Introduction to Vertex Algebras

1. Circle Algebras

1. Basic Notions

Let V be a vector space. The space of Laurent series with coefficients in V is denoted $V((z))$, Taylor series $V[[z]]$, Fourier (biinfinite) series $V[[z,z^{-1}]]$, polynomials $V[z]$, and Laurent polynomials $V[z,z^{-1}]$.

The formal differentiation $\partial = \frac{\partial}{\partial z}$ operates on each one of these five spaces. For example, $\partial : V[[z,z^{-1}]] \to V[[z,z^{-1}]]$, $\sum a(n)z^{-n-1} \mapsto \sum(-n-1)a(n)z^{-n-1}$. We also have the formal integration Res_z that takes any formal power series (or polynomial) in z and returns the coefficient of z^{-1}. Since z^{-1} can never be in image of ∂, it follows that

$$Res_z \partial a(z) = 0$$

for any power series $a(z)$. This is the formal analogue of the integration by parts formula.

Definition 1.1. *A circle operator is a linear map $a(z) : V \to V((z))$. The vector space of circle operators is denoted $CO(V)$.*

The formal variable z is to be interpreted as a parameter on the circle S^1 or a disk (hence the term circle operator.)

Definition 1.1'. *A circle operator is a $\mathbf{C}[z,z^{-1}]$-linear map $a(z) : V[z,z^{-1}] \to V((z))$.*

Definition 1.1". *A circle operator is a biinfinite formal series $a(z) = \sum_{n\in\mathbf{Z}} a(n)z^{-n-1}$ with $a(n) \in End(V)$ such that for any $v \in V$, we have $a(n)v = 0$ for $n >> 0$.*

The three definitions above are clearly equivalent. The first one emphasize the similarity between $End(V)$ and $CO(V)$. The second one thinks of a circle operator as a densely defined operator on the space $V((z))$, thought of as a completion of $V[z,z^{-1}]$. The third one thinks of an operator valued Fourier series. For now the variable z is only a book keeping device. Later it will be interpreted as point in $\mathbf{C}$. When we eventually globalize to Riemann surfaces, z will be a local coordinate. The notion of $CO(V)$ in the setting of vertex operator theory is meant to replace the notion of $End(V)$ in linear algebra, although this view point was not clear at the early developmental stage of vertex operator theory. This transition from linear operators to vertex operators has only become clear in the work (1996) of Lian-Zuckerman and independently Li.

Clearly it does not makes sense in general to multiply two circle operators $a(z)$, $b(z)$ pointwise. However the space $End(V)$ can be viewed as a subspace of $CO(V)$, where a linear map A is regarded as the constant series $\sum A\delta_{n,-1}z^{-n-1}$. The space $End(V)$ is an associative algebra in a canonical way. Is there a natural way to extend the product in $End(V)$ to all of $CO(V)$? One such (nonassociative) extension is known as the Wick product. Define

$$:a(z)b(w):=\sum_{n<0}a(n)z^{-n-1}b(w)+b(w)\sum_{n\geq 0}a(n)z^{-n-1}. \tag{1}$$

Exercise 1.2. *Check that when $w=z$, $:a(z)b(z):$ makes sense. Also when restricted to $End(V)$, the Wick product coincides with the natural product in $End(V)$.*

Here we introduce a notation for iterated Wick products which we will use later. Let $a_1(z),...,a_n(z)$ be circle operators. Their Wick product is defined inductively as follows:

$$:a_1(z)\cdots\ a_n(z):=:a_1(z)(:a_2(z)\cdots a_n(z):):\tag{2}$$

We now introduce the important notion of the circle products. This is a family of products (of which the Wick product is one) which measure formally the singularity of the formal product $a(z)b(w)$ as z "approaches" w.

Convention: Let z, w be two formal variables and α be any complex number. The expressions $(z+w)^\alpha$, $(w+z)^\alpha$ will respectively mean the following formal binomial expansions:

$$\begin{aligned}(z+w)^\alpha =& z^\alpha(1+\alpha\frac{w}{z}+\frac{\alpha(\alpha-1)}{2!}\frac{w^2}{z^2}+\cdots)\\ (w+z)^\alpha =& w^\alpha(1+\alpha\frac{z}{w}+\frac{\alpha(\alpha-1)}{2!}\frac{z^2}{w^2}+\cdots),\end{aligned} \tag{3}$$

and similar definitions for $(\pm z \pm w)^\alpha$, $(\pm w \pm z)^\alpha$. When working over characteristic p, α will only be an integer. Thus $(1+x)^n$ for $n \geq 0$ is defined in an obvious way, while $(1+x)^{-n}$ is defined to be the formal power series inverse of $(1+x)^n$. Given a formal series $A(z)$ in z with whatever coefficients, $Res_z\ A(z)$ will be the coefficient of z^{-1}; and $A(z)_+ := \sum_{n\geq 0} A(n)z^{-n-1}$, $A(z)_- := A(z) - A(z)_+$. The coefficients $A(n)$ are called the Fourier modes of $A(z)$. If S is a set of formal series, then $S_\pm = \{A(z)_\pm | A(z) \in S\}$, and $Mode(S)$ is the $\mathbf{C}$-span of the Fourier modes of all $A(z) \in S$. We sometimes write A instead of $A(z)$ when the context is clear. $\partial A(z)$ means the formal derivative $\frac{d}{dz}A(z)$.

Definition 1.3. *For each integer n we define the n^{th} circle product on $CO(V)$:*

$$a(w) \circ_n b(w) = Res_z a(z)b(w)(z-w)^n - Res_z b(w)a(z)(-w+z)^n. \tag{4}$$

Exercise 1.4. *Check that the circle products are well-defined. (Hint: $Res_z a(z)(z-w)^n \in End(V)[[w]]$ and $Res_z a(z)(-w+z)^n \in End(V)[w, w^{-1}]$.)*

Exercise 1.5. *Check that $\circ_{-1}$ is the Wick product.*

Lemma 1.6. *In any characteristic, $(\partial a(z)) \circ_n b(z) = -na(z) \circ_{n-1} b(z)$. Thus for $n < 0$, $(-n-1)!a(z) \circ_n b(z) =: \partial^{-n-1}a(z)\ b(z) :$. Moreover, ∂ is a derivation of each circle product.*

Proof: Using integration by parts, we get

$$\begin{aligned}Res_z\partial_z a(z)b(w)(z-w)^n =& -nRes_z a(z)b(w)(z-w)^{n-1}\\ Res_z b(w)\partial_z a(z)(-w+z)^n =& -nRes_z b(w)a(z)(-w+z)^{n-1}.\end{aligned} \tag{5}$$

Subtracting the second from the first equation, we get the first assertion. By induction we get $(-n-1)!a(z)\circ_n b(z) = \partial^{-n-1}a(z)\circ_{-1} b(z)$.

By Definition 1.3, we find that

$$\partial(a\circ_n b) = a\circ_n \partial b - na\circ_{n-1} b.$$

Since $-na\circ_{n-1} b = \partial a\circ_n b$, this shows that ∂ satisfies the Leibniz rule with respect to $\circ_n$. $\square$

Lemma 1.7. *(Commutator Lemma) For $m,n\geq 0$, we have*

$$\begin{aligned}
[a(m),b(w)] &= \sum_{k\geq 0}\binom{m}{k} a(w)\circ_k b(w)w^{m-k}\\
a(w)\circ_n b(w) &= \sum_{k=0}^{n}\binom{n}{k}(-w)^{n-k}[a(k),b(w)].
\end{aligned} \tag{6}$$

Proof: By definition,

$$\begin{aligned}
a(w)\circ_n b(w) =& Res_z(z-w)^n[a(z),b(w)]\\
[a(m),b(w)] =& Res_z z^m[a(z),b(w)].
\end{aligned} \tag{7}$$

Now multiply the first equation in (7) by $\binom{m}{n} w^{m-n}$, sum over n, and use the identity $\sum_{n=0}^{m}\binom{m}{n} w^{m-n}(z-w)^n = z^m$, then we get the first desired result. Now multiply the second equation in (7) by $\binom{n}{m}(-w)^{n-m}$, sum over m, and use the identity $\sum_{m=0}^{n}\binom{n}{m}(-w)^{n-m}z^m = (z-w)^n$, then we get the second desired result. $\square$

Corollary 1.8. *Suppose that there exists $C^0(w), C^1(w), ...$ such that*

$$[a(m),b(w)] = \sum_{n\geq 0}\binom{m}{n} C^n(w)w^{m-n} \tag{8}$$

for all $m \geq 0$*. Then* $C^n = a \circ_n b$*.*

Lemma 1.9. *(Operator Product Expansion) For two circle operators* $a(z), b(z)$*, the following equality of formal power series in two variables holds:*

$$a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w)(z-w)^{-n-1} + :a(z)b(w):. \tag{9}$$

Proof: By definition, $a(z)b(w) - :a(z)b(w): := [a(z)_+, b(w)]$. Using the first equation in 1.7 to compute this RHS, we get the desired result. □

In this sense $:a(z)b(w):$ is the nonsingular part of the operator product expansion, while $a(w) \circ_n b(w)(z-w)^{-n-1}$ is the polar part of order $-n-1$.

Definition 1.10. *A subspace* $\mathcal{A}$ *of* $CO(V)$ *containing the identity operator and closed under all the circle products is called a circle algebra.*

Lemma 1.11. *The formal differentiation* ∂ *is a derivation of every circle product. A circle algebra is closed under* ∂*.*

Proof: The first part follows from formal integration by parts using Res_w. The second part follows from the formula $a(z) \circ_{-2} 1 = \partial a(z)$. □

We note that none of the products $\circ_n$ is associative in general. However it clearly makes sense to speak of the left, right or two sided ideals in a circle algebra and they are defined in an obvious way. Given a set S of circle operators, the circle algebra generated by S is the smallest circle algebra $\mathcal{A}$ containing S. An element of $\mathcal{A}$ is a finite sum of words whose letters come from S and the circle products. We also denote such a circle algebra by $\langle S \rangle$.

Exercise 1.12. *Show that if* $A(z) \in CO(V)$ *and* $B(z) \in CO(U)$, *then* $A(z) \otimes B(z)$ *is a well defined element of* $CO(V \otimes U)$. *Thus the tensor product of two circle algebras over* **C** *makes sense.*

Definition 1.13. *A morphism of circle algebras is a linear map* $f : \mathcal{A} \to \mathcal{B}$ *which sends 1 to 1, and respects the circle products, ie.* $f(a \circ_n b) = f(a) \circ_n f(b)$.

Suppose that $V = \oplus_{\lambda \in \mathbf{C}} V[\lambda]$ is a graded vector space. We say that a vector $v \in V[\lambda]$ is homogeneous of weight $wt\ v = \lambda$. A circle operator $a(z) = \sum_{n \in \mathbf{Z}} a(n) z^{-n-1}$ is said to have weight $\mu \in \mathbf{C}$ if $a(n)V[\lambda] \subset V[\lambda + \mu - n - 1]$. In this case, if we formally assign $wt\ z = -1$, then $a(z)$ is a sum of homogeneous terms of weight μ. Thus we write $wt\ a(z) = \mu$. We say that a circle algebra algebra $O \subset CO(V)$ is graded if O is spanned by weight homogeneous elements. We say that a morphism of graded circle algebra $f : \mathcal{A} \to \mathcal{B}$ is homogeneous of weight ν if $wt\ f(a) = wt\ a + \nu$ for all weight homogeneous element a.

