Chiral Equivariant Cohomology III

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This paper is dedicated to the memory of our friend and colleague Jerome P. Levine

ABSTRACT. This is the third of a series of papers on a new equivariant cohomology that takes values in a vertex algebra. In the second paper, we defined the chiral equivariant cohomology $\mathbf{H}^*_G(\mathcal{A})$ of an $\mathfrak{sg}[t]$-algebra $\mathcal{A}$, where G is a compact Lie group and $\mathfrak{sg}$ is the superization of the Lie algebra $\mathfrak{g}$ of G. Our main examples of $\mathfrak{sg}[t]$-algebras are the chiral de Rham complex $\mathcal{Q}(M)$ of a G-manifold M, and the subalgebra $\mathcal{Q}'(M) \subset \mathcal{Q}(M)$ generated by the weight-zero subspace. Both $\mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M))$ are "chiralizations" of the classical equivariant cohomology $H^*_G(M)$. The main results in this paper are the existence of Mayer-Vietoris sequences, the invariance of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ under equivariant homotopy, and the existence of a quasi-conformal structure on $\mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M))$ for any G and M. Using these results, we describe $\mathbf{H}^*_G(\mathcal{Q}'(G/H))$ for any closed subgroup $H \subset G$. We prove a vanishing theorem for $\mathbf{H}^*_G(\mathcal{Q}(M))$ whenever G acts effectively on M. Finally, for any simple G we construct compact G-manifolds M and N together with a smooth, G-equivariant map $f : M \to N$ which induces a ring isomorphism $H^*_G(N) \to H^*_G(M)$, such that $\mathbf{H}^*_G(\mathcal{Q}'(M)) \neq \mathbf{H}^*_G(\mathcal{Q}'(N))$.

Contents

1. Introduction

Let G be a compact Lie group with complexified Lie algebra $\mathfrak{g}$. In [10], the chiral equivariant cohomology $\mathbf{H}^*_G(\mathcal{A})$ of an $O(\mathfrak{sg})$-algebra $\mathcal{A}$ was introduced. This is a functor from the category of $O(\mathfrak{sg})$-algebras to the category of vertex algebras. Examples of $O(\mathfrak{sg})$-algebras include the semi-infinite Weil algebra $\mathcal{W}(\mathfrak{g})$ and the chiral de Rham complex $\mathcal{Q}(M)$ of a smooth G-manifold M. In [11], this functor was extended to the larger categories of $\mathfrak{sg}[t]$-algebras and $\mathfrak{sg}[t]$-modules. Our main example of an $\mathfrak{sg}[t]$-algebra which is *not* an $O(\mathfrak{sg})$-algebra is the subalgebra $\mathcal{Q}'(M) \subset \mathcal{Q}(M)$ generated by the weight-zero subspace. $\mathbf{H}^*_G(\mathcal{Q}(M))$ and $\mathbf{H}^*_G(\mathcal{Q}'(M))$ are both "chiralizations" of the classical equivariant cohomology $H^*_G(M)$, that is, vertex algebras equipped with a weight grading by the non-negative integers, which contain $H^*_G(M)$ as the subspace of weight zero. In the case $M = pt$, $\mathcal{Q}(M) = \mathcal{Q}'(M) = \mathbf{C}$, and $\mathbf{H}^*_G(\mathbf{C})$ plays the role of $H^*_G(pt) = S(\mathfrak{g}^*)^G$ in the classical theory.

We briefly recall these constructions, following the notation in [10][11]. First of all, a differential vertex algebra (DVA) is a degree graded vertex algebra $\mathcal{A}^* = \oplus_{p\in\mathbf{Z}}\mathcal{A}^p$ equipped with a vertex algebra derivation d of degree 1 such that $d^2 = 0$. A DVA will be called *degree-weight graded* if it has an additional $\mathbf{Z}_{\geq 0}$-grading by weight, which is compatible with the degree in the sense that $\mathcal{A}^p = \oplus_{n\geq 0}\mathcal{A}^p[n]$.

There is an auxiliary structure on a DVA which is analogous to the structure of a G^*-algebra in [7]. Associated to $\mathfrak{g}$ is a Lie superalgebra $\mathfrak{sg} := \mathfrak{g} \triangleright \mathfrak{g}^{-1}$ with bracket $[(\xi,\eta),(x,y)] = ([\xi,x],[\xi,y]-[x,\eta])$, which is equipped with a differential $d : (\xi,\eta) \mapsto (\eta, 0)$. This differential extends to the loop algebra $\mathfrak{sg}[t,t^{-1}]$, and gives rise to a vertex algebra derivation on the corresponding current algebra $O(\mathfrak{sg}) := O(\mathfrak{sg}, 0)$. Here 0 denotes the zero bilinear form on $\mathfrak{sg}$.

Definition 1.1. *An $O(\mathfrak{sg})$-algebra is a degree-weight graded DVA $\mathcal{A}$ equipped with a DVA homomorphism $\rho : O(\mathfrak{sg}) \to \mathcal{A}$, which we denote by $(\xi,\eta) \to L_\xi + \iota_\eta$.*

Although this definition makes sense for any Lie algebra $\mathfrak{g}$, we will assume throughout this paper that $\mathfrak{g}$ is the Lie algebra of a compact group G, and we require $\mathcal{A}$ to admit an action $\hat{\rho}: G \to Aut(\mathcal{A})$ of G by vertex algbera automorphisms which is compatible with the $O(\mathfrak{sg})$-structure in the following sense:

$$\frac{d}{dt}\hat{\rho}(exp(t\xi))|_{t=0} = L_{\xi}(0), \tag{1.1}$$

$$\hat{\rho}(g)L_{\xi}(n)\hat{\rho}(g^{-1}) = L_{Ad(g)(\xi)}(n), \tag{1.2}$$

$$\hat{\rho}(g)\iota_{\xi}(n)\hat{\rho}(g^{-1}) = \iota_{Ad(g)(\xi)}(n), \tag{1.3}$$

$$\hat{\rho}(g)d\hat{\rho}(g^{-1}) = d, \tag{1.4}$$

for all $\xi \in \mathfrak{g}$, $g \in G$, and $n \in \mathbf{Z}$. These conditions are analogous to Equations (2.23)-(2.26) of [7]. In order for (1.1) to make sense, we must be able to differentiate along appropriate curves in $\mathcal{A}$, which is the case in our main examples $\mathcal{A} = \mathcal{Q}(M)$ and $\mathcal{A} = \mathcal{Q}'(M)$.

In [11], we observed that the chiral equivariant cohomology functor can be defined on the larger class of spaces which carry only a representation of the Lie subalgebra $\mathfrak{sg}[t]$ of $\mathfrak{sg}[t, t^{-1}]$.

Definition 1.2. *An $\mathfrak{sg}[t]$ module is a degree-weight graded complex $(\mathcal{A}, d_{\mathcal{A}})$ equipped with a Lie algebra homomorphism $\rho : \mathfrak{sg}[t] \to End\ \mathcal{A}$, such that for all $x \in \mathfrak{sg}[t]$ we have*

- $\rho(dx) = [d_{\mathcal{A}}, \rho(x)]$
- *$\rho(x)$ has degree 0 whenever x is even in $\mathfrak{sg}[t]$, and degree -1 whenever x is odd, and has weight $-n$ if $x \in \mathfrak{sg}t^n$.*

As above, we will always assume that $\mathfrak{g}$ is the Lie algebra of G, and $\mathcal{A}$ has an action of G which is compatible with the $\mathfrak{sg}[t]$-structure, ie, (1.1)-(1.4) hold for $n \geq 0$.

Definition 1.3. *Given an $\mathfrak{sg}[t]$-module $(\mathcal{A}, d)$, we define the chiral horizontal, invariant and basic subspaces of $\mathcal{A}$ to be respectively*

$$\mathcal{A}_{hor} = \{a \in \mathcal{A} | \rho(x)a = 0\ \forall x \in \mathfrak{g}^{-1}[t]\}$$
$$\mathcal{A}_{inv} = \{a \in \mathcal{A} | \rho(x)a = 0\ \forall x \in \mathfrak{g}[t],\ \hat{\rho}(g)(a) = a\ \forall g \in G\}$$
$$\mathcal{A}_{bas} = \mathcal{A}_{hor} \cap \mathcal{A}_{inv}.$$

An $\mathfrak{sg}[t]$-module $(\mathcal{A}, d)$ is called an $\mathfrak{sg}[t]$-algebra if it is also a DVA such that $\mathcal{A}_{hor}, \mathcal{A}_{inv}$ are both vertex subalgebras of $\mathcal{A}$, and G acts by DVA automorphisms.

When we are working with multiple groups $G, H, \dots$ we will use the notations $\mathcal{A}_{G-hor}$, $\mathcal{A}_{G-inv}$, $\mathcal{A}_{G-bas}$, etc., to avoid confusion. Given an $\mathfrak{sg}[t]$-module $(\mathcal{A}, d)$, $\mathcal{A}_{inv}$, $\mathcal{A}_{bas}$ are both subcomplexes of $\mathcal{A}$, but $\mathcal{A}_{hor}$ is not a subcomplex of $\mathcal{A}$ in general. The Lie algebra $\mathfrak{sg}[t]$ is not required to act by derivations on a DVA $\mathcal{A}$ to make it an $\mathfrak{sg}[t]$-algebra. If $(\mathcal{A}, d)$ is an $O(\mathfrak{sg})$-algebra, any subDVA $\mathcal{B}$ which is closed under the operators $(L_\xi + \iota_\eta)\circ_p$, $p \geq 0$, is an $\mathfrak{sg}[t]$-algebra.

Definition 1.4. *For any $\mathfrak{sg}[t]$-module $(\mathcal{A}, d_\mathcal{A})$, we define its chiral basic cohomology $\mathbf{H}^*_{bas}(\mathcal{A})$ to be $H^*(\mathcal{A}_{bas}, d_\mathcal{A})$. We define its chiral equivariant cohomology $\mathbf{H}^*_G(\mathcal{A})$ to be $\mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g})\otimes\mathcal{A})$. The differential on $\mathcal{W}(\mathfrak{g})\otimes\mathcal{A}$ is $d_\mathcal{W}\otimes 1 + 1\otimes d_\mathcal{A}$, where $d_\mathcal{W} = J(0) + K(0)$, as in [10].*

In this paper, our main focus is on the cases where $\mathcal{A}$ is either the chiral de Rham complex $\mathcal{Q}(M)$ or the subcomplex $\mathcal{Q}'(M)$ introduced in [11], for a smooth G-manifold M. Recall from [11] that for each $m \geq 0$, $\mathcal{Q}_M[m]$ is a sheaf of vector spaces on M, and $\mathcal{Q}(M)$ is the space of global sections of the weak sheaf of vertex algebras $\mathcal{Q}_M = \oplus_{m\geq 0}\mathcal{Q}_M[m]$ on M. Similarly, $\mathcal{Q}'_M[m]$ is the subsheaf of $\mathcal{Q}_M[m]$ generated by the weight-zero subspace, and $\mathcal{Q}'(M)$ is the space of global sections of the weak sheaf of abelian vertex algebras $\mathcal{Q}'_M = \oplus_{m\geq 0}\mathcal{Q}'_M[m]$. In this terminology, a *weak sheaf* is a presheaf in which

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \tag{1.5}$$

is exact for *finite* open covers $\{U_i\}$ of an open set U (see Section 1.1 of [11]). Whenever we need to construct a global section of $\mathcal{Q}_M$ or $\mathcal{Q}'_M$ by gluing together local sections, these sections are always homogeneous of finite weight, so we may work inside the sheaf $\mathcal{Q}_M[m]$ or $\mathcal{Q}'_M[m]$ for some m.

