantine approximation and complex hyperbolic dynamics. Namely, consider $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, and let $g_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$. Then it is not hard to show (see [Da1, Da3]) that $\alpha \in \mathbb{R}$ is badly approximable iff the trajectory $\left\{ g_t \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \Gamma \;\middle|\; t \ge 0 \right\}$ is bounded in G/Γ. From this and the preceding statement one easily concludes that the set of points x in G/Γ with bounded trajectories $\{g_t x \mid t \ge 0\}$ has full Hausdorff dimension at any point of G/Γ (note that this set has measure 0 by ergodicity of $\{g_t\}$-action on G/Γ). With a little more work (cf. [Da3]) one can prove the same for two-sided orbits. In other words, with the notation $F = \{g_t \mid t \in \mathbb{R}\}$,

$$(*) \qquad \begin{aligned} &\text{for any nonempty open subset } W \text{ of } G/\Gamma \\ &\dim(\{x \in W \mid Fx \text{ is bounded}\}) = \dim(G/\Gamma)\,. \end{aligned}$$

One might ask whether a similar statement holds for more general G and Γ. In 1985, S.G. Dani proved (*) in the following two cases:

(cf. [Da1]) $G = SL_n(\mathbb{R})$, $\Gamma = SL_n(\mathbb{Z})$, and $g_t = \operatorname{diag}(e^{-t}, \dots, e^{-t}, e^{\lambda t}, \dots, e^{\lambda t})$, where λ is such that the determinant of g_t is 1;

(cf. [Da2]) G is a connected semisimple Lie group of $\mathbb{R}$-rank 1, Γ a lattice in G, and F is *not quasiunipotent* (that is, $\operatorname{Ad} g_1$ has an eigenvalue with modulus different from 1).

This suggested the following

Conjecture (A) [Ma]**.** *Let G be a Lie group, Γ a lattice in G, F a nonquasiunipotent one-parameter subgroup of G. Then (*) holds.*

Observe that the restriction on the one-parameter subgroup not to be quasiunipotent is essential. Indeed, if $\{g_t\}$ is unipotent and G semisimple, from M. Ratner's results on closures of unipotent orbits [R, see also DM, Theorem 3] it follows that the

set
$\{x \in G/\Gamma \mid Fx$ is not uniformly distributed$\}$ belongs to a countable union of proper submanifolds of G/Γ, and therefore its Hausdorff dimension is strictly less than the dimension of the homogeneous space.

Surprisingly, this observation can be used to produce a family of trivial counterexamples to the above conjecture. Indeed, take $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, and $g_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \times \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then G/Γ is a direct product of two copies of $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, and the projection of the set of points with bounded orbits onto the second component has small Hausdorff dimension, which makes (*) impossible.

We now state the theorem which implies Conjecture (A) under certain restrictions on G and Γ, and then give necessary and sufficient conditions for (*) to hold, based on the reduction to the case of this theorem.

2. The main result

Theorem 1. *Let G be a connected semisimple Lie group without compact factors, Γ an irreducible lattice in G, $F = \{g_t \mid t \in \mathbb{R}\}$ a one-parameter nonquasiunipotent subgroup of G, and $Z \subset G/\Gamma$ a closed F-invariant set of (Haar) measure 0. Then for any nonempty open subset W of G/Γ*

$$\dim\big(\{x \in W \mid Fx \text{ is bounded and } \overline{Fx} \cap Z = \varnothing\}\big) = \dim(G/\Gamma)\,.$$

We will sketch the main ideas of the proof; details are contained in [KM]. One can define subgroups H^-, H^0 and H of G such that the multiplication map $H^- \times H^0 \times H \to G$ is locally 1-1 and bi-Lipschitz, and the orbits Hx (resp. H^0x, H^-x) are expanding (resp. neutral, contracting) leaves relative to the action of g_t, $t > 0$, for any $x \in G/\Gamma$. In other words, H (resp. H^-) is a horospherical subgroup relative to g_{-1} (resp. g_1), which means that the conjugation map $\Phi_t : g \to g_t g g_{-t}$, $t > 0$, is expanding on H and contracting on H^-. This makes it possible to reduce Theorem 1 to the following statement:

Theorem 2. *For any $x \in G/\Gamma \setminus Z$ and for any neighborhood V of identity in H, the Hausdorff dimension of the set*

$$\mathbf{A} = \{h \in V \mid F^+hx \text{ is bounded and } \overline{F^+hx} \cap Z = \varnothing\}$$

(here F^+ stands for $\{g_t \mid t \geq 0\}$) is equal to the dimension of H.

Note that the original problem about subsets of a homogeneous space of a "very non-Abelian" Lie group is reduced to that dealing with subsets of an "almost Abelian" group (indeed, H is a connected simply connected nilpotent Lie group). This gives one a chance to consider subsets of H which look like Sierpinski gaskets in $\mathbb{R}^2$, and then imbed a sequence of such sets with increasing dimensions into the set $\mathbf{A}$.