Note that in the above definition the elements of $\mathcal{A}, \mathcal{B}$ do not have to be defined on the same vector space.

2. Finite dimensional circle algebras

Lemma 1.14. *If a circle algebra* O *is finite dimensional, then* $\forall a(z) \in O$, *one has* $a(z)_+ = 0$, *i.e.* $a(z)$ *is a Taylor series.*

Proof: For if $a(z)_+ = \sum_{n \geq 0} a(n) z^{-n-1} \neq 0$ then there is a smallest nonnegative n such that $a(n) \neq 0$. The leading *negative* power of z in $\partial^i a(z)_+$ will be a nonzero multiple of $a(n) z^{-n-i-1}$. This shows that $a(z), \partial a(z), \partial^2 a(z), ...$ are all linearly independent. $\square$

Corollary 1.15. *If a circle algebra* O *is finite dimensional, then* $\forall a(z), b(z) \in O$ *one has* $a(z) \circ_n b(z) = 0$ *for all* $n \geq 0$

Proof: This follows from the preceeding lemma $a(m) = 0$ for all $m \geq 0$. Now the desired result follows from the formula 1.7. $\square$

Lemma 1.16. *If a circle algebra O consists of only Taylor series, then the Wick product is the pointwise product, hence associative.*

Proof: This is immediate. □

Theorem 1.17. *Every finite dimensional circle algebra O is canonically an (unital) associative algebra equipped with a derivation ∂. Conversely, given a finite dimensional associative algebra A equipped with a derivation D, there is a circle algebra $O \subset CO(A)$ which is canonically isomorphic to (A, D).*

Proof: The first statement follows from the preceding lemma and the fact that ∂ is a derivation of the Wick product $\circ_{-1}$.

Let A be a finite dimensional associative algebra with derivation D. Put

$$O = \{Y(a, z) := \sum_{k \geq 0} \frac{1}{k!} l_{D^k a} z^k \in CO(A) | a \in A\} \subset CO(A).$$

Note that

$$Y(a, z)b = (e^{zD}a) \cdot b.$$

In particular $Y(a, z)1 = e^{zD}a$, which is zero iff $a = 0$. Thus $Y : A \to O$ is a linear isomorphism.

Since the $Y(a, z)$ are Taylor series, their positive circle products are all zero. A straightforward computation (applying Leibniz' rule) gives

$$\begin{aligned} &(i)\ \partial Y(a, z) = [D, Y(a, z)] = Y(Da, z) \\ &(ii)\ Y(a, z)Y(b, z) = Y(a \cdot b, z). \end{aligned} \tag{10}$$

Thus (ii) shows that the space $Y(A, z)$ of Taylor series is closed under the Wick product $\circ_{-1}$. Now (i) shows that the space $Y(A, z)$ is closed under ∂. But each negative circle product is expressible in terms of ∂ and $\circ_{-1}$. This shows that $Y(A, z)$ is closed under each negative circle product, hence it is a circle algebra. Note that (i)–(ii) also show that $Y : (A, D) \to (O, \partial)$ is an algebra isomorphism. □

2. Commutative Circle Algebras

Definition 2.1. *We say that the circle operators $a(z), b(z)$ circle commute if for some nonnegative integer N, $(z-w)^N[a(z), b(w)] = 0$. A commutative circle algebra O is one in which any two elements circle commute.*

Warning. You cannot divide the circle commutativity condition by $(z-w)^N$ and conclude that $[a(z), b(w)] = 0$. The reason is that the vector space of power series in two variables (with whatever coefficients) is NOT a module over the ring of power series in general; $(z-w)^{-N}$ is a power series. On the other hand the same vector space is certainly a module over the polynomial ring $\mathbf{C}[z, w]$. Thus it is legitimate to multiply $[a(z), b(w)]$ by $(z-w)^N \in \mathbf{C}[z, w]$.

Exercise 2.2. *Show that the power series $\delta(z, w) = (z-w)^{-1} + (w-z)^{-1}$ is nonzero, but it is annihilated by $z - w \in \mathbf{C}[z, w]$. Note that the two terms in $\delta(z, w)$ are two geometric series that don't cancel, even though they would have cancel if they were analytic functions! It can be shown that in some sense all power series annihilated by positive powers of $z-w$ can be expressed in terms of $\delta(z, w)$. See Lemma 2.14 below.*

Loosely speaking, an operator that circle commutes with itself is in some sense a operator-valued distribution. In fact, under rather general condition, the matrix coefficients of such an operator and their products will be an nice function with well-understood singularities. The circle commutativity condition can be interpreted as saying that $a(z), b(w)$ commute except along the diagonal $z = w$.

Lemma 2.3. *A commutative circle algebra is local, i.e.* $a \circ_n b = 0$ *for* $n >> 0$.

Proof: For $n \geq 0$, $a \circ_n b = Res_z(z-w)^n[a(z), b(w)]$. If $(z-w)^N[a(z), b(w)] = 0$, then $a \circ_n b = 0$ for $n \geq N$. □

Example 2.4. *Commutative algebras.* Let A be any commutative associative algebra. Using the left multiplication, we can embed A into $End(A)$. If we now regard each element of A as a constant circle operator, then A becomes a commutative circle algebra. All circle products, except $\circ_{-1}$, are zero. The nonzero product is nothing but the original product on A.

If a, b, c are linear operators on V such that c commutes with both a, b, then of course c commutes with $a \cdot b$. Something similar but slightly weaker holds for circle operators.

Lemma 2.5. *(Circle Commutativity Lemma) If* $a(z), b(z), c(z)$ *are pairwise circle commuting, then for all* n $a(z) \circ_n b(z)$ *circle commutes with* $c(z)$.

Proof: For a positive integer N, $(z-w)^{2N}$ is a binomial sum of terms $(z-x)^i(x-w)^{2N-i}$, $i = 1, .., 2N$. So $(z-w)^{N+2N}(a(z) \circ_n b(z))c(w)$ is a binomial sum of terms

$$Res_x \left((z-w)^N(z-x)^i(x-w)^{2N-i}(a(x)b(z)(x-z)^n - b(z)a(x)(-z+x)^n)c(w)\right). \quad (1)$$

We want to show that for large enough N, and for $0 \leq i \leq 2N$, term by term we have

$$\begin{aligned}&(z-w)^N(z-x)^i(x-w)^{2N-i}(a(x)b(z)(x-z)^n - b(z)a(x)(-z+x)^n)c(w)\\ &= (z-w)^N(z-x)^i(x-w)^{2N-i}c(w)(a(x)b(z)(x-z)^n - b(z)a(x)(-z+x)^n).\end{aligned} \quad (2)$$

Consider two cases: $i \geq N$ and $i < N$. Because $a(z), b(z)$ circle commute, we have $(z-x)^k(a(x)b(z)(x-z)^n - b(z)a(x)(-z+x)^n) = 0$ for all large enough k. So for large enough N, (2) holds for $i \geq N$. Similarly for $i < N$, we have

$$\begin{aligned}&(z-w)^N(x-w)^{2N-i}(a(x)b(z)(x-z)^n - b(z)a(x)(-z+x)^n)c(w)\\ &= (z-w)^N(x-w)^{2N-i}c(w)(a(x)b(z)(x-z)^n - b(z)a(x)(-z+x)^n)\end{aligned} \quad (3)$$

because $c(z)$ circle commutes with $a(z)$ and with $b(z)$. □

Lemma 2.6. *If S is a set of circle commuting operators, then $\langle S\rangle$ a commutative circle algebra.*

Proof: The algebra $\langle S\rangle$ consists of linear sums of words whose letters come from S, 1, and the circle products. By induction and applying the Circle Commutativity Lemma, every $a(z)\in S$ circle commutes with every element in $\langle S\rangle$. Similarly any two elements in $\langle S\rangle$ circle commute. □

We shall now use this lemma to construct some examples of nontrivial commutative circle algebras.

Example 2.7. *The Heisenberg system.* Let $\tilde{g}=\mathbf{C}[t,t^{-1}]$ regarded as an abelian Lie algebra. Define a one dimensional central extension $\hat{g}:=\tilde{g}\oplus\mathbf{C}c$ by the bracket formula:

$$[A(t),B(t)]=Res_t A'(t)B(t)c\in\mathbf{C}c. \tag{4}$$

Let $V:=\mathbf{C}[\alpha(-1),\alpha(-2),...]$, and let $\hat{g}$ act on V as follows: $c\in\hat{g}$ acts by id; for $n<0$, t^n acts by left multiplication by $\alpha(n)$; for $n\geq 0$, t^n acts by $\alpha(n):=n\frac{\partial}{\partial\alpha(-n)}$.

Exercise 2.8. *Check that the above defines a $\hat{g}$-module structure on V.*

Exercise 2.9. *Let $\alpha(z):=\sum\alpha(n)z^{-n-1}\in CO(V)$. By direct computations show that $[\alpha(z)_+,\alpha(w)]=(z-w)^{-2}$, $[\alpha(z)_-,\alpha(w)]=-(w-z)^{-2}$, and hence $(z-w)^2[\alpha(z),\alpha(w)]=0$. Conclude that the circle algebra generated by $\alpha(z)$ is commutative.*

Later we shall prove that as a vector space $\langle\alpha(z)\rangle$ is canonically isomorphic to V.

If V is a super vector space, ie. a $\mathbf{Z}/2\mathbf{Z}$-graded vector space, then all the above basic notions can be generalized to the case *circle superalgebras* with the following minor modifications. We can speak of an even circle operator: $a(z):V_i\to V_i$, $i\in\mathbf{Z}/2\mathbf{Z}$; and an odd circle operator: $a(z):V_i\to V_{i+1}$. In this case, $CO(V)$ is defined to be the space of linear sums of homogeneous circle operators. Thus each circle operator has an odd and an even part. The definition of the n^{th} circle product of $a(z),b(z)$ has an extra sign $(-1)^{|a||b|}$ in front of the second term. Here $|a|:=1$ if $a(z)$ is odd, 0 if even. In the definition of circle commutativity, the commutator is replaced by graded commutator. More generally, any definition involving a permutation of symbols will be modified by a sign corresponding to the parities of the symbols involved.

Example 2.10. *Spin-$\frac{1}{2}$ Clifford system.* This will be the super analogue of the Heisenberg system above. Let c and ψ_s, $s \in \mathbf{Z} + \frac{1}{2}$, be the basis of a Lie superalgebra g with c even, the ψ_s odd; the bracket is given by $[\psi_r, \psi_s] = \delta_{r+s,0}c$, $[c, \psi_s] = 0$. Let V be the polynomial superalgebra generated by the odd symbols $\psi(-\frac{1}{2}), \psi(-\frac{3}{2}), \ldots$ (thus they all square to zero and are pairwise anticommuting). Let g act on V as follows: c acts by id; for $s < 0$, ψ_s acts by left multiplication by $\psi(s)$; for $s > 0$, ψ_s acts by by $\frac{\partial}{\partial \psi(-s)}$.