Since $\mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M))$ are both chiralizations of $H^*_G(M)$, we have the linear decompositions

$$\mathbf{H}^*_G(\mathcal{Q}(M)) = H^*_G(M) \oplus \mathbf{H}^*_G(\mathcal{Q}(M))_+, \quad \mathbf{H}^*_G(\mathcal{Q}'(M)) = H^*_G(M) \oplus \mathbf{H}^*_G(\mathcal{Q}'(M))_+.$$

In this notation,

$$\mathbf{H}^*_G(\mathcal{Q}(M))_+ = \bigoplus_{n>0} \mathbf{H}^*_G(\mathcal{Q}(M))[n], \qquad \mathbf{H}^*_G(\mathcal{Q}'(M))_+ = \bigoplus_{n>0} \mathbf{H}^*_G(\mathcal{Q}'(M))[n].$$

Ultimately, we would like to understand what kind of geometric information is contained in $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ and $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$. In [10][11] we proved a number of structural results about these two cohomology theories, which we recall below.

- The functor $\mathbf{H}^*_G(\mathcal{Q}'(-))$ is contravariant in M for any G. For fixed M, $\mathbf{H}^*_{(-)}(\mathcal{Q}'(M))$ is not functorial in G in general, but is contravariant with respect to abelian groups.

- For any G, $\mathbf{H}^*_G(\mathbf{C})$ contains nonzero classes in every positive weight. If G is semisimple, $\mathbf{H}^*_G(\mathbf{C})$ is a conformal vertex algebra with Virasoro element $\mathbf{L}$ of central charge zero.

- For any $\mathfrak{sg}[t]$-algebra $\mathcal{A}$, the canonical map

$$\kappa_G : \mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g})) = \mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_G(\mathcal{A})$$

  induced by $\mathcal{W}(\mathfrak{g}) \hookrightarrow \mathcal{W}(\mathfrak{g}) \otimes \mathcal{A}$ is called the *chiral Chern-Weil map* of $\mathcal{A}$. For $\mathcal{A} = \mathcal{Q}(M)$ or $\mathcal{A} = \mathcal{Q}'(M)$, this map extends the classical Chern-Weil map $H^*_G(pt) \to H^*_G(M)$.

- For any G, if M has a G-fixed point, $\kappa_G : \mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_G(\mathcal{Q}'(M))$ is injective. If G is semisimple, $\mathbf{H}^*_G(\mathcal{Q}'(M))$ is then a conformal vertex algebra with Virasoro element $\kappa_G(\mathbf{L})$ of central charge zero.

- For any G, if the action of G on M is locally free, $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = 0$ and $\mathbf{H}^*_G(\mathcal{Q}(M))_+ = 0$.

- If G is a torus T, $\mathbf{H}^*_T(\mathcal{Q}'(M))_+ = 0$ if and only if the action of T is locally free. The converse fails in general. For example, if G is simple and $T \subset G$ is a torus, $\mathbf{H}^*_G(\mathcal{Q}'(G/T))_+ = 0$ even though the action of G on G/T is not locally free.

- If G is semisimple and V is a faithful linear representation of G, $\mathbf{H}^*_G(\mathcal{Q}(V))_+ = 0$.

1.1. Outline of main results

In this paper, we continue the study of $\mathbf{H}^*_G(\mathcal{Q}(M))$ and $\mathbf{H}^*_G(\mathcal{Q}'(M))$. There are three basic results we need to establish. First, for G-invariant open sets $U, V \subset M$, there exist Mayer-Vietoris sequences

$$\cdots \to \mathbf{H}^{k-1}_G(\mathcal{Q}(U\cap V)) \to \mathbf{H}^k_G(\mathcal{Q}(U\cup V)) \to \mathbf{H}^k_G(\mathcal{Q}(U))\oplus\mathbf{H}^k_G(\mathcal{Q}(V)) \to \mathbf{H}^k_G(\mathcal{Q}(U\cap V)) \to \cdots,$$

$$\cdots \to \mathbf{H}^{k-1}_G(\mathcal{Q}'(U\cap V)) \to \mathbf{H}^k_G(\mathcal{Q}'(U\cup V)) \to \mathbf{H}^k_G(\mathcal{Q}'(U))\oplus\mathbf{H}^k_G(\mathcal{Q}'(V)) \to \mathbf{H}^k_G(\mathcal{Q}'(U\cap V)) \to \cdots.$$

As usual, these sequences are useful tools for passage from local to global information.

Second, $\mathbf{H}^*_G(\mathcal{Q}'(-))$ is invariant under G-equivariant homotopy. That is, if M and N are G-manifolds and $\phi_0, \phi_1 : M \to N$ are equivariantly homotopic G-maps, the induced maps $\phi_0^*, \phi_1^* : \mathbf{H}^*_G(\mathcal{Q}'(N)) \to \mathbf{H}^*_G(\mathcal{Q}'(M))$ are the same. In particular, if M is equivariantly contractible to a submanifold M', $\mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_G(\mathcal{Q}'(M'))$.

Third, for any G and M, $\mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M))$ have *quasi-conformal structures*. That is, they admit an action of the subalgebra of the Virasoro algebra generated by $\{L_n |\ n \geq -1\}$, such that L_{-1} acts by ∂ and L_0 acts by $n\ id$ on the subspace of weight n. When G is semisimple, this quasi-conformal structure coincides with the quasi-conformal structure given by $\{\kappa_G(\mathbf{L})_n |\ n \geq -1\}$. In the semisimple case, the vanishing of $\kappa_G(\mathbf{L})$ is equivalent to the vanishing of $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$. Likewise, for general G and M, the quasi-conformal structure on $\mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M))$ provides a vanishing criterion for $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ and $\mathbf{H}^*_G(\mathcal{Q}(M))_+$; it suffices to show that L_0 acts by zero.

Using these three basic tools, our goal will be to give a *relative* description of $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ and $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ in terms of the vertex algebras $\mathbf{H}^*_K(\mathbf{C})$ for various connected normal subgroups K of G, together with geometric data about M. If K is abelian, Theorem 6.1 of [10] gives a complete description of $\mathbf{H}^*_K(\mathbf{C})$, but if K is non-abelian $\mathbf{H}^*_K(\mathbf{C})$ is still a rather mysterious object. Computer calculations in the cases $K = SU(2)$ and $K = SU(3)$ indicate that $\mathbf{H}^*_K(\mathbf{C})$ has a rich structure and contains many non-classical elements beyond the Virasoro element $\mathbf{L}$.

Since G-manifolds locally look like vector bundles over homogeneous spaces G/H, a basic problem is to compute $\mathbf{H}^*_G(\mathcal{Q}'(G/H))$ for any closed subgroup $H \subset G$.

Theorem 1.5. *For any compact, connected G and closed subgroup $H \subset G$,*

$$\mathbf{H}^*_G(\mathcal{Q}'(G/H)) \cong \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes H^*_{G'}(G/H), \tag{1.6}$$

where K_0 is the identity component of $Ker(G \to Diff(G/H))$ and $G' = G/K_0$. Here $H^*_{G'}(G/H)$ is regarded as a vertex algebra in which all products are trivial except $\circ_{-1}$, and (1.6) is an isomorphism of vertex algebras.

A consequence of this result is that for compact M, the degree p and weight n subspace $\mathbf{H}^p_G(\mathcal{Q}'(M))[n]$ is finite-dimensional for all $p \in \mathbf{Z}$ and $n \geq 0$, which extends a well-known classical result in the case $n = 0$. Hence the generating function

$$\chi(M, G) = \sum_{p,n} dim\ \mathbf{H}^p(\mathcal{Q}'(M))[n]\ z^p q^n$$

is a well-defined invariant of M.

Next, we study $\mathbf{H}^*_G(\mathcal{Q}(M))$ via the map $\mathbf{H}^*_G(\mathcal{Q}'(M)) \to \mathbf{H}^*_G(\mathcal{Q}(M))$ induced by the inclusion $\mathcal{Q}'(M) \hookrightarrow \mathcal{Q}(M)$.

Theorem 1.6. *For any G-manifold M,*

$$\mathbf{H}^*_G(\mathcal{Q}(M)) \cong \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes H^*_{G'}(M),$$

where K_0 denotes the identity component of $Ker(G \to Diff(M))$ and $G' = G/K_0$. In particular, $\mathbf{H}^*_G(\mathcal{Q}(M))_+ = 0$ whenever the action of G is effective up to a finite group.

Thus $\mathbf{H}^*_G(\mathcal{Q}(M)))_+$ depends only on K_0, so it carries no other geometric information about M. An important consequence is that for any G and M, $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ and $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ are *vertex algebra ideals*, ie, they are closed under $\alpha \circ_n$ and $\circ_n \alpha$ for all $n \in \mathbf{Z}$ and α in $\mathbf{H}^*_G(\mathcal{Q}(M))$, $\mathbf{H}^*_G(\mathcal{Q}'(M))$, respectively.

Next, we study $\mathbf{H}^*_G(\mathcal{Q}'(M))$, which in contrast to $\mathbf{H}^*_G(\mathcal{Q}(M))$, carries non-trivial geometric information about M beyond weight zero. We focus on three special cases: G simple, $G = G_1 \times G_2$ where G_1, G_2 are simple, and G abelian. For simple G, the following theorem describes $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ in terms of $\mathbf{H}^*_G(\mathbf{C})$ and classical data.

Theorem 1.7. *For any simple group G and G-manifold M, the map*

$$\mathbf{H}^*_G(\mathcal{Q}'(M))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(M^G))_+$$

induced by the inclusion $M^G \hookrightarrow M$, is a linear isomorphism. It follows that $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ \cong \mathbf{H}^*_G(\mathbf{C})_+ \otimes H^*(M^G)$ as linear spaces.