Indeed, fix a large compact set K in G/Γ and a neighborhood V of identity in H of a special kind (a tesselation domain of H, see [KM, §3]). Our first purpose is to prove that, roughly speaking, for large enough T and for any $x \in K$, there exists $t(x) \in [T, 2T]$ such that the most part of the set $g_{t(x)}Vx = \Phi_{t(x)}(V)g_{t(x)}x$ lies in K, the quantitative meaning of "the most" being uniform in $x \in K$ (see

[KM, Proposition 2.5]). After that we divide H (up to a set of measure 0) into pieces which are right translations of V by elements γ of H. For any $x \in K$ we choose all the translations γ such that $V\gamma \subset \Phi_{t(x)}(V)$ and $V\gamma g_{t(x)}x \subset K$, and define a compact subset $\mathbf{A}_1(x)$ of $\overline{V}$ to be the union of $\Phi_{-t(x)}(\overline{V}\gamma)$ over all γ as above. By iteration of this procedure, a sequence of compact sets $\overline{V}x \supset \mathbf{A}_1(x) \supset \mathbf{A}_2(x) \supset \dots \mathbf{A}_j(x) \supset \dots$ ($\mathbf{A}_{j+1}(x)$ being the union of $\Phi_{-t(x)}\big(\mathbf{A}_j(\gamma g_{t(x)}x)\gamma\big)$ over all γ as above) is constructed. The *limit set* $\mathbf{A}_\infty = \bigcap_{i=1}^\infty \mathbf{A}_j(x)$ then consists of elements h such that the F^+-orbit of hx lies in a certain compact subset of G/Γ; thus $\mathbf{A}_\infty \subset \mathbf{A}$. The detailed description of this construction is given in [KM, §4]; a result of C. McMullen and M. Urbanski (cf. [U]) allows one to derive a lower estimate on the Hausdorff dimension of $\mathbf{A}_\infty$ from the information about the quantity of pieces $V\gamma g_{t(y)}y$ chosen at each stage for each $y \in K$.

To prove Proposition 2.5 mentioned above, we consider a thickening $U = V^-V^0V$ of V, where V^- and V^0 are small neighborhoods of identity in H^- and H^0 respectively. From *mixing properties* (cf. [Mo]) of the action of F on G/Γ it follows that for large enough t, the most part of the set g_tUx should lie in K. However, $g_tUx = \Phi_t(V^-V^0)\Phi_t(V)g_tx$ is close to $g_tVx = \Phi_t(V)g_tx$ only if the map Φ_t is *nonexpanding* on H^-H^0; this is the case if the elements g_t are semisimple.

In general, one can only say that $\Phi_t|_{H^-H^0}$ is *at most polynomially expanding*. But this is not too bad, provided one can apply the exponential decay estimates [KS, §3] on matrix coefficients of smooth functions. These estimates require a special condition (EM) (see [KM, §2.4]), valid, in particular, when there exists a simple factor of G of $\mathbb{R}$-rank greater than 1 such that the projection of the semisimple part of $\{g_t\}$ onto this factor is not relatively compact. The remaining case can be reduced to that of semisimple elements g_t by using nondivergence property [DM, Theorem 6.1] of orbits of unipotent flows.

3. Generalizations and open questions

One can use Theorem 1 to show that all the possible counterexamples to Conjecture (A) look more or less like the one described in the introduction.

Theorem 3 [KM, Theorem 5.2]**.** *Let G be a connected Lie group, Γ a lattice in G, F a one-parameter subgroup of G. Then (*) is equivalent to*

(Q) *for any connected normal subgroup N of of G with the quotient map $p : G \to G' \stackrel{\text{def}}{=} G/N$ such that G' is semisimple without compact factors and $p(\Gamma)$ is an irreducible lattice in G', at least one of the following three conditions is satisfied:*

(Q1) *$p(\Gamma)$ is cocompact in G';*

(Q2) *$Ad\,p(F)$ is relatively compact;*

(Q3) *$p(F)$ is not quasiunipotent.*

The proof of (*)⇒(Q) is a generalization of the argument we used when discussing $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$-counterexample in the introduction. The opposite direction is done by reduction to the case of Theorem 1 using certain properties of lattices in Lie groups.

Using the same argument, one can also prove that for an F-invariant subset Z of G/Γ, the set

$$\{x \in G/\Gamma \mid Fx \text{ is bounded and } \overline{Fx} \cap Z = \varnothing\}$$

has full Hausdorff dimension at any point of G/Γ whenever for any $p : G \to G'$ as above the following two conditions hold: (a) $p(F)$ is not quasiunipotent, and (b) the closure of $\bar{p}(Z)$ (here $\bar{p}$ is the induced map of homogeneous spaces $G/\Gamma \to G'/p(\Gamma)$) has Haar measure 0.

We conjecture that the above statement is still true if the cumbersome condition (b) is replaced by "the closure of Z has measure 0" as in Theorem 1.

Another class of open questions is obtained by relaxing the assumption of F-invariance of the set Z to be avoided. For example, the case $Z = \{x_1, \dots, x_n\}$ is the subject of Conjecture (B) from [Ma].

Finally let us mention a special case of the flows studied above: the geodesic flow on the unit tangent bundle SM of a Riemannian manifold M of constant negative curvature and finite volume. Since the ambient group in this case has $\mathbb{R}$-rank 1, the fact that the set of points in SM with bounded geodesics has full Hausdorff dimension follows from the result of Dani [Da2]. In that paper it was asked whether an analogous statement would be true for manifolds of variable negative curvature. A certain progress in this direction has been recently obtained by D. Dolgopiat [Do]. This (as well as the paper [U] where related problems were considered for Anosov flows on compact Riemannian manifolds) suggests that our results on flows on homogeneous spaces may have their nonhomogeneous analogues.

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