Exercise 2.11. *Check that the above defines a $\hat{g}$-module structure on V.*

Exercise 2.12. *Let $\psi(z) := \sum \psi_{n+\frac{1}{2}} z^{-n-1} \in CO(V)$. By direct computations show that $[\psi(z)_+, \psi(w)] = (z-w)^{-1}$, $[\psi(z)_-, \psi(w)] = (w-z)^{-1}$, and hence $(z-w)[\psi(z), \psi(w)] = 0$. Conclude that the super circle algebra generated by $\psi(z)$ is (super) commutative.*

1. Wick's Calculus

Lemma 2.13. *For any linear operator on V, the commutator $[A, -]$ is a derivation of each of the circle products. Thus $a(z) \circ_0 b(z) = [a(0), b(z)]$, gives a derivation $a(z)\circ_0$ of all circle products.*

Lemma 2.14. *(Commutator Lemma) Let $N \geq 0$. The following are equivalent:*

(a) $(z-w)^N [a(z), b(w)] = 0$.

(b)

$$
\begin{aligned}
[a(z)_+, b(w)] &= \sum_{n=0}^{N-1} a(w) \circ_n b(w) (z-w)^{-n-1} \quad \textit{and} \\
[a(z)_-, b(w)] &= -\sum_{n=0}^{N-1} a(w) \circ_n b(w) (-w+z)^{-n-1}.
\end{aligned}
\tag{5}
$$

(c)

$$\begin{aligned}[a(z), b(w)] =& \sum_{n=0}^{N-1} a(w) \circ_n b(w) \frac{(-1)^n}{n!} \partial_z^n \delta(z, w) \quad \textit{where} \\ \delta(z, w) =& (z-w)^{-1} - (-w+z)^{-1}.\end{aligned} \tag{6}$$

(d) For all m (cf. lemma 1.7),

$$[a(m), b(w)] = \sum_{n=0}^{N-1} \binom{m}{n} a(w) \circ_n b(w) w^{m-n}. \tag{7}$$

(e) For all m, p,

$$[a(m), b(p)] = \sum_{n=0}^{N-1} \binom{m}{n} (a \circ_n b)(m-n+p). \tag{8}$$

Proof: (a) $\Longrightarrow$ (b): We saw that (a) implies $a(z) \circ_n b(w) = 0$ for all $n \geq N$. Thus the first equation in (b) follows from lemma 1.9. Also we have

$$\begin{aligned}(-w+z)^N [a(z)_-, b(w)] =& (z-w)^N [a(z)_-, b(w)] \quad \textit{(we can do so because } N \geq 0) \\ =& -(z-w)^N [a(z)_+, b(w)] \quad \textit{by (a)} \\ =& -\sum_{n=0}^{N-1} a(w) \circ_n b(w) (z-w)^{N-n-1} \\ =& -\sum_{n=0}^{N-1} a(w) \circ_n b(w) (-w+z)^{N-n-1} \quad \textit{(again } N \geq n+1)\end{aligned} \tag{9}$$

Now observe that $[a(z)_-, b(w)] \in M[[z]]$ where $M = (End\ V)[[w, w^{-1}]]$ and that $(-w+z)^{-N} \in R[[z]]$ where $R = \mathbf{C}[w, w^{-1}]$. Since M is an R-module, $M[[z]]$ is an $R[[z]]$-module. Thus it makes sense to multiply the above both sides by $(-w+z)^{-N}$. Moreover, each summand of (9) lies in $M[[z]]$. This gives our second equation in (b).

(b) $\Longrightarrow$ (a): Obviously $(End\ V)[[z, w, z^{-1}, w^{-1}]]$ is a $\mathbf{C}[z, w]$-module by left multiplication. Thus multiplying $[a(z)_\pm, b(w)]$ by $(z-w)^N$ makes sense. Then it follows from the two equations in (b) that $(z-w)^N [a(z), b(w)] = 0$.

(b) $\Longrightarrow$ (c): This is obtained by adding the two equations in (b).

(c) $\Longrightarrow$ (b): Break up both sides of (c) into two sums – one involving only nonnegative powers of z, the other only negative powers of z. This gives the two equations in (b).

(c) $\Longleftrightarrow$ (d): Multiply both sides of (c) by z^m, take Res_z, and apply integration by parts. Then use the fact that $\delta(z,w)z^{m-n} = \delta(z,w)w^{m-n}$, and that $Res_z\delta(z,w) = 1$, we get (d). Reversing the process, we get (c) from (d).

(d) $\Longleftrightarrow$ (e): Multiply both sides of (d) by w^p and take Res_w. Then we get (e). To reverse, multiply (e) by w^{-p-1} and sum over p. $\Box$

Remark 2.15. *It is illegal to multiply, in the last step above, by $(z-w)^{-N} \in z^{-N}\mathbf{C}[[\frac{w}{z}]]$ because $CO(V)[[z]]$ is not a $\mathbf{C}[[\frac{w}{z}]]$-module! More concretely, here is an example. We have $(z-w)\cdot(-w+z)^{-1} = 1$. So*

$$(z-w)^{-1}\cdot[(z-w)\cdot(-w+z)^{-1}] = (z-w)^{-1}. \tag{10}$$

But we have

$$[(z-w)^{-1}\cdot(z-w)]\cdot(-w+z)^{-1} = (-w+z)^{-1}. \tag{11}$$

The two LHS above are not equal as formal power series. This shows the nonassociative nature of series multiplication. The reason: $(z-w)^{-1}\cdot(-w+z)^{-1}$ makes no sense as a power series. Also note that when the integers $m < 0, n \geq 0$, we define $\binom{m}{n} := (-1)^n\binom{n-m-1}{n}$.

Corollary 2.16. *Let $Mode(O) \subset End(V)$ be the linear span of the Fourier modes of all the elements of the commutative circle algebra O. Then $Mode(O)$ is a closed under commutator, hence is a Lie algebra.*

Theorem 2.17. *If a circle algebra O is commutative, so is its homomorphic image. In particular if a,b circle commute, so do their homomorphic images.*

This proof is involves verifying a series of circle product identities. We omit it here.

1.1. Matrix coefficients

We now define the notion of matrix coefficients for (products) of circle operators. To do this, we shall need an additional structure on V. We assume that V is $\mathbf{Z}$-graded: $V = \oplus_r V[r]$. A formal variable z is assigned $wt\ z = -1$, and a circle operator $a(z)$ will be a finite sum of homogeneous circle operators. Let $V^\vee$ denote the restricted dual of V. When a circle algebra $\mathcal{A}$ comes equipped with such a grading, we shall call it a *graded circle algebra*. In this case *the n^{th} circle product will be homogeneous of weight* $-n-1$.

We denote by $V^\vee = \oplus_r Hom(V[r], \mathbf{C})$.

Exercise 2.18. *Show that for $\nu \in V^\vee$, $v \in V$, and a, b homogeneous circle operators on V, $\langle \nu, : a(z)b(w) : v\rangle$ is a Laurent polynomial in z, w.*

Lemma 2.19. *(Rational matrix coefficient) Suppose a, b circle commute. Then $\forall \nu \in V^\vee, \forall v \in V$, the series $\langle \nu, a(z)b(w)v\rangle$ converges to a rational function $f(z,w)$ in the domain $|z| > |w|$. Moreover for $|w| > |z - w|$,*

$$f(z,w) = \sum_{n \in \mathbf{Z}} \langle \nu, a(w) \circ_n b(w)x\rangle (z-w)^{-n-1}. \tag{12}$$

Proof: : For simplicity we shall drop ν, v in the following notations. The Laurent polynomial $\langle : a(z)b(w) :\rangle$ in the above region is just $\sum_{i \geq 0} \frac{1}{i!} \langle : (\partial^i a(w))b(w) :\rangle (z-w)^i$. Now apply Lemma 1.6. $\square$

Corollary 2.20. *(cf. Definition 1.3) Let C be a small circle around a point w. Then we have*

$$\langle a(w) \circ_n b(w)\rangle = \oint_C f(z,w)(z-w)^n dz \tag{13}$$

.

Set $F_1 = \mathbf{C}[z_1, z_1^{-1}]$, and for $k \geq 2$, let F_k be the ring of rational functions on $\mathbf{C}^k$ with possible poles along $z_i = z_j$, $i \neq j$, and $z_i = 0$. Let R_k be its image under the linear map $F_k \to \mathbf{C}[[z_1,..,z_k,z_1^{-1},..,z_k^{-1}]]$ determined by $(z_i - z_j)^p \mapsto z_i^p \sum_{p\geq 0} \begin{pmatrix} p \\ n \end{pmatrix} (-\frac{z_j}{z_i})^n$, for $i > j$. Thus this map sends a function to its power series expansion in the domain $|z_1| > \cdots > |z_k|$. A basic result in *Several Complex Variables* says that the map $F_k \to R_k$ is an isomorphism, hence an element of R_k determines a unique rational function in F_k.

Throughout this subsection, let $a_1(z),..,a_k(z)$ be any homogeneous circle operators, not necessarily commuting.

Lemma 2.21. *If $a_i(n) = 0$ for $n >> 0$, then for any ν, v, $\langle \nu, a_1(z_1)\cdots a_k(z_k)v\rangle$ is a Laurent polynomial in $z_1,..,z_k$.*

Proof: Without loss of generality, we may assume that ν, v are homogeneous vectors. Then there is a constant N such that $\langle \nu, a_1(n_1)\cdots a_k(n_k)v\rangle \neq 0$ implies that $n_1 + \cdots + n_k = N$. But since the $a_i(n) = 0$ for $n >> 0$, there are only finitely many choices of $n_1,..,n_k$ for which $\langle \nu, a_1(n_1)\cdots a_k(n_k)v\rangle \neq 0$. $\square$

Corollary 2.22. *For any ν, v, $\langle \nu, a_1(z_1)_- \cdots a_{k-1}(z_{k-1})_- a_k(z_k)v\rangle \in R_k$.*

Proof: We can break this up into two terms using $a_k(z_k) = a_k(z_k)_- + a_k(z_k)_+$. The first term will be in R_k by the preceeding lemma. So is the second term because $a_k(z_k)_+ v$ is a Laurent polynomial. $\square$

Lemma 2.23. *(Matrix Coefficient Rationality) Let $a_1(z),..,a_k(z)$ be any homogeneous commuting circle operators. For any ν, v, we have for any $j = 0,1,..,k-1$, $\langle \nu, a_1(z_1)_- \cdots a_{j-1}(z_{j-1})_- a_j(z_j)\cdots a_k(z_k)v\rangle \in R_k$.*

Proof: The case $k = 1$ is easy and will be left as an exercise. We shall assume that $k \geq 2$. It is clear that when $j = k-1$, the claim follows from the preceeding corollary. Suppose the claim holds for $j = k-1, k-2,..,p+1 > 1$. We want to show that it holds for $j = p$, ie. $\langle \nu, a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- a_p(z_p)\cdots a_k(z_k)v\rangle \in R_k$. We shall first expand $a_p(z_p)a_{p+1}(z_{p+1})$ using lemma 1.9. By circle commutativity (which implies that $a_p(z_{p+1}) \circ_n a_{p+1}(z_{p+1}) = 0$ for $n >> 0$.), we have

$$\begin{aligned} &\langle \nu, a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- a_p(z_p)\cdots a_k(z_k)v\rangle \\ &\equiv \langle \nu, a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- : a_p(z_p)a_{p+1}(z_{p+1}) : \cdots a_k(z_k)v\rangle \ mod\ R_k. \end{aligned} \tag{14}$$