We also describe the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$, and show that it encodes certain classical geometric data such as the ring structure of $H^*(M^G)$ and the map $H^*_G(M) \to H^*_G(M^G)$. Using a theorem of R. Oliver which describes the fixed-point submanifolds of group actions on disks, we construct compact G-manifolds M and N together with a smooth, G-equivariant map $f : M \to N$ which induces a ring isomorphism $H^*_G(N) \to H^*_G(M)$ (with $\mathbf{Z}$-coefficients), such that $\mathbf{H}^*_G(\mathcal{Q}'(M)) \neq \mathbf{H}^*_G(\mathcal{Q}'(N))$. *Hence $\mathbf{H}^*_G(\mathcal{Q}'(-))$ is a strictly stronger invariant than $H^*_G(-)$ on the category of compact G-manifolds.*

Similarly, in the case $G = G_1 \times G_2$ where G_1, G_2 are simple, we describe $\mathbf{H}^*_G(\mathcal{Q}'(M))$ in terms of the vertex algebras $\mathbf{H}^*_{G_1}(\mathbf{C})$, $\mathbf{H}^*_{G_2}(\mathbf{C})$ and the rings $H^*(M^G)$, $H^*_{G_1}(M^{G_2})$, and $H^*_{G_2}(M^{G_1})$.

The case where G is the circle S^1 is analogous to the case of simple G.

Theorem 1.8. *For any S^1-manifold M, the map $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ \to \mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1}))_+$ induced by the inclusion $M^{S^1} \hookrightarrow M$, is a linear isomorphism. Hence $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ \cong \mathbf{H}^*_{S^1}(\mathbf{C})_+ \otimes H^*(M^{S^1})$.*

When G is a general torus T, $\mathbf{H}^*_T(\mathcal{Q}'(M))_+$ will typically depend on the family of rings $H^*_{T/T'}(M^{T'})$ for all subtori $T' \subset T$ for which $M^{T'}$ is non-empty, and can be quite complicated. As an example, we compute $\mathbf{H}^*_T(\mathcal{Q}'(\mathbf{CP}^2))$, where $T = S^1 \times S^1$ and $\mathbf{CP}^2$ is equipped with the usual linear action.

We conclude with a few remarks about $\mathbf{H}^*_G(\mathbf{C})$. In [3], Feigin-Frenkel suggest that the semi-infinite Weil complex $\mathcal{W}(\mathfrak{g})$ should play a role in semi-infinite geometry analogous to the role of the classical Weil complex $W(\mathfrak{g})$. Note that $H^*_G(pt) = S(\mathfrak{g}^*)^G$ can be regarded either as the basic cohomology $H^*_{bas}(W(\mathfrak{g}))$, or as the Lie algebra cohomology

$H^0(\mathfrak{g}, S(\mathfrak{g}^*))$. The analogue of $H^0(\mathfrak{g}, S(\mathfrak{g}^*))$ is the *semi-infinite cohomology* $H^{\infty+*}(\hat{\mathfrak{g}}, \mathcal{S}(\mathfrak{g}))$ where $\mathcal{S}(\mathfrak{g})$ is the semi-infinite symmetric algebra on $\mathfrak{g}$, whereas the analogue of $H^*_{bas}(W(\mathfrak{g}))$ is $\mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g})) = \mathbf{H}^*_G(\mathbf{C})$. In contrast to the classical case, $\mathbf{H}^*_G(\mathbf{C})$ does *not* coincide with this semi-infinite cohomology. It would be interesting to construct an equivariant cohomology theory for manifolds in which $H^{\infty+*}(\hat{\mathfrak{g}}, \mathcal{S}(\mathfrak{g}^*))$ plays the role of $H^*_G(pt)$ (as suggested in [3]), and compare it to our theory.

Various aspects of the chiral de Rham complex and its generalizations have been studied and developed in recent years. In [12], the sheaf cohomology was considered in terms of representation theory of affine Lie algebras. In [6], cohomological obstructions to the existence of certain subsheaves and global sections were considered, while in [4], the chiral de Rham complex was studied in the context of finite group actions and orbifolds. More recently, chiral differential operators have been interpreted in terms of certain twisted two-dimensional supersymmetric sigma models [16]. It has been suggested that chiral equivariant cohomology for G-manifolds could be a gauged version of a sigma model.[1]

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1.2. General remarks about group actions on manifolds

Let G be a compact Lie group with identity component G_0. If M is a G-manifold, the finite group $\Gamma = G/G_0$ acts on the complex $(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas}$ by DVA automorphisms, and we have

$$(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G-bas} = ((\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas})^G = ((\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas})^\Gamma.$$

Since the differential $d_{\mathcal{W}} + d_{\mathcal{Q}}$ commutes with the action of Γ on $(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M))_{G_0-bas}$, Γ acts on $\mathbf{H}^*_{G_0}(\mathcal{Q}(M))$, and we have $\mathbf{H}^*_G(\mathcal{Q}(M)) = \mathbf{H}^*_{G_0}(\mathcal{Q}(M))^\Gamma$. Similarly, $\mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_{G_0}(\mathcal{Q}'(M))^\Gamma$. *Hence there is essentially no new content in studying* $\mathbf{H}^*_G(-)$ *for disconnected groups, so for the remainder of this paper, we will only consider the functor* $\mathbf{H}^*_G(-)$ *for connected* G.

[1] We thank E. Witten for drawing our attention to this development.

We say that G acts *effectively* on M if $K = Ker(G \to Diff(M))$ is trivial, and we say that G acts *almost effectively* if K is finite. Let K_0 denote the identity component of K and let $G' = G/K_0$.

The next lemma shows that up to tensoring with $\mathbf{H}^*_{K_0}(\mathbf{C})$, it is enough to consider only actions which are almost effective.

Lemma 1.9. *Let M be a G manifold, and suppose that $K = Ker(G \to Diff(M))$ has positive dimension. Then G' acts almost effectively on M and*

$$\mathbf{H}^*_G(\mathcal{Q}(M)) = \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes \mathbf{H}^*_{G'}(\mathcal{Q}(M)), \qquad \mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes \mathbf{H}^*_{G'}(\mathcal{Q}'(M)).$$

Proof: Clearly $K/K_0 = Ker(G' \to Diff(M))$, which is finite because K is compact. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{g}'$ where $\mathfrak{k}$ and $\mathfrak{g}'$ are the Lie algebras of K and G', respectively, we have $\mathcal{W}(\mathfrak{g}) = \mathcal{W}(\mathfrak{k}) \otimes \mathcal{W}(\mathfrak{g}')$. Then

$$(\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M))_{G-bas} = \mathcal{W}(\mathfrak{k})_{K_0-bas} \otimes (\mathcal{W}(\mathfrak{g}') \otimes \mathcal{Q}'(M))_{G'-bas}.$$

Note that the differential $d = d_{\mathcal{W}(\mathfrak{g})} + d_{\mathcal{Q}}$ of $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ can be written as $d_{\mathcal{W}(\mathfrak{k})} + d_{\mathcal{W}(\mathfrak{g}')} + d_{\mathcal{Q}}$, and these three terms pairwise commute. Since $d_{\mathcal{W}(\mathfrak{k})}$ only acts on $\mathcal{W}(\mathfrak{k})_{K_0-bas}$ and $d_{\mathcal{W}(\mathfrak{g}')} + d_{\mathcal{Q}}$ only acts on $\mathcal{W}(\mathfrak{g}') \otimes \mathcal{Q}(M))_{G'-bas}$, the claim follows. $\square$

Any compact connected G has a finite cover of the form $\tilde{G} = G_1 \times \cdots \times G_k \times T$, where the G_i are simple and T is a torus. If M is a G-manifold, the action can be lifted to $\tilde{G}$ so that $\Gamma \subset Ker(\tilde{G} \to Diff(M))$. The next lemma shows that without loss of generality, we may always assume that G is of this form.

Lemma 1.10. *Suppose M is a G manifold, $K = Ker(G \to Diff(M))$, and Γ is a finite subgroup of K. Then G/Γ acts on M and*

$$\mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_{G/\Gamma}(\mathcal{Q}'(M)), \qquad \mathbf{H}^*_G(\mathcal{Q}(M)) = \mathbf{H}^*_{G/\Gamma}(\mathcal{Q}(M)).$$

In particular, if G, $\tilde{G}$, and Γ are as above, we have

$$\mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_{\tilde{G}}(\mathcal{Q}'(M)), \qquad \mathbf{H}^*_G(\mathcal{Q}(M)) = \mathbf{H}^*_{\tilde{G}}(\mathcal{Q}(M)).$$

Proof: Γ acts trivially on $\mathcal{Q}(M)$ since it acts trivially on M. The adjoint and coadjoint actions of Γ on $\mathfrak{g}$ and $\mathfrak{g}^*$ are trivial, so Γ also acts trivially on $\mathcal{W}(\mathfrak{g})$. Thus the G invariance and the G/Γ invariance conditions on $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M)$ and $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ are the same. Since G and G/Γ have the same Lie algebra, the $\mathfrak{sg}[t]$ basic condition is also the same. □

2. Mayer-Vietoris Sequences

In this section, we show that for G-invariant open sets $U, V \subset M$, there exist Mayer-Vietoris sequences

$$\cdots \to \mathbf{H}^p_G(\mathcal{Q}'(U \cup V)) \to \mathbf{H}^p_G(\mathcal{Q}'(U)) \oplus \mathbf{H}^p_G(\mathcal{Q}'(V)) \to \mathbf{H}^p_G(\mathcal{Q}'(U \cap V)) \to \cdots,$$

$$\cdots \to \mathbf{H}^p_G(\mathcal{Q}(U \cup V)) \to \mathbf{H}^p_G(\mathcal{Q}(U)) \oplus \mathbf{H}^p_G(\mathcal{Q}(V)) \to \mathbf{H}^p_G(\mathcal{Q}(U \cap V)) \to \cdots.$$

Lemma 2.1. *Let M be a manifold and let $\mathfrak{h}$ be a Lie algebra. Suppose that $\mathcal{F}$ a sheaf of C^∞ modules and of $\mathfrak{h}$ modules on M, where two module structures are compatible, i.e. $\mathfrak{h}$ acts compatibly on the sheaf C^∞ by derivations. Assume that U, V are open sets, and that $\phi_U, \phi_V \in C^\infty(M)$ form an $\mathfrak{h}$ invariant partition of unity for the cover $\{U, V\}$ of $U \cup V$. If the invariant functor $(-)^{\mathfrak{h}}$ is applied to the standard exact sequence*

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

the result is an exact sequence.