We can express the RHS as a sum of two terms using

$$: a_p(z_p)a_{p+1}(z_{p+1}) := a_p(z_p)_- a_{p+1}(z_{p+1}) + a_{p+1}(z_{p+1})a_p(z_p)_+. \tag{15}$$

The first term will be in R_k by the inductive hypothesis of the case $j = p+1$. So we want to show that the second term

$$\langle \nu, a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- a_{p+1}(z_{p+1})a_p(z_p)_+ \cdots a_k(z_k)v\rangle \in R_k. \tag{16}$$

We can write (dropping ν, v in the notation)

$$\begin{aligned} &\langle a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- a_{p+1}(z_{p+1})a_p(z_p)_+ \cdots a_k(z_k)\rangle \\ &= \langle a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- a_{p+1}(z_{p+1}) \cdots a_k(z_k)a_p(z_p)_+\rangle \\ &+ \sum_{i=p+2}^{k} \langle a_1(z_1)_- \cdots a_{p-1}(z_{p-1})_- a_{p+1}(z_{p+1}) \cdots [a_p(z_p)_+, a_i(z_i)] \cdots a_k(z_k)\rangle \end{aligned} \tag{17}$$

On the RHS, the first term is a Laurent polynomial in z_p with coefficients having just $k-1$ circle operators inserted. Thus by inductive hypothesis on k, this term is in R_k. Applying lemma 1.9 again, we get

$$[a_p(z_p)_+, a_i(z_i)] = \sum_{n\geq 0} a_p(z_i) \circ_n a_i(z_i)(z_p - z_i)^{-n-1} \tag{18}$$

which is a finite sum by circle commutativity. Using this we can replace the second term above by a *finite* sum of matrix coefficients multiplied by some power $(z_p - z_i)^{-n-1}$, and those matrix coefficients will now only have $k-1$ insertion of operators. This shows that the second term too is in R_k. □

Lemma 2.24. *(Permutation Invariance) Let σ be a permutation of k letters. Let $a_1(z), .., a_k(z)$ be any homogeneous commuting circle operators. For any ν, v, the following matrix coefficients converge to the same rational function (in different domains):*

$$\langle \nu, a_1(z_1) \cdots a_k(z_k)v\rangle, \quad \langle \nu, a_{\sigma(1)}(z_{\sigma(1)}) \cdots a_{\sigma(k)}(z_{\sigma(k)})v\rangle. \tag{19}$$

Proof: Following the preceding lemma, we name the rational functions these matrix coefficients converge to f, f^σ. Put $g := \prod_{i<j}(z_i - z_j)$. Since $f \in F_k$, we can find a large enough N so that

$$g^N f = g^N \langle \nu, a_1(z_1) \cdots a_k(z_k) v \rangle$$

is a Laurent polynomial. On the other hand, by circle commutativity, we can choose a large enough even N so that the RHS is invariant under a permutation of the subscript indices. In other words, the RHS is equal to $g^N f^\sigma$. Thus we have an identity

$$g^N f = g^N f^\sigma$$

in F_k. But F_k is a subring of the field of rational functions in $\mathbf{C}^k$. Thus we can divide both sides by g^N. $\square$

Example 2.25. *n-point functions.* The module $V = \mathbf{C}[\alpha(-1), \alpha(-2), ...]$ has a canonical nondegenerate bilinear (or Hermitian) form. It is determined uniquely by the condition that $\langle 1, 1 \rangle = 1$ and that $\alpha(-n)^t = n \frac{\partial}{\partial \alpha(-n)}$. This allows us to identify $V^\vee$ with V. We shall compute the matrix coefficients: $\langle 1, \alpha(z_1) \cdots \alpha(z_n) 1 \rangle$. For simplicity we shall drop 1 from our notation below. We claim that the matrix coefficient is zero if n is odd, and is

$$\begin{aligned} &\langle \alpha(z_1) \cdots \alpha(z_n) \rangle \\ &= \frac{1}{2^{\frac{1}{2}n} (\frac{1}{2}n)!} \sum_{\sigma \in S_n} (z_{\sigma(1)} - z_{\sigma(2)})^{-2} \cdots (z_{\sigma(n-1)} - z_{\sigma(n)})^{-2} \end{aligned} \tag{20}$$

if n is even.

First we can write

$$\begin{aligned} &\langle \alpha(z_1) \cdots \alpha(z_n) \rangle \\ &= \langle \alpha(z_1)_- \alpha(z_2) \cdots \alpha(z_n) \rangle + \langle \alpha(z_2) \cdots \alpha(z_n) \alpha(z_1)_+ \rangle \\ &+ \sum_{i=2}^{n} \langle \alpha(z_2) \cdots [\alpha(z_1)_+, \alpha(z_i)] \cdots \alpha(z_n) \rangle \end{aligned} \tag{21}$$

By construction, we have $\alpha(n)^t \cdot 1 = 0$ for $n < 0$. So on the RHS, the first term is zero. Similar $\alpha(n) \cdot 1 = 0$ for $n \geq 0$ implies that the second term is also zero. In the third term, we substitute $[\alpha(z_1)_+,\alpha(z_i)]=(z_1-z_i)^{-2}$. The case of $n=1,2$ is now clear. By induction see immediately that the matrix coefficient is zero when n is odd. Thus assume n is even. By induction again, we get

$$\begin{aligned}
&\langle\alpha(z_1)\cdots\alpha(z_n)\rangle\\
&=\sum_{i=2}^{n}\langle\alpha(z_2)\cdots\hat{\alpha}(z_i)\cdots\alpha(z_n)\rangle(z_1-z_i)^{-2}\\
&=f_{n-2}\sum_{odd\ i>1}\sum_{\sigma\in S_{n-2}}(z_{\sigma(2)}-z_{\sigma(3)})^{-2}\cdots(z_{\sigma(i-1)}-z_{\sigma(i+1)})^{-2}\cdots(z_{\sigma(n-1)}-z_{\sigma(n)})^{-2}(z_1-z_i)^{-2}\\
&+f_{n-2}\sum_{even\ i>1}\sum_{\sigma\in S_{n-2}}(z_{\sigma(2)}-z_{\sigma(3)})^{-2}\cdots(z_{\sigma(i-2)}-z_{\sigma(i-1)})^{-2}\cdots(z_{\sigma(n-1)}-z_{\sigma(n)})^{-2}(z_1-z_i)^{-2},
\end{aligned}\tag{22}$$

where $f_n=\frac{1}{2^{\frac{1}{2}n}(\frac{1}{2}n)!}$. In the last two sums, we regard S_{n-2} as the subgroup of S_n which fixes $1,i$. If we now regard S_{n-1} as the subgroup which fixes just 1, then those two sums above combine to

$$f_{n-2}\sum_{\sigma\in S_{n-1}}(z_1-z_{\sigma(2)})^{-2}(z_{\sigma(3)}-z_{\sigma(4)})^{-2}\cdots(z_{\sigma(n-1)}-z_{\sigma(n)})^{-2}\tag{23}$$

Now if we permute 1 as well in the summand by S_n, we would over count by a factor of n. Thus replacing S_{n-1} by S_n, 1 by $\sigma(1)$, and multiplying the sum by $1/n$, we get the desired formula.

Exercise 2.26. *Compute* $\langle:\alpha(z)\alpha(z):\alpha(w)\rangle$ $\langle:\alpha(z)\alpha(z)::\alpha(w)\alpha(w):\rangle$.

Exercise 2.27. *Carry out the whole parallel construction for the Clifford system mimiking the case of the Heisenberg system: the basis lemma for* $\langle\psi(z)\rangle$, *n-point functions etc.*

2. Circle Algebras on a Based Space

We shall develop a mild but very useful specialization of the theory we have discuss thus far. It is based on the notion of a based space.

Definition 2.28. *A based space* $(V, \mathbf{1}, D)$ *is a vector space* V *equipped with a distinguished nonzero vector* $\mathbf{1}$ *and a operator* D *with* $D\mathbf{1} = 0$. *When no confusion arises, we denote a based space simply by the underlying space* V.

Definition 2.29. $CO(V, \mathbf{1}, D)$ *is space of circle operators* $A(z)$ *satisfying* $[D, A(z)] = \partial A(z)$. *Such an operator will be called a based circle operator. When it is understood that* V *is a based space which comes equipped with the data* $\mathbf{1}, D$, *we will sometimes omit* $\mathbf{1}, D$ *in the above notation.*

Lemma 2.30. *If* A, B *are based circle operators, so are* $A \circ_n B$ *for all* n. *Thus the circle algebra generated by based circle operators is based.*

Proof: The first assertion follows immediately from the fact that both $[D, -]$ and ∂ are derivations of the circle products. The second assertion follows from the first and the fact that 1 is a based circle operator. □

Definition 2.31. *Given* $(V, \mathbf{1}, D)$, *a based circle algebra will mean a circle algebra in* $CO(V, \mathbf{1}, D)$.

Lemma 2.32. *Let* $A(z)$ *be a based circle operator. Then* $A(z)_+\mathbf{1} = 0$. *Moreover if* $A(-1)\mathbf{1} = 0$ *then* $A(z)\mathbf{1} = 0$.

Proof: We have $D \cdot A(z)\mathbf{1} = \partial A(z)\mathbf{1}$, implying that

$$D \cdot A(n)\mathbf{1} = -nA(n-1)\mathbf{1}. \tag{24}$$

By induction, this implies our second assertion. Now we know that $A(n)\mathbf{1} = 0$ for $n >> 0$. By induction again, this and (24) imply our first assertion. □

Definition 2.33. *The map* $\chi : CO(V, \mathbf{1}, D) \to V$ *is defined by* $A(z) \mapsto A(-1)\mathbf{1}$. *We call it the creation map. If* O *is any based circle algebra, we denotes its image* $\chi(O)$. *We write*

$V^\chi := \chi(CO(V,\mathbf{1},D))$. *Note that this is a subspace of* V *canonically attached to the data* $\mathbf{1}, D$.

Lemma 2.34. *For any based circle algebra* O, $(\chi(O),\mathbf{1},D)$ *is a based space.*

Proof: It's obvious that $\mathbf{1}\in\chi(O)$. So we need to check that D stabilizes $\chi(O)$. But this follows from $D\cdot A(z)\mathbf{1} = \partial A(z)\mathbf{1}$. □

Lemma 2.35. *Let* O *be any based circle algebra, and* $A(z)\in O$. *Then* $A(z)$ *stabilizes the subspace* $\chi(O)\subset V$, *i.e.* $A(z):\chi(O)\rightarrow\chi(O)((z))$. *In particular, we have a canonical restriction map*

$$O\rightarrow CO(\chi(O),\mathbf{1},D),\quad A(z)\mapsto A(z)|_{\chi(O)}. \tag{25}$$

Proof: Let $A(z),B(z)\in O$. We want to show that $A(z)B(-1)\mathbf{1}\in\chi(O)((z))$. It is enough to show that

$$A(n)B(-1)\mathbf{1} = (A\circ_n B)(-1)\mathbf{1} \tag{26}$$

because O is closed under $\circ_n$ and hence the RHS is in $\chi(O)$. Now $Res_z A(z)B(w)(z-w)^n\mathbf{1} = Res_z A(z)(z-w)^n B(w)_-\mathbf{1} = A(n)B(-1)\mathbf{1}+o(w)$. Similarly $Res_z B(w)A(z)\mathbf{1}(-w+z)^n = 0$. Eqn. (26) now follows. □

Corollary 2.36. *Let* $A(z),B(z)$ *be any based circle operators. Then* $A(z)\circ_n B(z)\mathbf{1} = 0$ *for* $n >> 0$.

Proof: Since $A(z)$ is a circle operator, we have $A(n)B(-1)\mathbf{1} = 0$, $n >> 0$. By (26), it follows that $(A\circ_n B)(-1)\mathbf{1} = 0$, $n >> 0$. This means that $A(z)\circ_n B(z)\mathbf{1} = 0$, $n >> 0$, by lemma 2.32. □

Note that the canonical restriction map $O\rightarrow CO(\chi(O),\mathbf{1},D)$ above is not injective in general.

Theorem 2.37. *If $O \subset CO(V, \mathbf{1}, D)$ is a commutative circle algebra, then the map $O|_{\chi(O)} \to \chi(O)$, $A(z) \mapsto A(-1)\mathbf{1}$ is a linear isomorphism. In particular if $\chi(O) = V$, then χ itself is a linear isomorphism $O \to V$.*

Proof: Call the map χ'. By definition χ' is surjective. Suppose $A(-1)\mathbf{1} = 0$, then $A(z)\mathbf{1} = 0$ by lemma 2.32. We claim that $A(z)B(-1)\mathbf{1} = 0$ for all $B(z) \in O$, ie. $A(z)|_{\chi(O)} = 0$. Now we have some $N \geq 0$ with $(z-w)^N A(z)B(w)\mathbf{1} = (z-w)^N B(w)A(z)\mathbf{1} = 0$. By lemma 2.32again, $B(w)\mathbf{1} = B(w)_-\mathbf{1}$. So we can take the limit $w \to 0$ and get $z^N A(z)B(-1)\mathbf{1} = 0$. $\square$.

Corollary 2.38. *Let $(V, \mathbf{1}, D)$ is a based space. Suppose $Y : V \to CO(V, \mathbf{1}, D)$, $a \mapsto a(z)$, is a linear map such that for all $a, b \in V$,*

(i) $a(-1)\mathbf{1} = a$

(ii) for all $m, n \in \mathbf{Z}$,

$$[a(m), b(n)] = \sum_{p \geq 0} \binom{m}{p} (a(p)b)(m+n-p).$$

Then $Im\ Y$ is a commutative circle algebra.

Proof: It suffices to show that for $a, b \in V$, the based circle operators $a(z), b(z)$ circle commute. For then we let O to be the circle algebra generated by the set $Im\ Y$. Then $O \subset CO(V, \mathbf{1}, D)$ is a based commutative circle algebra. By assumption (i), $\chi \circ Y = id_V$ so that $\chi : O \to V$ is surjective. By the preceding theorem, $\chi : O \to V$ is a isomorphism. This shows that $Im\ Y \subset O$ must be all of O.

We now prove that $a(z), b(z)$ circle commute. By our commutator characterization of circle commutativity before, it suffices to show that

$$(*) \qquad (a(p)b)(w) = (a \circ_p b)(w)$$

for all $p \geq 0$. For then formula (ii), which holds for *all* m, n, implies that $a(z), b(z)$ circle commutes. Now we prove (*). From (ii), we get

$$[a(m), b(w)] = \sum_{p \geq 0} \binom{m}{p} (a(p)b)(w) w^{m-p}.$$

On the other hand, for $m \geq 0$, we also know that

$$[a(m), b(w)] = \sum_{p \geq 0} \binom{m}{p} (a \circ_p b)(w) w^{m-p}.$$

Comparing the two RHS above for $m = 0$, we see that $(a(0)b)(w) = a \circ_0 b$. For $m = 1$, and using the $m = 0$ result, we see that $(a(1)b)(w) = a \circ_1 b$. Continuing this way, we see that (*) holds for all $p \geq 0$. $\square$

Condition (i) is equivalent to that $\chi \circ Y = id_V$, i.e. Y is a section of the creation map. This condition can be weakened by replacing it with $\chi(Im\ Y) = V$. The conclusion is false in general without this condition. For example, Y can be the zero map, in which case $Im\ Y$ does not even contain 1.

Definition 2.39. *A vertex algebra is a based space $(V, \mathbf{1}, D)$ equipped with a map Y satisfying axioms (i) and (ii).*

Thus the preceding corollary says that every vertex algebra gives rise to a commutative circle algebra canonically. Later, we will show that the converse is also true that every commutative circle algebra arises this way. Note that axiom (ii) is equivalent to that $Y(a, z), Y(b, z)$ circle commute for any $a, b \in V$.

Corollary 2.40. *Under the same assumption as preceding corollary, we have $Y(a(-1)b, z) = a \circ_{-1} b$, $Y(\mathbf{1}, z) = 1$, $Y(Da, z) = \partial Y(a, z)$. In particular these identities hold for a vertex algebra.*

Proof: Now both $Y(a(-1)b, z)$ and $a \circ_{-1} b$ are elements of $Im\ Y = O$ by the preceding theorem. Their images under χ are both $a(-1)b$. Since χ is an isomorphism, it follows that the circle operators are the same. Likewise both $Y(\mathbf{1}, z)$ and 1 are in $Im\ Y = O$. And they have the same image under χ.

Finally, $\chi Y(Da, z) = Da = Da(-1)\mathbf{1}$ by (i). On the other hand, $[D, a(z)] = Da(z)\mathbf{1} = \partial a(z)\mathbf{1}$ implies that $Da(-1)\mathbf{1} = a(-2)\mathbf{1} = \chi \partial a(z)$. This shows that $\chi Y(Da, z) = Da = \chi \partial a(z) = \chi \partial Y(a, z)$. By injectivity of χ, we have $Y(Da, z) = \partial Y(a, z)$. $\square$

2.1. An application

Proposition 2.41. *The circle algebra* $\langle \alpha(z) \rangle$ *on* $V = \mathbf{C}[t(-1), t(-2), ..]$ *has a basis consisting of the monomials:*

$$: \partial^{n_1} \alpha(z) \cdots \partial^{n_k} \alpha(z) :, \tag{27}$$

where $n_1 \geq \cdots \geq n_k \geq 0$*.*

Proof: Let $D : V \to V$ be the linear operator defined by $[D, t(n)] = -nt(n-1)$ and $D \cdot 1 = 0$. Then it is trivial to check that $(V, 1, D)$ is a based space and that α is a based circle operator. Thus $\langle \alpha \rangle$ is a based commutative circle algebra. Thus we can consider the creation map $\chi : \langle \alpha \rangle$, $A \mapsto A(-1)1$. The image of (27) form a basis of V. By theorem 2.37 χ is a linear isomorphism. But since the span of the monomials (27) already maps *onto* V, this span must be all of $\langle \alpha \rangle$. $\square$.

3. Representation theory

Throughout this chapter, g will be a Lie algebra over the complex numbers. An associative algebra A is assumed unital and defined over the complex numbers. It is always assumed that when A is regarded as a Lie algebra, the bracket is the commutator in A: $[x, y] = xy - yx$.

1. The Enveloping Algebra

Because the bracket $[,]$ of a Lie algebra g is a nonassociative product, it is convenient to have an associative algebra that encodes the very same information of g. The enveloping algebra Ug of g is the smallest such algebra.

Definition 3.1. *An enveloping algebra A of g is an associative algebra equipped with a Lie algebra homomorphism $i : g \to A$, and has the following universal property. If B is any associative algebra, then any Lie algebra homomorphism $\rho : g \to B$ factors through i uniquely, i.e. there exists a unique algebra homomorphism $\rho' : A \to B$ such that $\rho' \circ i = \rho$.*

As usual, once exists, an enveloping algebra of g is unique up to algebra isomorphisms. We now construct the enveloping algebra. Let $T^0 g = \mathbf{C}$, $T^1 g = g$, $T^2 g = g \otimes g$, and so on. Let Tg be the tensor algebra on g, i.e.

$$Tg := \oplus_{n \geq 0} T^n g$$

equipped with the (associative) product

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m.$$

Let I be the two-sided ideal (the relation ideal of g) in Tg generated by the elements

$$x \otimes y - y \otimes x - [x, y], \qquad for\ x, y \in g.$$

Define

$$Ug := Tg/I.$$

Since Tg is an associative algebra generated by $x \in g$, it follows that Ug is an associative algebra generated by $x + I$, $x \in g$. We denote the induced product simply by XY for $X, Y \in Ug$. The map

$$g = T^1 g \to Ug, \quad x \mapsto x + I$$

is Lie algebra homomorphism because

$$[x, y] \mapsto [x, y] + I \equiv x \otimes y - y \otimes x + I = (xy - yx) + I = (x + I)(y + I) - (y + I)(x + I),$$

and $x \mapsto x + I$ and $y \mapsto y + I$.

Exercise 3.2. *Show that Ug has the requisite universal property. Hence Ug is the enveloping algebra of g. (Hint: You will have to use the universal property of the tensor algebra Tg.)*

Let M be a Ug-module, say $\rho : Ug \to End\ M$. Then composing this with the canonical Lie homomorphism $i : g \to Ug$, we get a Lie homomorphism $i \circ \rho : g \to End\ M$. This makes M a g-module. Conversely, if M is a g-module equipped with Lie homomorphism $\rho' : g \to End\ M$, then by the universal property of Ug, ρ' factors through Ug via a unique algebra homomorphism $Ug \to End\ M$. This makes M canonically a Ug-module. This shows that module theories for the Lie algebra g and for its enveloping algebra Ug are essentially the same. Any question on g-modules can be readily translated to a question on Ug-modules, and vice versa. From now on we will make no distinction between a g-module and a Ug-module.

1.1. The linear structure of Ug

Put $T_m := T^0g \oplus \cdots \oplus T^mg$. Then $T_mT_n \subset T_{m+n}$, and they form an increasing filtration of the vector space Tg, i.e. $T_0 \subset T_1 \subset \cdots Tg$ and $\cup_m T_m = Tg$. Put

$$U_m := Im(T_m \to Ug) \subset Ug.$$

Then $U_mU_n \subset U_{m+n}$ and $\mathbf{C} = U_0 \subset U_1 \subset \cdots Ug$ is a filtration of the vector space Ug. Put

$$G := \oplus_{m\geq 0} G^m, \quad G^m := U_m/U_{m-1}, \quad U_{-1} := 0.$$

Define $G^m \times G^n \to G^{n+m}$, $a + U_{m-1}, b + U_{n-1} \mapsto ab + U_{m+n-1}$. Using $U_mU_n \subset U_{n+m}$, it is straightforward to check that this defines a graded associative algebra structure on G. Consider the composition maps $\phi_m : T^mg \to U_m \to G^m = U_m/U_{m-1}$. Combining them we get a (graded) linear map $\phi : Tg \to G$.