Proof: Note that the left exactness of the standard sequence is just the sheaf axiom, but the surjectivity of the last map is not true for a general sheaf unless it is a fine sheaf (i.e. has the partition of unity property for sheaves.) Since $\mathcal{F}$ is assumed to be a C^∞ sheaf, the exactness of our standard sequence is guaranteed.

Since the invariant functor $(-)^{\mathfrak{h}}$ is left exact, applying it to the standard sequence yields a left exact sequence. So it remains to show that

$$\mathcal{F}(U)^{\mathfrak{h}} \oplus \mathcal{F}(V)^{\mathfrak{h}} \to \mathcal{F}(U \cap V)^{\mathfrak{h}} \tag{2.1}$$

is onto. Since the C^∞ and $\mathfrak{h}$ structures on $\mathcal{F}$ are compatible, the map is a C^∞ module map, i.e. it is compatible with multiplications by functions. Let $a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ and let $\{\phi_U, \phi_V\}$ be an $\mathfrak{h}$-invariant partition of unity subordinate to the cover $\{U, V\}$ of $U \cup V$.

We claim that there is an extension of $(\phi_U|U \cap V)a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ by zero to all of V (but not to all of U). We have

$$(U \cap V) \cup (V \backslash supp(\phi_U)) = V.$$

For if $x \in V\backslash(U \cap V)$ then $x \notin supp(\phi_U) \subset U$, and so x lies on the left side. Now to see that $(\phi_U|U \cap V)a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ and the zero $0 \in \mathcal{F}(V\backslash supp(\phi_U))^{\mathfrak{h}}$ glue together to form a section in $\mathcal{F}(V)^{\mathfrak{h}}$, it is enough to check that $(\phi_U|U \cap V)a$ restricts to zero on the overlap $W = (U \cap V) \cap (V\backslash supp(\phi_U))$. This restriction is equal to $(\phi_U|W)(a|W)$ since function multiplication commutes with restriction. But $\phi_U|W = 0$ because $W \cap supp(\phi_U) = \varnothing$. This proves our claim. Call this extension $a_V \in \mathcal{F}(V)^{\mathfrak{h}}$. Likewise let $a_U \in \mathcal{F}(U)^{\mathfrak{h}}$ be the extension of $(\phi_V|U \cap V)a \in \mathcal{F}(U \cap V)^{\mathfrak{h}}$ by zero to all of U.

Now under (2.1) we have

$$(a_V, -a_U) \mapsto (a_V|U\cap V)+(a_U|U\cap V) = (\phi_U|U\cap V)a+(\phi_V|U\cap V)a = [(\phi_U+\phi_V)|U\cap V]a = a$$

since $\phi_U + \phi_V = 1$. This proves that (2.1) is onto. $\square$

Remark 2.2. *This result holds if we replace "sheaf" with "weak sheaf" since the reconstruction axiom (1.5) holds for finite covers.*

Theorem 2.3. *Let M be a G-manifold and let U, V be G-invariant open sets in M. Then $U \cap V \rightrightarrows U \coprod V \to U \cup V$ induces a long exact sequence*

$$\cdots \to \mathbf{H}^p_G(\mathcal{Q}'(U \cup V)) \to \mathbf{H}^p_G(\mathcal{Q}'(U)) \oplus \mathbf{H}^p_G(\mathcal{Q}'(V)) \to \mathbf{H}^p_G(\mathcal{Q}'(U \cap V)) \to \cdots.$$

Proof: Regard $\mathcal{W} = \mathcal{W}(\mathfrak{g})$ as a constant sheaf of vector spaces over M. Then $\mathcal{W} \otimes \mathcal{Q}'$ is a weak sheaf of C^∞ modules where functions act only on the right factor. It is a weak sheaf of modules over the Lie algebra $\mathfrak{sg}[t]$ as shown in Section 3 of [11]. Choose a partition of unity ϕ_U, ϕ_V of $U \cap V$ as before. Since U, V are G invariant sets, by averaging over G, we can assume that the two functions are G-invariant. Note that even though $\mathfrak{sg}[t]$ does not act by derivations on a general element of $\mathcal{W} \otimes \mathcal{Q}'$, $\mathfrak{sg}[t]$ does act by derivations on weight zero elements. Moreover, any G-invariant function f, regarded as $1 \otimes f \in \mathcal{W} \otimes \mathcal{Q}'$, is chiral basic (i.e. $\mathfrak{sg}[t]$-invariant). So ϕ_U, ϕ_V form a $\mathfrak{sg}[t]$-invariant partition of unity. Hence the preceding lemma can be applied to $\mathcal{F} = \mathcal{W} \otimes \mathcal{Q}'$ and $\mathfrak{h} = \mathfrak{sg}[t]$. The invariant functor applied to the standard sequence for $\mathcal{F}$ yields an exact sequence

$$0 \to \mathcal{F}(U \cup V)_{bas} \to \mathcal{F}(U)_{bas} \oplus \mathcal{F}(V)_{bas} \to \mathcal{F}(U \cap V)_{bas} \to 0,$$

which induces the corresponding long exact sequence for chiral equivariant cohomology. $\square$

The theorem holds if we replace $\mathcal{Q}'$ by $\mathcal{Q}$. The only tricky part here is that $\mathcal{Q}$ is no longer a C^∞ module because the Wick product on $\mathcal{Q}$ is not associative. But we still have a $\mathbf{C}$ bilinear operation $C^\infty \times \mathcal{Q} \to \mathcal{Q}$, $(f, a) \mapsto: fg :$ which is a homomorphism of weak sheaves. Moreover, even though $\mathcal{Q}$ is not functorial under general smooth mappings, it is functorial with respect to open inclusions. A partition of unity argument shows that the standard sequence for $\mathcal{Q}$ is still exact. The proof of the preceding lemma then carries over to the case $\mathcal{F} = \mathcal{W} \otimes \mathcal{Q}$.

3. Homotopy Invariance of $\mathbf{H}^*_G(\mathcal{Q}'(M))$

Let M and N be G-manifolds, and let $\phi_0, \phi_1 : M \to N$ be G-equivariant maps. Let I denote the interval $[0, 1]$, which we regard as a G-manifold equipped with the trivial action.

Definition 3.1. *A G-equivariant homotopy from ϕ_0 to ϕ_1 is a smooth G-equivariant map $\Phi : M \times I \to N$ such that for all $x \in M$, $\Phi(x, 0) = \phi_0(x)$ and $\Phi(x, 1) = \phi_1(x)$. For each $t \in I$, $\phi_t : M \to N$ will denote the map $\phi_t(x) = \Phi(x, t)$.*

The main result in this section is

Theorem 3.2. *Let M and N be G-manifolds, and let $\phi_0, \phi_1 : M \to N$ be G-equivariant maps. If there exists a G-equivariant homotopy Φ from ϕ_0 to ϕ_1 as above, the induced maps $\phi_0^*, \phi_1^* : \mathbf{H}^*_G(\mathcal{Q}'(N)) \to \mathbf{H}^*_G(\mathcal{Q}'(M))$ are the same.*

We first define an appropriate notion of chiral chain homotopy in the category of $s\mathfrak{g}[t]$-modules, and show that two morphisms of $s\mathfrak{g}[t]$-modules which are chiral chain homotopic induce the same map in chiral equivariant cohomology. In the geometric setting, if Φ is a G-equivariant homotopy between $\phi_0, \phi_1 : M \to N$, we will construct a chiral chain homotopy between the induced maps $\phi_0^*, \phi_1^* : \mathcal{Q}'(N) \to \mathcal{Q}'(M)$.

An immediate consequence of Theorem 3.2 is that if M is G-equivariantly contractible to a submanifold M', then $\mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_G(\mathcal{Q}'(M'))$. By contrast, the functor $\mathbf{H}^*_G(\mathcal{Q}(-))$ does *not* have this property. Let G be simple and let V be a faithful linear representation of G. Then V is G-equivariantly contractible to the origin $o \in V$, but $\mathbf{H}^*_G(\mathcal{Q}(V))_+ = 0$ whereas $\mathbf{H}^*_G(\mathcal{Q}(o))_+ = \mathbf{H}^*_G(\mathbf{C})_+ \neq 0$.

3.1. Chiral chain homotopies

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $s\mathfrak{g}[t]$-modules. A *chiral chain homotopy* from $\mathcal{A}$ to $\mathcal{B}$ is a linear map $P : \mathcal{A} \to \mathcal{B}$, homogeneous of weight 0 and degree -1, which is G-equivariant and satisfies

$$P\iota^{\mathcal{A}}_\xi(k) + \iota^{\mathcal{B}}_\xi(k)P = 0, \qquad PL^{\mathcal{A}}_\xi(k) - L^{\mathcal{B}}_\xi(k)P = 0, \tag{3.1}$$

for all $\xi \in \mathfrak{g}$ and $k \geq 0$.

Lemma 3.3. *If $P : \mathcal{A} \to \mathcal{B}$ be a chiral chain homotopy, the map $\tau = Pd_{\mathcal{A}} + d_{\mathcal{B}}P$ is a morphism of $s\mathfrak{g}[t]$-modules.*

Proof: The argument is similar to the proof of Proposition 2.3.1 in [7]. First note that $d_{\mathcal{B}}\tau = d_{\mathcal{B}}Pd_{\mathcal{A}} = \tau d_{\mathcal{A}}$. From the assumption $L^{\mathcal{B}}_\xi(k)P = PL^{\mathcal{A}}_\xi(k)$, it is immediate that

$$L^{\mathcal{B}}_\xi(k)\tau = \tau L^{\mathcal{A}}_\xi(k).$$

Finally,

$$\iota^{\mathcal{B}}_\xi(k)\tau = \iota^{\mathcal{B}}_\xi(k)d_{\mathcal{B}}P + \iota^{\mathcal{B}}_\xi(k)Pd_{\mathcal{A}} = \iota^{\mathcal{B}}_\xi(k)d_{\mathcal{B}}P - P\iota^{\mathcal{A}}_\xi(k)d_{\mathcal{A}}$$

$$= -d_{\mathcal{B}}\iota^{\mathcal{B}}_{\xi}(k)P + L^{\mathcal{B}}_{\xi}(k)P + Pd_{\mathcal{A}}\iota^{\mathcal{A}}_{\xi}(k) - PL^{\mathcal{A}}_{\xi}(k) = (d_{\mathcal{B}}P + Pd_{\mathcal{A}})\iota^{\mathcal{A}}_{\xi}(k) = \tau\iota^{\mathcal{A}}_{\xi}(k). \square$$

Two $\mathfrak{sg}[t]$-module homomorphisms $\phi_0, \phi_1 : \mathcal{A} \to \mathcal{B}$ are said to be chiral chain homotopic if there is a chiral chain homotopy $P : \mathcal{A} \to \mathcal{B}$ such that $\phi_1 - \phi_0 = \tau$. This clearly implies that ϕ_0, ϕ_1 induce the same map from $\mathbf{H}^*_{bas}(\mathcal{A}) \to \mathbf{H}^*_{bas}(\mathcal{B})$.