Lemma 3.3. *ϕ is an algebra homomorphism. Moreover, it factors through the relations $x \otimes y - y \otimes x$, hence descends to $\phi : Sg \to G$.*

Proof: Clearly $\phi(1) = 1$. Let $X \in T^mg$, $Y \in T^ng$. Then $\phi(XY) = XY + U_{m+n-1} = (X + U_{m-1})(Y + U_{n-1}) = \phi(X)\phi(Y)$. Thus ϕ is an algebra homomorphism.

For $x, y \in g$, we have $\phi(x \otimes y) = xy + U_1 = (yx + [x, y]) + U_1 = yx + U_1 = \phi(y \otimes x)$. $\square$

Theorem 3.4. *(Poincare-Birkhoff-Witt) $\phi : Sg \to G$ is an algebra isomorphism.*

You can find a proof of this in Humphrey, “Introduction to Lie Algebras and Representation Theory”.

Corollary 3.5. *Let W be a linear subspace of T^mg. If $\sigma : T^mg \to S^mg$ sends W isomorphically onto S^mg, then $\pi : T^mg \to U_m$ sends W isomorphically onto a complement* *of* U_{m-1} *in* U_m*, hence we have a direct sum* $U_m = \pi(W) + U_{m-1}$*.*

Proof: Consider the diagram of canonical maps:

$$\begin{array}{ccccc} & & U_m & & \\ & \nearrow & & \searrow & \\ T^m & & & & G^m \\ & \searrow & & \nearrow & \\ & & S^m & & \end{array}$$

By the preceding lemma, this is a commutative diagram. By PBW, $S^m \to G^m$ is an isomorphism. So the bottom arrows send $W \subset T^m$ isomorphically onto G^m. Now reverting back to the top arrows, we see that $\pi : W \to U_m$ must be injective. Since $ker(U_m \twoheadrightarrow G^m) = U_{m-1}$, we have a direct sum $U_m = \pi(W) + U_{m-1}$. □

Corollary 3.6. *(PBW II) Let* $\{X_\alpha\}_{\alpha \in S}$ *be an ordered basis of* g*. Then* Ug *has a basis consisting of the unit* $1 \equiv 1 + I$*, and the monomials*

$$X_{\alpha_1} \cdots X_{\alpha_m}, \quad \alpha_1 \leq \cdots \leq \alpha_m. \tag{1}$$

Proof: Let $W \subset T^m$ be the linear subspace spanned of $X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_m}$, $\alpha_1 \leq \cdots \leq \alpha_m$. Clearly, $T^m \to S^m$ sends W isomorphically onto S^m. So we have $\pi : W \hookrightarrow U_m$ and a direct sum $U_m = \pi(W) + U_{m-1}$, by the preceding corollary. The direct sum property means that the monomials (1) span Ug. The injectivity property means that the monomials are linearly independent. □

Corollary 3.7. *The canonical map* $i : g \to Ug$ *is an inclusion.*

Proof: This follows immediately from PBW II. □

Corollary 3.8. *If* $h \subset g$ *is a Lie subalgebra, then the induced algebra homomorphism* $Uh \to Ug$ *is an inclusion.*

Proof: Choose a basis of h and a basis of g containing the first basis. By PBW II, Ug has a PBW basis which contains a PBW basis of Uh. □

Corollary 3.9. *Suppose that h, b are two Lie subalgebras of g such that $g = h + b$ is a direct sum of vector spaces. Then $Ug \cong Uh \otimes_{\mathbf{C}} Ub$, as (Uh, Ub)-bimodules.*

Proof: We regard $Uh, Ub \subset Ug$, and define a linear map by $Uh \otimes Ub \to Ug$, $a \otimes b \mapsto a \cdot b$. It is straightforward to check that this is a bimodule homomorphism. By PBW II, it maps a basis to a basis, hence it is an isomorphism. □

Corollary 3.10. *Let W be any left b-module. The left g-module $Ug \otimes_{Ub} W$ is isomorphic to $Uh \otimes_{\mathbf{C}} W$ as Uh-modules.*

Proof: This follows from the preceding corollary. □

Corollary 3.11. *The ring Ug is a domain, i.e. if $a, b \in Ug$ and $ab = 0$ then $a = 0$ or $b = 0$.*

Proof: Suppose that $a, b \neq 0$. Let m, n be the smallest integers such that $a \in U_m$, $b \in U_n$. Then $a + U_{m-1} \in G^m$, $b + U_{n-1} \in G^n$ are nonzero elements. Their product in G is $ab + U_{m+n-1} \in G^{m+n}$ and is nonzero because $G \cong Sg$ and Sg is a domain. It follows that $ab \neq 0$ in U_{n+m}. □

Lemma 3.12. *Ug has a nontrivial two-sided ideal generated by g, called the augmentation ideal.*

Proof: Let $J = \oplus_{n \geq 1} T^n g$ which is clearly a nontrivial 2-sided ideal generated by g in Tg. It contains the relation ideal I, hence the image of J under the natural map $Tg \to Ug$ is a 2-sided ideal in Ug. Since $\mathbf{C} \cong Tg/J$, this ideal is nontrivial. □

Exercise 3.13. *Show that $Ug \otimes_{Ub} W$ has the following universal property: let M be any g-module, and $W \to M$ be a b-homomorphism. Then there exists a unique g-homomorphism $Ug \otimes_{Ub} W \to M$ extending $W \to M$.*

Remark 3.14. *All of the above has an obvious generalization to the case of Lie superalgebra. For example as a vector space Ug is isomorphic to the polynomial superalgebra* *generated by a basis of the even part g_0, and a basis of the odd part g_1.*

Example 3.15. *The Heisenberg system revisited.* Recall that we have constructed a module V over the Heisenberg Lie algebra $\hat{g} = \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c$. If we put $\hat{g}' := t^{-1}\mathbf{C}[t^{-1}]$, $b = \mathbf{C}[t] \oplus \mathbf{C}c$, and W be the one dimensional b-module in which t^n, $n \geq 0$, acts by zero and c acts by id, then $Ug \otimes_{Ub} W \cong$ is isomorphic to V. They is a one-parameter family of similar modules: by letting the central element t^0 acts by a scalar p rather than 0 on W. The resulting g-module is denoted $F(p)$.

Exercise 3.16. *Show that each $F(p)$ is irreducible as a g-module.*

Example 3.17. *The Virasoro algebra.* Let $Witt$ be the Lie algebra of meromorphic vector fields on $\mathbf{P}^1$ with poles only at $0, \infty$. It has a basis given by $t^{n+1}\frac{d}{dt}$, $n \in \mathbf{Z}$. The Lie bracket is given by the commutator:

$$[t^{n+1}\frac{d}{dt}, t^{m+1}\frac{d}{dt}] = -(n-m)t^{n+m+1}\frac{d}{dt}. \tag{2}$$

A theorem of Gel'fand-Fuchs states that $Witt$ has a one dimensional central extension, which is denoted as Vir,

$$0 \to \mathbf{C}\kappa \to Vir \to Witt \to 0 \tag{3}$$

with the following universal property: if

$$0 \to K \to W' \to Witt \to 0 \tag{4}$$

is any one dimensional central extension, then there is a unique map $Vir \to W'$ sending κ into K, and the map is compatible with $Witt$. To see this, recall that a central extension is determined by a K-valued 2-cocyle $c : Witt^{\otimes 2} \to K$ (2-cocyle means that the resulting W' must satisfy the Lie-Jacobi identity). By direct computation, we can show that c must be a multiple of (up to adding 2-coboundary $Witt \wedge Witt \to K$),

$$c(t^{n+1}\frac{d}{dt}, t^{m+1}\frac{d}{dt}) = \frac{1}{12}(n^3 - n)\kappa\ \delta_{n+m,0}. \tag{5}$$

To write down the bracket relation for Vir, let κ and L_n, $n \in \mathbf{Z}$, together be a basis of Vir such that $Vir \to Witt$ sends $L_n \mapsto -t^{n+1}\frac{d}{dt}$. Then the bracket on Vir is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\kappa}{12}(n+1)n(n-1)\delta_{n+m,0}. \tag{6}$$

It is weight graded in an obvious way $wt\ L_n := n$, $wt\ \kappa := 0$. We let b be the span of weight $n \geq -1$ elements, and h be the span of weight $n < -1$ elements. It is clear that they are both Lie subalgebras of Vir, and that we have a vector space direct sum $Vir = h + b$. Note also that we have a Lie algebra direct sum $b = b' + \mathbf{C}\kappa$, where b' is the span of the L_n, $n \geq -1$. Let $\mathbf{C}_c$ be the one dimensional b-module in which the b' acts by zero, and κ acts by the scalar c. Fix a nonzero element $v_c \in \mathbf{C}_c$. Then $UVir \otimes_{Ub} \mathbf{C}_c$ is canonically isomorphic to $UVir_-$ as vector spaces. The module has a PBW basis given by

$$L_{-2}^{n_2} L_{-3}^{n_3} \cdots \otimes v_c \tag{7}$$

where all but finitely many n_i are zeros. We shall denote this Vir-module by $N(c)$. *The number c is called the central charge.* Note the $\mathbf{Z}$-grading of Vir induces a grading on $N(c)$. From the PBW basis above, it is obvious that there is no negative weight elements in $N(c)$. Since $[L_0, L_n] = -nL_n$ and $L_0 v_c = 0$, it follows that L_0 acts diagonalizably and that the weight of a homogeneous vector $v \in N(c)$ coincides with the eigenvalue of $-L_0$ on v.

Example 3.18. *The Virasoro circle algebra.* It follows from the above construction that for any $v \in N(c)$, $L_n v = 0$ for $n >> 0$. Thus we have a weight 2 operator $L(z) := \sum L_{n-1} z^{-n-1}$ on the vector space $N(c)$.

Lemma 3.19. *$L(z) \in CO(N(c))$ circle commutes with itself. More generally for any Vir-module V, if $L(z) \in CO(V)$ then $L(z)$ circle commutes with itself.*

Proof: A direct computation gives

$$\begin{aligned} [L(z)_+, L(w)] &= \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1} \\ [L(z)_-, L(w)] &= -\frac{c}{2}(-w+z)^{-4} - 2L(w)(-w+z)^{-2} - \partial L(w)(-w+z)^{-1}. \end{aligned} \tag{8}$$

By lemma 2.14, we have

$$(z-w)^4[L(z), L(w)] = 0. \quad \square \tag{9}$$

In particular $\langle L(z)\rangle$ is a commutative circle algebra in $CO(N(c))$.