Lemma 3.4. *Chiral chain homotopic maps $\phi_0, \phi_1 : \mathcal{A} \to \mathcal{B}$ induce the same map from $\mathbf{H}^*_G(\mathcal{A}) \to \mathbf{H}^*_G(\mathcal{B})$.*

Proof: This is analogous to Proposition 2.4.1 of [7]. The map $id \otimes P : \mathcal{W} \otimes \mathcal{A} \to \mathcal{W} \otimes \mathcal{B}$ is a chiral chain homotopy between $id \otimes \phi_0$, $id \otimes \phi_1$. Hence $id \otimes \phi_0$, $id \otimes \phi_1$ induce the same map from $\mathbf{H}^*_G(\mathcal{A}) = \mathbf{H}^*_{bas}(\mathcal{W} \otimes \mathcal{A}) \to \mathbf{H}^*_{bas}(\mathcal{W} \otimes \mathcal{B}) = \mathbf{H}^*_G(\mathcal{B})$. $\square$

Suppose that $\phi_0, \phi_1 : M \to N$ are G-equivariantly homotopic via $\Phi : M \times I \to N$. We recall the classical construction of a chain homotopy $P : \Omega^*(N) \to \Omega^{*-1}(M)$ between the maps $\phi_0^*, \phi_1^* : \Omega^*(N) \to \Omega^*(M)$, following [7]. For fixed $x \in M$, consider the curve in N given by $s \mapsto \phi_s(x)$, and let $\xi_t : M \to TN$ be the map which assigns to x the tangent vector to this curve at $s = t$. Consider the map

$$f_t : \Omega^*(N) \to \Omega^{*-1}(M), \qquad \sigma \mapsto \phi_t^*(\iota_{\xi_t}(\sigma)). \tag{3.2}$$

At each $x \in M$, this map is defined by

$$\phi_t^*(\iota_{\xi_t}(\sigma))(\eta_1, \dots, \eta_k) = \sigma(\xi_t(x), d\phi_t(\eta_1), \dots, d\phi_t(\eta_k)), \tag{3.3}$$

given vectors $\eta_1, \dots, \eta_k \in TM_x$.

Let $\phi : A \to B$ be a map of G^*-algebras. A degree-homogeneous map $f : A \to B$ is said to be a ϕ-derivation if

$$f(ab) = f(a)\phi(b) + (-1)^{(deg\ f)(deg\ a)}\phi(a)f(b), \tag{3.4}$$

for all homogeneous $a, b \in \mathcal{A}$. Taking $A = \Omega(N)$, $B = \Omega(M)$, $\phi = \phi_t^*$, the map f_t is a ϕ_t^*-derivation.

A well-known formula (see [7]) asserts that

$$\frac{d}{dt}\phi_t^*\sigma = \phi_t^*(\iota_{\xi_t}(d\sigma) + d(\phi_t^*(\iota_{\xi_t}(\sigma)). \tag{3.5}$$

Define $P : \Omega^*(N) \to \Omega^{*-1}(M)$ by $P\sigma = \int_0^1 \phi_t^*(\iota_{\xi_t}(\sigma))\, dt$. Integrating (3.5) over I shows that $Pd+dP = \phi_1^* - \phi_0^*$, so P is a chain homotopy. Since ϕ_0, ϕ_1, Φ are G-equivariant maps, f_t is G-equivariant and satisfies $f_t\iota_\xi^N + \iota_\xi^M f_t = 0$, for all $t \in I$. It follows that P is also G-equivariant and satisfies $P\iota_\xi^N + \iota_\xi^M P = 0$. Hence P is a chain homotopy in the sense of [7], and ϕ_0, ϕ_1 induce the same maps in equivariant cohomology.

We need to show that P extends to a linear map $\mathbf{P} : \mathcal{Q}'(N) \to \mathcal{Q}'(M)$ which is a chiral chain homotopy between $\phi_0^*, \phi_1^* : \mathcal{Q}'(N)) \to \mathcal{Q}'(M))$. By Lemma 3.2 of [11], for each $m \geq 0$, we may regard $\mathcal{Q}'_M[m]$ as a smooth vector bundle over M of finite rank, which has a local trivialization induced from a collection of charts on M. Given a coordinate open set $U \subset M$ with coordinates $\gamma^1, \dots, \gamma^n$, $U \times V$ is a local trivialization of $\mathcal{Q}'_M$, where V is the vector space with basis consisting of all nonzero monomials of the form

$$\partial^{r_1}\gamma^{i_1} \cdots \partial^{r_k}\gamma^{i_k}\partial^{s_1}c^{j_1} \cdots \partial^{s_l}c^{j_l}, \tag{3.6}$$

where $r_1 + \cdots + r_k + s_1 + \cdots + s_l = m$, and each $r_i > 0$ and $s_i \geq 0$. Here c^i denotes the coordinate one-form $d_{\mathcal{Q}}\gamma^i$.

Let $\pi : M \times I \to M$ be the projection onto the first factor. We can pull back $\mathcal{Q}'_M[m]$ to a vector bundle $\pi^*(\mathcal{Q}'_M[m]) \to M \times I$. Let $\Gamma[m] = \Gamma(M \times I, \pi^*(\mathcal{Q}'_M[m]))$ denote the space of smooth sections

$$\sigma : M \times I \to \mathcal{Q}'_M[m], \qquad \sigma(x,t) = (x, v(x,t)),$$

where in local coordinates $v(x,t) = \sum_{j\in J} f_j(\gamma^1, \dots, \gamma^n, t)\mu_j$. Here the set J indexes all monomials of the form (3.6), and each f_j is a smooth function on $U \times I$.

Note that $\Gamma = \oplus_{m\geq 0}\Gamma[m]$ is an $\mathfrak{sg}[t]$-algebra, and that $\frac{d}{dt}$ and ∂ are derivations on Γ. It is clear from the local description of $\sigma(x,t)$ that

$$\frac{d}{dt}\partial(\sigma(x,t)) = \partial(\frac{d}{dt}\sigma(x,t)).$$

Furthermore, the (fiberwise) integral $\int_0^1 \sigma(x,t)dt$ is a well-defined map from $\Gamma \to \Gamma(M, \mathcal{Q}'_M)$, and $\int_0^1 \partial\sigma(x,t)dt = \partial\int_0^1 \sigma(x,t)dt$.

Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism of $\mathfrak{sg}[t]$-algebras. A degree-weight homogeneous linear map $f : \mathcal{A} \to \mathcal{B}$ will be called a ϕ-derivation if

$$f(a \circ_n b) = f(a) \circ_n \phi(b) + (-1)^{(deg\ f)(deg\ a)} \phi(a) \circ_n f(b), \tag{3.7}$$

for all homogeneous $a, b \in \mathcal{A}$ and $n \in \mathbf{Z}$. Clearly $f(1) = 0$ and $f(\partial a) = \partial f(a)$ for all $a \in \mathcal{A}$. If $\mathcal{A}, \mathcal{B}$ are abelian vertex algebras, to check that f is a ϕ-derivation, it is enough to show that for all $a, b \in \mathcal{A}$,

$$f(: ab :) = \, : f(a)\phi(b) : \ + (-1)^{(deg\ f)(deg\ a)} : \phi(a) f(b) : , \qquad f(\partial a) = \partial f(a). \tag{3.8}$$

Remark 3.5. *A ϕ-derivation f is determined by its values on a set of generators of $\mathcal{A}$. In the case $\mathcal{A} = \mathcal{Q}'(N)$, $\mathcal{B} = \mathcal{Q}'(M)$, $\phi = \phi_t^*$, since $\mathcal{Q}'(N)$ is generated by $\Omega(N)$, any two ϕ_t^*-derivations which agree on $\Omega(N)$ must agree on all of $\mathcal{Q}'(N)$.*

Remark 3.6. *Suppose that f is a ϕ-derivation and $\delta_{\mathcal{A}}, \delta_{\mathcal{B}}$ are vertex algebra derivations on $\mathcal{A}, \mathcal{B}$, respectively, which are homogeneous of degree d and satisfy $\phi \circ \delta_{\mathcal{A}} = \delta_{\mathcal{B}}$. Then $f \circ \delta_{\mathcal{A}} - (-1)^{(deg\ f)(d)} \delta_{\mathcal{B}} \circ f$ is also a ϕ-derivation.*

For example, for any $\xi \in \mathfrak{g}$, the operators $\iota_\xi^{\mathcal{A}}(0)$, $\iota_\xi^{\mathcal{B}}(0)$ are vertex algebra derivations of degree -1, and $L_\xi^{\mathcal{A}}(0)$, $L_\xi^{\mathcal{B}}(0)$ are vertex algebra derivations of degree 0. Hence

$$f \circ \iota_\xi^{\mathcal{A}}(0) - (-1)^{(deg\ f)} \iota_\xi^{\mathcal{B}}(0) \circ f, \qquad f \circ L_\xi^{\mathcal{A}}(0) - L_\xi^{\mathcal{B}}(0) \circ f$$

are ϕ_t^*-derivations. If $\mathcal{A}$ and $\mathcal{B}$ are abelian vertex algebras, the operators $\iota_\xi^{\mathcal{A}}(k)$, $\iota_\xi^{\mathcal{B}}(k)$, $L_\xi^{\mathcal{A}}(k)$, $L_\xi^{\mathcal{B}}(k)$ are vertex algebra derivations for all $k \geq 0$, so

$$f \circ \iota_\xi^{\mathcal{A}}(k) - (-1)^{(deg\ f)} \iota_\xi^{\mathcal{B}}(k) \circ f, \qquad f \circ L_\xi^{\mathcal{A}}(k) - L_\xi^{\mathcal{B}}(k) \circ f$$

are ϕ_t^*-derivations.

Lemma 3.7. *There is a unique extension of the map $f_t : \Omega^*(N) \to \Omega^{*-1}(M)$ defined by (3.2) to a linear map $F_t : \mathcal{Q}'^*(N) \to \mathcal{Q}'^{*-1}(M)$, which is a ϕ_t^*-derivation.*

Proof: We first construct F_t locally. On a coordinate open set $U \subset N$ with coordinates $\gamma^1, \dots, \gamma^n$, recall that $\mathcal{Q}'(U)$ is the abelian vertex algebra with generators $f \in \mathcal{C}^\infty(U)$ and c^i, $i = 1, \dots, n$, subject to the relations:

$$1(z) - 1, \qquad (fg)(z) - f(z)g(z), \qquad \partial f(z) - \frac{\partial f}{\partial \gamma^i}(z) \partial \gamma^i(z), \tag{3.9}$$

for all $f, g \in \mathcal{C}^\infty(U)$. We define F_t on generators by $F_t(f(\gamma^1,\dots,\gamma^n)) = 0$ and $F_t(c^i) = f_t(c^i)$, and then extend F_t to a linear map on all of $\mathcal{Q}'(U)$ using the ϕ_t^*-derivation property (3.8). Since the relations (3.9) are all homogeneous of degree 0 and F_t lowers degree by one, it is clear that F_t annihilates these relations, and hence is well-defined. Using the fact that $F_t(f(\gamma^1,\dots,\gamma^n)) = 0$ for any $f \in \mathcal{C}^\infty(U)$, it is easy to check that the definition of F_t is coordinate-independent.

Finally, cover N with coordinate open sets $\{U_i\}$, and define $F_t|_{U_i}$ as above. Fix a partition of unity $\{\psi_i\}$ subordinate to this covering. For each $m \geq 0$, we define $F_t : \mathcal{Q}'(N)[m] \to \mathcal{Q}'(M)[m]$ by $F_t = \sum_i \psi_i F_t|_{U_i}$, which is well-defined since $\mathcal{Q}'(N)[m]$ is a fine sheaf. Moreover, F_t still satisfies (3.8) on each $\mathcal{Q}'(N)[m]$ because $F_t(\psi_i) = 0$. Finally, since $\mathcal{Q}'(N) = \sum_{m\geq 0} \mathcal{Q}'(N)[m]$, this defines F_t on all of $\mathcal{Q}'(N)$. $\square$

Lemma 3.8. *Equation (3.5) holds for any σ in $\mathcal{Q}'(N)$, not just $\Omega(N)$.*

Proof: By the preceding lemma, both sides of (3.5) are well-defined. It suffices to show that it holds for $t = s$ for each $s \in I$. Let

$$g = \frac{d}{dt}\phi_t^*|_{t=s}(-), \qquad h = \phi_s^*(\iota_{\xi_s}(0)d(-)) + d\phi_s^*(\iota_{\xi_s}(0)(-)),$$

which are the maps from $\mathcal{Q}'(N) \to \mathcal{Q}'(M)$ appearing on the left and right sides of (3.5), evaluated at $t = s$. Clearly g and h are both ϕ_s^*-derivations, and since g and h agree on generators, they agree on all of $\mathcal{Q}'(N)$ by Remark 3.5. $\square$

We now define $\mathbf{P} : \mathcal{Q}'(N) \to \mathcal{Q}'(M)$ by $\mathbf{P}(\sigma) = \int_0^1 F_t(\sigma)dt$, which coincides with P at weight zero. Integration of (3.5) over I shows that $d\mathbf{P} + \mathbf{P}d = \phi_1^* - \phi_0^*$.

Finally, we need to show that the map $\mathbf{P}$ constructed above is in fact a chiral chain homotopy. Recall that for all $\sigma \in \Omega(N)$, $\xi \in \mathfrak{g}$, and $t \in I$, f_t satisfies

$$f_t L_\xi^N - L_\xi^M f_t = 0, \qquad f_t \iota_\xi^N + \iota_\xi^M f_t = 0. \tag{3.10}$$

For $\xi \in \mathfrak{g}$ and $k \geq 0$, consider the maps

$$R_{t,\xi,k} = F_t L_\xi^N(k) - L_\xi^M(k) F_t, \qquad S_{t,\xi,k} = F_t \iota_\xi^N(k) + \iota_\xi^M(k) F_t.$$

By Remark 3.6, $R_{t,\xi,k}$ and $S_{t,\xi,k}$ are ϕ_t^*-derivations from $\mathcal{Q}'(N) \to \mathcal{Q}'(M)$, which are homogeneous of weight $-k$ and degree -1 and -2, respectively. For $k > 0$, $R_{t,\xi,k}$ and $S_{t,\xi,k}$ both act by zero on $\mathcal{Q}'(N)[0]$ by weight considerations. For $k = 0$, $R_{t,\xi,k}$ and $S_{t,\xi,k}$ act by zero on $\mathcal{Q}'(N)[0]$ by (3.10). Since $R_{t,\xi,k}$ and $S_{t,\xi,k}$ are ϕ_t^*-derivations, it follows from Remark 3.5 that they act by zero on all of $\mathcal{Q}'(N)$.

Finally, since this holds for each $t \in I$, it is immediate that

$$\mathbf{P}L_\xi^N(k)(\sigma) - L_\xi^M(k)\mathbf{P}(\sigma) = \int_0^1 R_{t,\xi,k}(\sigma) = 0, \quad \mathbf{P}\iota_\xi^N(k)(\sigma) + \iota_\xi^M(k)\mathbf{P}(\sigma) = \int_0^1 S_{t,\xi,k}(\sigma) = 0,$$

for all $\xi \in \mathfrak{g}$, $k \geq 0$ and $\sigma \in \mathcal{Q}'(N)$. Hence $\mathbf{P}$ is a chiral chain homotopy, as desired. □

4. A Quasi-conformal Structure on $\mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathbf{H}_G^*(\mathcal{Q}(M))$

When G is semisimple, $\mathbf{H}_G^*(\mathbf{C})$ is a conformal vertex algebra with Virasoro element $\mathbf{L}$ of central charge 0. For any M, $\mathbf{H}_G^*(\mathcal{Q}'(M))$ is a conformal vertex algebra with Virasoro element $\kappa_G(\mathbf{L})$. The vanishing of $\kappa_G(\mathbf{L})$ is a necessary and sufficient condition for the vanishing of $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ since $\kappa_G(\mathbf{L}) \circ_1 \omega = n\omega$ for all $\omega \in \mathbf{H}_G^*(\mathcal{Q}'(M))[n]$. Unfortunately, $\mathbf{H}_G^*(\mathcal{Q}'(M))$ is not a conformal vertex algebra when G has a positive-dimensional center, so a priori we have no such vanishing criterion for general G.

In this section we show that for any G and M, both $\mathbf{H}_G^*(\mathcal{Q}'(M))$ and $\mathbf{H}_G^*(\mathcal{Q}(M))$ have a *quasi-conformal* structure, that is, an action of the subalgebra generated by $\{L_n | \, n \geq -1\}$ of the Virasoro algebra, such that $L_{-1} = L\circ_0$ acts by ∂ and $L_0\omega = L \circ_1 \omega = n\omega$ for $\omega \in \mathbf{H}_G^*(\mathcal{Q}'(M))[n]$. Thus we have a similar vanishing criterion for $\mathbf{H}_G^*(\mathcal{Q}'(M))_+$ and $\mathbf{H}_G^*(\mathcal{Q}(M))_+$ for any G; it suffices to show that $L\circ_1$ acts by zero.

We work in the setting of a general $O(\mathfrak{sg})$ topological vertex algebra (TVA), which we defined in [11]. Recall that an $O(\mathfrak{sg})$ TVA is a degree-weight graded DVA $(\mathcal{A}, d)$ equipped with an $O(\mathfrak{sg})$-structure $(\xi, \eta) \mapsto L_\xi^{\mathcal{A}} + \iota_\eta^{\mathcal{A}}$, a chiral horizontal element $g^{\mathcal{A}}$, such that $L^{\mathcal{A}} = dg^{\mathcal{A}}$ is a conformal structure, with respect to which the $L_\xi^{\mathcal{A}}$ and $\iota_\eta^{\mathcal{A}}$ are primary of weight one. We call $g^{\mathcal{A}}$ a chiral contracting homotopy of $\mathcal{A}$. Given an $O(\mathfrak{sg})$ TVA $(\mathcal{A}, d)$, a differential vertex subalgebra $\mathcal{B}$ is called a half $O(\mathfrak{sg})$ TVA if the nonnegative Fourier modes of the vertex operators $\iota_\xi^{\mathcal{A}}$ and $g^{\mathcal{A}}$ preserve $\mathcal{B}$. Note that the nonnegative Fourier modes of $L_\xi^{\mathcal{A}} = d\iota_\xi^{\mathcal{A}}$ and $L^{\mathcal{A}} = dg^{\mathcal{A}}$ automatically preserve $\mathcal{B}$ as well. In particular, the action of $\{L^{\mathcal{A}} \circ_n \mid n \geq 0\}$ is a quasi-conformal structure on $\mathcal{B}$. Since $[d, g^{\mathcal{A}}\circ_1] = L^{\mathcal{A}}\circ_1$ and $g^{\mathcal{A}}\circ_1$ acts on $\mathcal{B}_{bas}$, Theorem 4.8 of [11] shows that $\mathbf{H}^*_{bas}(\mathcal{B})$ vanishes beyond weight zero.

For a G-manifold M, $\mathcal{Q}(M)$ is our main example of an $O(\mathfrak{sg})$ TVA. In local coordinates,

$$g = g^M = b^i\partial\gamma^i, \qquad L = L^M = \beta^i\partial\gamma^i - b^i\partial c^i. \tag{4.1}$$

The subalgebra $\mathcal{Q}'(M)$ is then a half $O(\mathfrak{sg})$ TVA as above. Recall that the semi-infinite Weil algebra $\mathcal{W}$ is *not* an $O(\mathfrak{sg})$ TVA since there is no chiral horizontal element $g^{\mathcal{W}}$ satisfying $dg^{\mathcal{W}} = L^{\mathcal{W}}$.

Let $\mathcal{B}$ be a half $O(\mathfrak{sg})$ TVA inside some $O(\mathfrak{sg})$ TVA $\mathcal{A}$ as above. Then the non-negative Fourier modes of

$$L^{tot} = L^{\mathcal{A}} \otimes 1 + 1 \otimes L^{\mathcal{W}} \in \mathcal{A} \otimes \mathcal{W}$$

act on $\mathcal{B} \otimes \mathcal{W}$, giving $\mathcal{B} \otimes \mathcal{W}$ a quasi-conformal structure. Moreover,

$$L_\xi^{tot} = L_\xi^{\mathcal{A}} \otimes 1 + 1 \otimes L_\xi^{\mathcal{W}}, \qquad \iota_\xi^{tot} = \iota_\xi^{\mathcal{A}} \otimes 1 + 1 \otimes \iota_\xi^{\mathcal{W}}$$

are primary of weight one with respect to L^{tot}, and $dL^{tot} = 0$. By Theorem 4.8 of [11], it follows that $L^{tot}\circ_n$ operates on $\mathbf{H}^*_G(\mathcal{B})$ for $n \geq 0$, and gives $\mathbf{H}^*_G(\mathcal{B})$ a quasi-conformal structure. Note that if G is semisimple, $\mathbf{L}\circ_n = L^{tot}\circ_n$ as operators on $\mathbf{H}^*_G(\mathcal{B})$ for all $n \geq 0$ (see Theorem 4.17 of [11]).