Example 3.20. *n-point functions for $\langle L\rangle$.* We shall proceed in a way analogous to the case $\langle\alpha\rangle$. The module $N(c)$ has a canonical (up to scalar) possibly degenerate bilinear form determined by the conditions that $\langle v_c, v_c\rangle = 1$ and that $L_n^T = L_{-n}$. This is an example of what is called a Shapovalov bilinear form. Using this we can consider an n-point function to be the rational function to which the following series converges: We shall derive a system of recursive differential equations for them. Again to simplify notations, we shall drop v_c. As before, we first substitute $L(z_1) = L(z_1)_- + L(z_1)_+$. By construction we have $L(z_1)_-^T v_c = 0$ and $L(z_1)_+ v_c = 0$. Thus by leaving $L(z_1)_-$ in the first slot, this term contributes zero. By moving $L(z_1)_+$ the last slot, it too contributes zero, except for its commutators with the rest of the inserted $L(z_i)$. Thus applying the commutator formula,

$$\begin{aligned}
&\langle L(z_1)\cdots L(z_n)\rangle \\
&= \sum_{i=2}^{n} \langle L(z_2)\cdots [L(z_1)_+, L(z_i)]\cdots L(z_n)\rangle \\
&= \sum_{i=2}^{n} \frac{c}{2}(z_1 - z_i)^{-4}\langle L(z_2)\cdots \hat{L}(z_i)\cdots L(z_n)\rangle \\
&+ \left(\sum_{i=2}^{n}(2(z_1-z_i)^{-2} + (z_1-z_i)^{-1}\frac{\partial}{\partial z_i}\right)\langle L(z_2)\cdots\cdots L(z_n)\rangle
\end{aligned} \tag{10}$$

It is possible to write down a formula for the solution to this system of equations. But we shall no do it here.

Exercise 3.21. *Compute the* 2-, 3- *and* 4- *point functions.*

Exercise 3.22. *Compute* $\langle : L(z)L(z) : L(w)\rangle$.

Proposition 3.23. $\langle L\rangle$ *has a basis of the form* $:\partial^{n_1}L\cdots\partial^{n_k}L:$, $n_1 \geq \cdots \geq n_k \geq 0$.

Proof: It is word for word the same as the case of $\langle\alpha\rangle$. $\square$

2. Left Regular Action

Definition 3.24. *Let O be a circle algebra. An O-module is a vector space M equipped with homomorphism $O \to CO(M)$.*

For ordinary associative algebra, the underlying space is always a faithful module. We shall prove the analogous theorem for general commutative circle algebras.

Let O be a commutative circle algebra in $CO(V)$. For every $a(z)$, we can define a new operator acting on the vector space O, ie. an $A(\zeta) \in O$ by $A(\zeta) = \sum A(n)\zeta^{-n-1}$, where $A(n) : O \to O$ is defined by

$$A(n) \cdot u(z) = a(z) \circ_n u(z). \tag{11}$$

Note that this is zero for $n >> 0$ because of the locality assumption.

Definition 3.25. *The map above will be called the left regular action map, and be denoted by ρ_O or just ρ.*

To simplify notations *throughout this discussion, we shall use capital letters $A, B, ..$ to denote the respective images of $a, b, ... \in O$ under ρ. We shall also denote $A(n)$ as $(\rho a)(n)$.* Observe that ρ is injective; for if $\rho a = 0$, then $A(n) = 0$ for all n, and $0 = A(-1)1 = a \circ_{-1} 1 = a$ in particular. We shall eventually prove that for any commutative O, the map $\rho : O \to \rho(O)$ is a *circle algebra* isomorphism.

Lemma 3.26. *$(O, 1, \partial)$ is a based space, and every ρa is based.*

Proof: , Define $D : O \to O$ by $a(z) \mapsto \partial a(z)$. Obviously $D \cdot 1 = 0$. We need to check that $[D, (\rho a)(\zeta)] = \partial(\rho a)(\zeta)$.

$$\begin{aligned} D \cdot (\rho a)(n) \cdot u(z) - (\rho a)(n) \cdot D \cdot u(z) =& \partial(a \circ_n u)(z) - a(z) \circ_n \partial u(z) \\ =& (\partial a(z)) \circ_n u(z) \\ =& -na(z) \circ_{n-1} u(z) \\ =& -n\rho a(n-1) \cdot u(z). \quad \square \end{aligned} \tag{12}$$

Theorem 3.27. *(Left Regular Action) If $a(z), b(z)$ circle commute, then for all m, n,*

$$[(\rho a)(m), (\rho b)(n)] = \sum_{p\geq 0} \binom{m}{p} \rho(a \circ_p b)(m+n-p). \tag{13}$$

Proof: Put $A(z) = (\rho a)(z),\ B(z) = (\rho b)(z)$. Then we have

$$\begin{aligned}
&A(m)\cdot B(n)\cdot u(z) - B(n)\cdot A(m)\cdot u(z)\\
&= Res_{z_1} Res_{z_2} [a(z_2), b(z_1)] u(z)(z_2 - z)^m (z_1 - z)^n\\
&- Res_{z_2} Res_{z_1} u(z)[a(z_2), b(z_1)](-z+z_2)^m(-z+z_1)^n\\
&= Res_{z_1} Res_{z_2} \sum_{p\geq 0} (a \circ_p b)(z_1) u(z) \frac{(-1)^p}{p!} \left(\partial_{z_2}^p \delta(z_2, z_1)\right)(z_2 - z)^m (z_1 - z)^n\\
&- Res_{z_1} Res_{z_2} \sum_{p\geq 0} u(z)(a \circ_p b)(z_1) \frac{(-1)^p}{p!} \left(\partial_{z_2}^p \delta(z_2, z_1)\right)(-z+z_2)^m(-z+z_1)^n \qquad \textit{lemma } 2.14\\
&= \sum_p Res_{z_1} \binom{m}{p} (a \circ_p b)(z_1) u(z)(z_1 - z)^{m-p+n}\\
&- \sum_p Res_{z_1} \binom{m}{p} u(z)(a \circ_p b)(z_1)(-z+z_1)^{m-p+n}.
\end{aligned} \tag{14}$$

The last equality is obtained using integration by parts, and the fact that $Res_{z_2} f(z_1, z_2)\delta(z_2, z_1) = f(z_1, z_1)$ whenever $f(z_1, z_2)\delta(z_2, z_1)$ makes sense. Now we recognize that the RHS of (14) is

$$\begin{aligned}
RHS &= \sum_p \binom{m}{p} (a \circ_p b)(z) \circ_{m-p+n} u(z)\\
&= \sum_p \binom{m}{p} (\rho(a \circ_p b))(m-p+n)\cdot u(z). \qquad \square
\end{aligned} \tag{15}$$

Exercise 3.28. *Carry out the details of integration by parts above.*

Corollary 3.29. *For commutative circle algebra O, ρ_O preserves $\circ_n$, $n \geq 0$.*

Proof: From the preceeding lemma, we get for $m \geq 0$,

$$[(\rho a)(m), (\rho b)(w)] = \sum_{p\geq 0} \binom{m}{p} \rho(a \circ_p b)(w) w^{m-p}. \tag{16}$$

From lemma 1.7, we see that the $[(\rho a)(m), (\rho b)(w)]$ determines all the $(\rho a) \circ_n (\rho b)$ with $n \geq 0$ (by inverting the above equation). This shows that $\rho(a \circ_p b)$ in the equation above cannot be anything but $(\rho a) \circ_p (\rho b)$ for all $p \geq 0$. $\square$

Corollary 3.30. *If $a(z), b(z)$ circle commute, so do $(\rho a)(z), (\rho b)(z)$.*

Proof: Combining the preceeding corollary and lemma, we see that for *all* m, n

$$[(\rho a)(m), (\rho b)(n)] = \sum_{p \geq 0} ((\rho a) \circ_p (\rho b))(m + n - p) \tag{17}$$

which is equivalent to

$$[(\rho a)(m), (\rho b)(w)] = \sum_{p \geq 0} \binom{m}{p} (\rho a \circ_p \rho b)(w) w^{m-p} \tag{18}$$

for all m. So by lemma 2.14, our assertion follows. $\square$

Corollary 3.31. *$\rho(a \circ_n b) = \rho a \circ_n \rho b$ for all $a, b \in O$ and all n.*

Proof: Let $\hat{O}$ be the circle algebra generated by $\rho(O)$ in $CO(O)$. Now since $\rho(O)$ consists of based operators, it follows that $\hat{O}$ is based commutative circle algebra. The creation map $\chi : \hat{O} \to O$ is surjective because $\chi(\rho a) = \chi(A) = A(-1)1 = a$ for any $a \in O$. But this means that $\chi : \hat{O} \to O$ is an isomorphism (Chapter 2). Note also that ρ is the inverse of χ. Finally, we have

$$\chi(\rho a \circ_n \rho b) = (A \circ_n B)(-1)1 = A(n)B(-1)1 = A(n)b = a \circ_n b.$$

Now applying ρ to this, we get the asserted identity. $\square$

Corollary 3.32. *O is canonically an O-module.*

Corollary 3.33. *$(O, 1, \partial, \rho)$ is a vertex algebra.*

Corollary 3.34. *If a, b, c circle commute (with itself and others), then for all n, m, $(a \circ_n b) \circ_m c$ is a linear sum of terms of the forms $a \circ_k (b \circ_l c)$ and $b \circ_k (a \circ_l c)$ with universal rational coefficients (i.e. depend only on n, m, k, l).*

Proof: By writing out $\rho(a \circ_n b) \cdot c = (\rho a) \circ_n (\rho b) \cdot c$, the assertion follows immediately. $\square$

These universal relations were part of the axioms in the original mathematical definition of a vertex algebra.

Corollary 3.35. *Let O be a vector space of circle commuting operators. Let S be a set of circle commuting operators which also circle commute with O. If $S \circ_n O \subset O$ for all n, then $\langle S \rangle \circ_n O \subset O$.*

Proof: Without loss of generality we may assume $1 \in S$. We must show that $A \circ_n O \subset O$ for a word A of any length. We shall do induction on the length of A. The length 1 case is our assumption. Suppose the assertion holds up to length k. Let $A = A_1 \circ_n A_2$ where A_1, A_2 are words of length at most k. Then by the preceeding corollary, $A \circ_m O \subset \sum_{k.,l} A_1 \circ_k (A_2 \circ_l O) + \sum_{k,l} A_2 \circ_k (A_1 \circ_l O)$. By the inductive hypothesis both sums of the RHS are in O. $\square$.

Corollary 3.36. *Let O be a commutative circle algebra and $f : \langle S \rangle \to O$ be a linear map preserving* 1. *If $f(a \circ_n B) = fa \circ_n fB$ for all $a \in S$, $B \in \langle S \rangle$ and all n, then f is a circle algebra morphism.*

Proof: We must show that $f(A \circ_n B) = fA \circ_n fB$ for all words A in $\langle S \rangle$. This is done by induction on the length of A. The argument is similar to the preceeding corollary. $\square$

3. Modules

Lemma 3.37. *If O is a commutative circle algebra, and M an irreducible O-module, then for any $v(z) \in O$ with $v(z) \neq 0$ on M we have $v(z)m \neq 0$ for any nonzero $m \in M$.*

Proof: Given $v \in O$, put

$$M_v := \{m \in M | v(z)m = 0\}.$$

We claim that M_v is a submodule. Let $u \in O$. Then for some $N \geq 0$, we have

$$v(z)u(w) = \sum_{0 \leq n \leq N-1} v(w) \circ_n u(w)(z-w)^{-n-1} + : v(z)u(w) :$$

where $v \circ_n u = 0$ for $n \geq N$. Applying this to $m \in M_v$, and using the fact that $(z-w)^N[v(z), u(w)] = 0$ and that $v(z)m = 0$, we get

$$0 = \sum_{0 \leq n \leq N-1} v(w) \circ_n u(w) m(z-w)^{N-n-1} + :v(z)u(w): m(z-w)^N.$$

Note that on the RHS, setting $z = w$ makes sense *term wise.* Setting $z = w$, we get $(v \circ_{N-1} u)m = 0$. Differentiating WRT z once and setting $z = w$, we get $(v \circ_{N-2} u)m = 0$, etc. Thus $(v \circ_n u)m = 0$ for all $n \geq 0$. This shows that $v(z)u(w)m = 0$, i.e. $u(n)m \in M_v$ $\forall u \in V$.