4.1. A vanishing criterion

For any compact G, and any $\mathcal{A}$ and $\mathcal{B}$ as above, $L^{tot}\circ_1$ acts diagonalizably on both $\mathbf{H}^*_G(\mathcal{A})$ and $\mathbf{H}^*_G(\mathcal{B})$ with eigenvalues given by the weight grading on $\mathcal{A}$, $\mathcal{B}$, respectively. Hence the vanishing of $L^{tot}\circ_1$ on $\mathbf{H}^*_G(\mathcal{B})$ (resp. $\mathbf{H}^*_G(\mathcal{A})$) is equivalent to the vanishing of $\mathbf{H}^*_G(\mathcal{B})_+$ (resp. $\mathbf{H}^*_G(\mathcal{A})_+$). The next lemma gives a useful vanishing criterion for $L^{tot}\circ_1$.

Lemma 4.1. *Suppose that $\alpha \in \mathcal{W} \otimes \mathcal{B}$ is homogeneous of weight 2 and degree -1, is G-invariant, chiral horizontal, and satisfies*

$$L_\xi \circ_1 \alpha = \beta^\xi \otimes 1, \tag{4.2}$$

for all $\xi \in \mathfrak{g}$. *Then* $L^{tot}\circ_1$ *acts by zero on* $\mathbf{H}^*_G(\mathcal{B})$, *and we have* $\mathbf{H}^*_G(\mathcal{B})_+ = 0$.

Proof: Recall from [10] that $d(\beta^{\xi_i}\partial c^{\xi'_i}) = L^{\mathcal{W}}$, but $\beta^{\xi_i}\partial c^{\xi'_i}$ is not chiral horizontal since $\iota_\xi \circ_n (\beta^{\xi_i}\partial c^{\xi'_i}) = -\delta_{n,1}\beta^\xi$ for $n \geq 0$. Let $\omega_0 = \beta^{\xi_i}\partial c^{\xi'_i} + d\alpha$. Clearly $d\omega_0 = L^{\mathcal{W}}$ and

$$\iota_\xi \circ_n (d\alpha) = L_\xi \circ_n \alpha = \delta_{n,1}\beta^\xi \otimes 1 = -\iota_\xi \circ_n (\beta^{\xi_i}\partial c^{\xi'_i})$$

for $n \geq 0$, since α is chiral horizontal. It follows that $\iota_\xi \circ_n \omega_0 = 0$ for all $n \geq 0$, so ω_0 is chiral horizontal. In particular, the non-negative Fourier modes of ω_0 act on $(\mathcal{W}\otimes\mathcal{B})_{bas}$. Finally, let $\omega = \omega_0 + g^{\mathcal{A}} \in \mathcal{W}\otimes\mathcal{A}$. Since $dg^{\mathcal{A}} = L^{\mathcal{A}}$ we have $d\omega = L^{tot}$. The non-negative Fourier modes of ω clearly preserve $(\mathcal{W}\otimes\mathcal{B})_{bas}$ since both ω_0 and $g^{\mathcal{A}}$ have this property. In particular, $[d, \omega\circ_1] = L^{tot}\circ_1$, so $\omega\circ_1$ is a contracting homotopy for $L^{tot}\circ_1$, as desired. □

This lemma clearly holds if we replace $\mathcal{B}$ with $\mathcal{A}$. Note that when G is semisimple, the existence of α as above guarantees that $\kappa_G(\mathbf{L}) = 0$; take $\omega = \beta^{\xi_i}\partial c^{\xi'_i}\otimes 1 + d\alpha + \theta^{\xi_i}_{\mathcal{S}} b^{\xi_i}\otimes 1$. An OPE calculation shows that ω is chiral basic and $d\omega = \mathbf{L}$.

In [11], we considered two situations where we can construct α as above. First, if G acts locally freely on M, we have a map $\mathfrak{g}^* \to \Omega^1(M)$ sending $\xi' \to \theta^{\xi'}$, such that $\iota_\xi\theta^{\eta'} = \langle \eta', \xi\rangle$. The $\theta^{\xi'}$ are known as *connection one-forms.* Choose an orthonormal basis $\{\xi_i\}$ for $\mathfrak{g}$ relative to the Killing form, and let $\Gamma^{\xi'_i} = g \circ_0 \theta^{\xi'_i}$. Then

$$\alpha = \beta^{\xi_i} \otimes \Gamma^{\xi'_i} \in \mathcal{W}\otimes\mathcal{Q}'(M)$$

is G-invariant, chiral horizontal, and satisfies (4.2). This shows that $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = 0$ and $\mathbf{H}^*_G(\mathcal{Q}(M))_+ = 0$.

Second, let V be a faithful linear representation of G, and let $\rho : \mathfrak{g} \to End(V)$ denote the corresponding representation of $\mathfrak{g}$. The induced bilinear form $\langle \xi, \eta\rangle = Tr(\rho(\xi)\rho(\eta))$ is nondegenerate, so we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via $\langle , \rangle$ and fix an orthonormal basis ξ_i of $\mathfrak{g}$. Let x_k be a basis of V and x'_k the corresponding dual basis for V^*. Define

$$\alpha = \beta^{\xi_i} \otimes \Gamma^{\xi_i} - \beta^{\xi_i} b^{\xi_j} \otimes \iota_{\xi_j} \circ_0 \Gamma^{\xi_i} \in \mathcal{W}\otimes\mathcal{Q}(V), \tag{4.3}$$

where $\Gamma^{\xi_i} = \beta^{\rho(\xi_i)(x_k)}\gamma^{x'_k}$. An OPE calculation shows that α satisfies the conditions of Lemma 4.1, so $\mathbf{H}^*_G(\mathcal{Q}(V))_+ = 0$.

The next lemma shows that locally defined vertex operators α satisfying the conditions of Lemma 4.1 can be glued together.

Lemma 4.2. *Let M be a G-manifold and let $\{U_i|\ i \in I\}$ be a cover of M by G-invariant open sets. Suppose that $\alpha_i \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(U_i)$ satisfies the conditions of Lemma 4.1. Then $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = 0$.*

Proof: Let $\{\phi_i|\ i \in I\}$ be a G-invariant partition of unity subordinate to the cover. Let $\alpha = \sum_i \phi_i \alpha_i$, which is a well-defined global section of $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M)$. Moreover, since ϕ_i is basic, it follows that α remains G-invariant, G-chiral horizontal and satisfies (4.2), as desired. □

Remark 4.3. *Similarly, if $\alpha_i \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(U_i)$ satisfies the conditions of Lemma 4.1, $\alpha = \sum_i \phi_i \alpha_i \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ does as well, so that $\mathbf{H}^*_G(\mathcal{Q}(M))_+ = 0$.*

5. $\mathbf{H}^*_G(\mathcal{Q}'(G/H))$ for Homogeneous Spaces G/H

A basic fact about G-manifolds which can be found in [15] is that locally they look like vector bundles over homogeneous spaces G/H.

Theorem 5.1. *Let G be a compact Lie group and let M be a smooth G-manifold. For each point $x \in M$, the isotropy group G_x is a closed subgroup of G and the orbit Gx is G-diffeomorphic to G/G_x. Moreover, Gx has a G-invariant tubular neighborhood U_x which is G-diffeomorphic to the bundle $G \times_{G_x} V$ for some real G_x-representation V.*

By homotopy invariance, $\mathbf{H}^*_G(\mathcal{Q}'(U_x)) = \mathbf{H}^*_G(\mathcal{Q}'(G/G_x))$. Thus the problem of computing $\mathbf{H}^*_G(\mathcal{Q}'(G/H))$ for any closed subgroup $H \subset G$ is an important building block in the study of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ for general M.

Theorem 5.2. *For any compact, connected Lie group G and closed subgroup $H \subset G$,*

$$\mathbf{H}^*_G(\mathcal{Q}'(G/H)) \cong \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes H^*_{G'}(G/H), \tag{5.1}$$

where K_0 *is the identity component of* $K = Ker(G \to Diff(G/H))$*, and* $G' = G/K_0$*. Here* $H^*_{G'}(G/H)$ *is regarded as a vertex algebra in which all circle products are trivial except* $\circ_{-1}$*, and (5.1) is a vertex algebra isomorphism.*

Theorem 5.2 generalizes Corollary 6.14 of [11], which deals with the case where G is semisimple and H is a torus. Note that $H^*_{G'}(G/H) \cong H^*_{G'}(G'/H')$ where $H' = H/K_0$. Since G' acts almost effectively on G'/H', which is a homogeneous space for G', we may assume without loss of generality that K is finite. In this case, $\mathbf{H}^*_{K_0}(\mathbf{C}) = \mathbf{C}$, so it suffices to prove that $\mathbf{H}^*_G(\mathcal{Q}'(G/H))_+ = 0$.

We need a basic property of simple finite-dimensional complex Lie algebras $\mathfrak{g}$. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra of positive codimension. Via the Killing form, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and in particular we identify $\mathfrak{h}^\perp$ with $(\mathfrak{g}/\mathfrak{h})^*$. Note that $(\mathfrak{g}/\mathfrak{h})^*$ is a representation of $\mathfrak{h}$. Regarding $(\mathfrak{g}/\mathfrak{h})^*$ as a subspace of $\mathfrak{g}^*$, we may consider the subspace $ad^*(\mathfrak{g}/\mathfrak{h})^* \subset \mathfrak{g}^*$ under the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

Lemma 5.3. $ad^*(\mathfrak{g}/\mathfrak{h})^* = \{ad^*_\xi(\eta')|\ \xi \in \mathfrak{g}, \eta' \in (\mathfrak{g}/\mathfrak{h})^*\} = \mathfrak{g}^*$*.*

Proof: If $\mathfrak{h} \subset \mathfrak{h}' \subset \mathfrak{g}$ for some other subalgebra $\mathfrak{h}'$, we have $(\mathfrak{g}/\mathfrak{h}')^* \subset (\mathfrak{g}/\mathfrak{h})^*$. Then $ad^*(\mathfrak{g}/\mathfrak{h}')^* \subset ad^*(\mathfrak{g}/\mathfrak{h})^*$, so we may assume that $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$ for which $\mathfrak{h} \neq \mathfrak{g}$. If $ad^*(\mathfrak{g}/\mathfrak{h})^* \neq \mathfrak{g}^*$, there is some nonzero $\xi_0 \in \mathfrak{g}$ such that

$$\langle \xi_0, ad^*_{\xi_1}\eta'\rangle = 0, \quad \forall \xi_1 \in \mathfrak{g}, \quad \eta' \in (\mathfrak{g}/\mathfrak{h})^* \tag{5.2}$$

Let B denote the set of ξ_0 which satisfy (5.2). Then if $\xi_0 \in B$, for any $\xi_1 \in \mathfrak{g}$ and $\eta' \in (\mathfrak{g}/\mathfrak{h})^*$ we have

$$\langle ad_{\xi_1}\xi_0, \eta'\rangle = 0 \Longleftrightarrow ad^*_{\xi_1}\xi_0 \in \mathfrak{h}. \tag{5.3}$$

Suppose first that $\xi_0 \notin \mathfrak{h}$. Then $\mathfrak{h}' = \mathbf{C}\xi_0 \oplus \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is maximal, $\mathfrak{h}' = \mathfrak{g}$. It follows from (5.3) that $\mathfrak{h}$ is a nontrivial ideal of $\mathfrak{g}$, which contradicts the assumption that $\mathfrak{g}$ is simple. Hence $\xi_0 \in \mathfrak{h}$, so we have $B \subset \mathfrak{h}$.