If $v(z) \neq 0$, then $v(z)m \neq 0$ $\exists m \in M$. Therefore in this case, $M_v \neq M$. Since M is assumed irreducible, we must have $M_v = 0$, which means that $v(z)m = 0 \Longrightarrow m = 0$. □

Example 3.38. $\langle\alpha\rangle$*-modules.* Let $\mathcal{A} = \langle\alpha\rangle$ as introduced before. We shall give a complete description of all $\mathcal{A}$-modules. Let M be an $\mathcal{A}$-module with the map $\mathcal{A} \to CO(M)$. Let $a(z)$ be the image of the generator $\alpha(z)$. Applying corollary 2.14, we see that $[a(n), a(m)] = n\delta_{n+m,0}id$, i.e. the Fourier modes of $a(z)$ represent the Heisenberg algebra g on M, in which the central element $c \in g$ acts by id. Let g_+ be the Lie subalgebra of positive weight elements. Since $a(z) : M \to M((z))$, M is necessarily g_+-finite (i.e. for any given $v \in M$, we have $t^n \cdot v = 0$ for $n >> 0$). Thus in a canonical way, *every* $\mathcal{A}$*-module is a* g_+*-finite* g*-module in which* c *acts by* id*.*

We now prove the converse. Let M be a g_+-finite g-module $\pi : g \to End(M)$ in which c acts by id. Put

$$a(z) = \sum a(n) z^{-n-1} := \sum \pi(t^n) z^{-n-1}.$$

Since M is g_+-finite, $a \in CO(M)$. By the Heisenberg commutation relations, we see that $O_M := \langle a \rangle$ is a commutative circle algebra in $CO(M)$. By the left regular action $\rho_M : O_M \to CO(O_M)$, O_M is itself an O_M-module. By the Left Regular Action Theorem,

$$[(\rho_M a)(n), (\rho_M a)(m)] = n\delta_{n+m,0} id.$$

This shows that O_M is itself a g-module. Moreover $(\rho_M a)(n) \cdot 1 = 0$ for $n \geq 0$. By the universal property of the g-module $V = \mathbf{C}[\alpha(-1), \alpha(-2), ..]$ we have a unique g-module homomorphism $\varphi_M : V \to O_M$ with $1_V \mapsto 1_{O_M}$.

Specializing this to $M = V$, then $\rho_V : O_V = \langle \alpha \rangle \to CO(O_V)$ defines a g-module structure on O_V. Recall that the creation map $\chi : O_V = \langle \alpha \rangle \to V$ is a linear isomorphism. This is also a g-module homomorphism because

$$\chi((\rho_V \alpha)(n) \cdot b) = \chi(\alpha \circ_n b) = \alpha(n)b(-1)1 = \alpha(n)\chi(b).$$

Composing $\varphi_M : V \to O_M$ with $\chi : O_V \to V$, we get a g-module homomorphism $f : O_V \to O_M$. It remains to show that this is a circle algebra morphism, i.e. f preserves all circle products. By corollary 3.36, it suffices to show that for all n and X,

$$f(\alpha \circ_n X) = f(\alpha) \circ_n f(X) \tag{19}$$

First note that

$$f(\alpha) = \varphi_M \chi(\alpha) = \varphi_M(\alpha(-1)1_V) = (\rho_M a)(-1)1_{O_M} = a \circ_{-1} 1 = a.$$

Second, we have

$$\begin{aligned} f(\alpha \circ_n X) &= \varphi_M \chi(\alpha \circ_n X) \\ &= \varphi_M(\alpha(n)\chi(X)) \\ &= (\rho_M a)(n)\varphi_M \chi(X) \\ &= (\rho_M a)(n) f(X) \\ &= a \circ_n f(X) \\ &= f(\alpha) \circ_n f(X). \end{aligned}$$

Exercise 3.39. *Show that the correspondence between g_+-finite g-modules and $\langle \alpha \rangle$-modules above is an equivalence of categories.*

3.1. Universal Property of $\langle L \rangle$

It is obvious that if M is an $\langle L \rangle$-module, then M is naturally a Vir_+-finite Vir-module of central charge c. The exact same approach as in the preceeding example $\langle \alpha \rangle$ shows that this correspondence gives an equivalence of category. (You will need the basis lemma for $\langle L \rangle$ which can be proved easily using VA theory. See next chapter.) In particular, we have the following

Proposition 3.40. *If M is a a Vir_+-finite Vir-module of central charge c with $\pi : Vir \to End(M)$, then there is a unique circle algebra morphism $\langle L\rangle \to CO(M)$ sending $L \mapsto \pi L$.*

This raises the following interesting question: *Does $CO(M)$ itself become a Vir-module, not necessarily Vir_+-finite, via $\langle L\rangle \to CO(M)$?* We will answer in the affirmative.

4. Action of one commutative circle algebra on another

Let M be a Vir_+-finite Vir-module in which the central element κ acts by a scalar c. We have just seen that M is a $\langle L(z)\rangle$-module canonically. Let $X(z)$ be the image of $L(z)$ in $CO(M)$. Let O be any commutative circle algebra in $CO(M)$ containing $X(z)$ (e.g. $O = CO(M)$.). Define $L(n) : O \to O$ by

$$L(n) \cdot u(z) := X(z) \circ_n u(z). \tag{20}$$

Lemma 3.41. *$L_n \mapsto L(n+1)$ defines a Vir-module structure on O in which κ acts by $c \cdot id$. If O is local, then O is Vir_+-finite as a Vir-module.*

Proof: The second assertion follows immediately from the first. To see the first, we must show that

$$\begin{aligned} &L(m) \cdot L(n) \cdot u(z) - L(n) \cdot L(m) \cdot u(z) \\ &= (m-n)L(m+n-1) \cdot u(z) + \frac{\kappa}{12} m(m-1)(m-2)\delta_{n+m-2} u(z). \end{aligned} \tag{21}$$

We start from the LHS. By definition the circle products of $X(z)$ with itself is the same as that of $L(z)$. Using this and lemma 2.14, we get

$$\begin{aligned} LHS =& Res_{z_1} Res_{z_2} [X(z_2), X(z_1)] u(z)(z_2 - z)^m (z_1 - z)^n \\ &- Res_{z_1} Res_{z_2} u(z)[X(z_2), X(z_1)](-z + z_2)^m (-z + z_1)^n \\ =& Res_{z_1} Res_{z_2} (z_2 - z)^m (z_1 - z)^n \left(\frac{\kappa}{12}\partial_{z_1}^3 + 2X(z_1)\partial_{z_1} + \partial_{z_1} X(z_1)\right) \delta(z_1, z_2) u(z) \\ &- Res_{z_1} Res_{z_2} u(z)(-z + z_2)^m (-z + z_1)^n \left(\frac{\kappa}{12}\partial_{z_1}^3 + 2X(z_1)\partial_{z_1} + \partial_{z_1} X(z_1)\right) \delta(z_1, z_2) \end{aligned} \tag{22}$$

where $\delta(z_1,z_2) = (z_2 - z_1)^{-1} - (-z_1 + z_2)^{-1}$. For any Laurent polynomial $g(z_1,z_2)$, and any formal series $h(z_1,z_2)$, we have the identities

$$\begin{aligned} g(z_1,z_2)\delta(z_1,z_2) =& g(z_1,z_1)\delta(z_1,z_2) \\ Res_{z_1}\delta(z_1,z_2) =& 1 \\ Res_{z_1} g(z_1,z_2)\partial_{z_1}^k h(z_1,z_2) =& (-1)^k Res_{z_1}(\partial_{z_1}^k g(z_1,z_2))\ h(z_1,z_2). \end{aligned} \tag{23}$$

Applying these and continuing the above computation:

$$\begin{aligned} LHS =& Res_{z_1}\Big(-\frac{\kappa}{12}n(n-1)(n-2)(z_1-z)^{m+n-3} \\ & -2X(z_1)n(z_1-z)^{m+n-1} - 2\partial_{z_1}X(z_1)(z_1-z)^{m+n} + \partial_{z_1}X(z_1)(z_1-z)^{m+n}\Big)\, u(z) \\ & - Res_{z_1} u(z)\Big(-\frac{\kappa}{12}n(n-1)(n-2)(-z+z_1)^{m+n-3} \\ & -2X(z_1)n(-z+z_1)^{m+n-1} - 2\partial_{z_1}X(z_1)(-z+z_1)^{m+n} + \partial_{z_1}X(z_1)(-z+z_1)^{m+n}\Big) \\ =& -\frac{\kappa}{12}n(n-1)(n-2)1\circ_{m+n-3} u(z) + (m-n)X(m+n-1)\cdot u(z). \end{aligned} \tag{24}$$

This proves (21). □

Corollary 3.42. *The linear map $\langle L(z)\rangle \to N(c)$, $A(z) \mapsto A(-1)v_c$ is isomorphism of Vir-module.*

Proof: We have already seen that $(N(c), \mathbf{1}, D)$ and L is based circle operator. Thus by theorem 2.37 , we saw that the creation map $\chi : \langle L\rangle \to N(c)$ is a *linear isomorphism*. Now applying the preceeding lemma to the case $O = \langle L\rangle$, we get a Vir-module structure on O. Moreover O is a cyclic Vir-module generated by $1 \in O$. It is also easy to check that $L(n)\cdot 1 = 0$ for $n \geq 0$. By the universal property of $N(c)$, there is a unique Vir-map $N(c) \to O$ sending $v_c \mapsto 1$. Using the definition of Vir action on O, it is easy to check that this map sends the canonical BPW basis (with factor $n_1!\cdots n_k!$) to the monomial basis of O. By definition, this is the inverse $\chi : O \to N(c)$. □

Exercise 3.43. *In $O := \langle\alpha(z)\rangle$, put $X(z) = \frac{1}{2} : \alpha(z)\alpha(z) :$. Show that $X(z)$ has the same OPE as $L(z)$ with $c = 1$.*

Exercise 3.44. *Now regard both $\alpha(z)$ and $L(z)$ as graded, having weights 1, 2 respectively. For each fixed value of c, classify the circle maps $\langle L(z)\rangle \to \langle\alpha(z)\rangle$. (There are two such* *maps for each* $c \neq 1$, *and just one otherwise.) Show that the image of* $L(z)$ *is* $\frac{1}{2} : \alpha(z)\alpha(z) : \pm\epsilon\partial\alpha(z)$ *with* $c = 1 - 12\epsilon^2$. *This put a* Vir*-module structure on the* g*-module. Similarly the* g*-module which we call* $F(p)$ *before now acquires a* Vir*-module structure which is denoted as* $F(\epsilon, p)$. *It is known as a Feigin-Fuchs module. In physics, it is also called a free field module.*

Exercise 3.45. *In* $O := \langle\psi(z)\rangle$, *put* $X(z) =: \psi(z)\partial\psi(z) :$. *Show that* $X(z)$ *has the same OPE as* $L(z)$ *with* $c = \frac{1}{2}$.