For any $\xi_1 \in \mathfrak{g}$, we have the decomposition $\xi_1 = \xi_1^{\mathfrak{h}} + \xi_1^{\mathfrak{h}^\perp}$, where $\xi_1^{\mathfrak{h}} \in \mathfrak{h}$ and $\xi_1^{\mathfrak{h}^\perp} \in \mathfrak{h}^\perp$. Note that $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 = -ad_{\xi_0} \xi_1^{\mathfrak{h}^\perp} \in \mathfrak{h}^\perp$ since $\mathfrak{h}^\perp = (\mathfrak{g}/\mathfrak{h})^*$ is a representation of $\mathfrak{h}$. But $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 \in \mathfrak{h}$ by (5.3), so that $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 \in \mathfrak{h} \cap \mathfrak{h}^\perp$ and we have $ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 = 0$. Hence

$$ad_{\xi_1} \xi_0 = ad_{\xi_1^{\mathfrak{h}}} \xi_0 + ad_{\xi_1^{\mathfrak{h}^\perp}} \xi_0 = ad_{\xi_1^{\mathfrak{h}}} \xi_0.$$

We claim that for any $\xi_2 \in \mathfrak{g}$ and $\eta' \in (\mathfrak{g}/\mathfrak{h})^*$,

$$\langle ad_{\xi_1} \xi_0, ad^*_{\xi_2} \eta' \rangle = 0,$$

so by definition $ad_{\xi_1} \xi_0 \in B$. Hence B is a nontrivial ideal since $B \neq 0$ and $B \subset \mathfrak{h}$, which is impossible.

We have

$$\langle ad_{\xi_1^{\mathfrak{h}}} \xi_0, ad^*_{\xi_2} \eta' \rangle = -\langle \xi_0, ad^*_{[\xi_1^{\mathfrak{h}}, \xi_2]} \eta' \rangle - \langle \xi_0, ad^*_{\xi_2} ad^*_{\xi_1^{\mathfrak{h}}} \eta' \rangle.$$

The first term above is zero since $\xi_0 \in B$. The second term is also zero since $ad^*_{\xi_1^{\mathfrak{h}}} \eta' \in (\mathfrak{g}/\mathfrak{h})^*$, since $(\mathfrak{g}/\mathfrak{h})^*$ is a representation of $\mathfrak{h}$. □

Remark 5.4. *Lemma 5.3 remains true if $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ does not contain any simple component of $\mathfrak{g}$.*

Proof of Theorem 5.2: We need to show that the operator $L^{tot} \circ_1$ coming from the quasi-conformal structure on $\mathbf{H}^*_G(\mathcal{Q}'(G/H))$ acts by zero. By Lemma 4.1, it suffices to construct a G-invariant, G-chiral horizontal element $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ satisfying $L^{tot}_\xi \circ_1 \alpha = \beta^\xi \otimes 1$ for all $\xi \in \mathfrak{g}$.

We first prove Theorem 5.2 in the case where G is semisimple, and we deal with the general case later. Since G acts almost effectively on G/H, $\mathfrak{h}$ does not contain any simple component of $\mathfrak{g}$, so the conclusion of Lemma 5.3 holds, by the preceding remark. Fix a basis $\{\xi_1, \dots, \xi_n\}$ of $\mathfrak{g}$ and a corresponding dual basis $\{\xi'_1, \dots, \xi'_n\}$ of $\mathfrak{g}^*$ (relative to the Killing form), such that $\xi_1, \dots, \xi_h$ is a basis of $\mathfrak{h}$.

In order to study G/H as a G space under left multiplication, it is convenient to regard G as a $G \times H$-space, on which G acts on the left and H acts on the right. The right H-action induces compatible actions of H and $\mathfrak{sh}[t]$ on $\mathcal{Q}'(G)$ which commute with the actions of G and $\mathfrak{sg}[t]$ coming from the left G-action. By Lemma 3.9 of [11], the projection $\pi : G \to G/H$ induces an isomorphism of vertex algebras: $\pi^* : \mathcal{Q}'(G/H)) \to \mathcal{Q}'(G)_{H-bas}$. Moreover, by declaring that H and $\mathfrak{sh}[t]$ act trivially on $\mathcal{W}(\mathfrak{g})$, we may extend the actions of H and $\mathfrak{sh}[t]$ to $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)$. We identify the complexes $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G/H)$ and $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)_{H-bas}$ and regard $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G/H)_{H-bas}$ as a subcomplex of $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)$. Thus in order to prove Theorem 5.2, it suffices to find a G-invariant, G-chiral horizontal element $\alpha \in \mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)_{H-bas}$ satisfying $L^{tot}_{\xi}\circ_1 \alpha = \beta^{\xi}\otimes 1$ for all $\xi\in\mathfrak{g}$. In order to deal with all the operators $L^{tot}_{\xi}\circ_1$ simultaneously, it is convenient to define a new operator

$$\mathcal{L} : \mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G) \to \mathfrak{g}\otimes\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)$$

sending $\omega \mapsto \xi_k \otimes L^{tot}_{\xi_k}\circ_1\omega$. Clearly $\mathcal{L}$ is G-equivariant, and the condition $L^{tot}_{\xi}\circ_1\alpha = \beta^{\xi}\otimes 1$ for all $\xi\in\mathfrak{g}$ is equivalent to $\mathcal{L}(\alpha) = \xi_k\otimes\beta^{\xi_k}\otimes 1$.

Let $f \in \mathcal{C}^{\infty}(G)$ be a smooth function, and fix $\zeta\in\mathfrak{g}$, $\xi\in\mathfrak{g}$, and $\eta'\in\mathfrak{g}^*$. Then for $k = 1,\dots,n$ we have

$$L^{tot}_{\xi_k}\circ_1(\beta^{\zeta}b^{\xi}c^{\eta'}\otimes f) = L^{\mathcal{W}}_{\xi_k}\circ_1(\beta^{\zeta}b^{\xi}c^{\eta'}\otimes f) = \beta^{\zeta}\otimes\langle[\xi_k,\xi],\eta'\rangle f = \beta^{\zeta}\otimes\langle\xi_k, ad^*_{\xi}\eta'\rangle f. \quad (5.4)$$

By Lemma 5.3, there exist elements $\chi_i\in\mathfrak{g}$ and $\eta'_i\in(\mathfrak{g}/\mathfrak{h})^*$ for which $ad^*_{\chi_i}\eta'_i = \xi'_i$, for $i = 1,\dots,n$. Then

$$L^{tot}_{\xi_k}\circ_1(\sum_i\beta^{\xi_i}b^{\chi_i}c^{\eta'_i}\otimes 1) = \beta^{\xi_i}\otimes\langle\xi_k,\xi'_i\rangle = \beta^{\xi_k}\otimes 1,$$

so that

$$\mathcal{L}(\sum_i\beta^{\xi_i}b^{\chi_i}c^{\eta'_i}\otimes 1) = \xi_k\otimes\beta^{\xi_k}\otimes 1. \quad (5.5)$$

However, $\sum_i\beta^{\xi_i}b^{\chi_i}c^{\eta'_i}\otimes 1$ is not G-invariant. We seek a G-invariant element

$$\alpha_0 = \sum_{jkl}\beta^{\xi_j}b^{\xi_k}c^{\xi'_l}\otimes f_{jkl}$$

which also satisfies (5.5).

We will construct α_0 using the connections one-forms coming from both the left and right actions of G on itself, which we denote by $\theta^{\xi'}$, $\bar{\theta}^{\xi'}$, respectively, for $\xi'\in\mathfrak{g}^*$. We denote the $\mathfrak{sg}[t]$-algebra structure on $\mathcal{Q}'(G)$ coming from the *right* G-action by $(\xi,\eta)\mapsto \bar{L}_\xi+\bar{\iota}_\eta$. Evaluating the functions $\iota_\xi\circ_0\bar{\theta}^{\xi'}$ and $\bar{\iota}_\xi\circ_0\theta^{\xi'}$ at the identity $e\in G$, we have

$$\iota_\xi\circ_0\bar{\theta}^{\xi'}|_e=\langle\xi,\xi'\rangle=\bar{\iota}_\xi\circ_0\theta^{\xi'}|_e. \tag{5.6}$$

Define

$$\alpha_0=\sum_{i,j,k,l}\beta^{\xi_j}b^{\xi_k}c^{\xi'_l}\otimes\bar{\iota}_{\xi_i}(\theta^{\xi'_j})\bar{\iota}_{\chi_i}(\theta^{\xi'_k})\iota_{\xi_l}(\bar{\theta}^{\eta'_i}).$$

Clearly α_0 is G-invariant, and $\alpha_0|_e=\sum_i\beta^{\xi_i}b^{\chi_i}c^{\eta'_i}\otimes 1$, by (5.6). Acting by $\mathcal{L}$ we see that $(\mathcal{L}(\alpha_0))|_e=\xi_k\otimes\beta^{\xi_k}\otimes 1$. Finally, since α_0 is G-invariant and the operator $\mathcal{L}$ is G-equivariant, it follows that $\mathcal{L}(\alpha_0)=\xi_k\otimes\beta^{\xi_k}\otimes 1$ at every point of G, as desired.

Our next step is to correct α_0 to make it G-chiral horizontal without destroying G-invariance or condition (5.5). Note that for $n\geq 0$,

$$\iota^{tot}_{\xi_t}\circ_n\alpha_0=b^{\xi_t}\circ_n\alpha_0=\delta_{n,0}\sum_{i,j,k}\beta^{\xi_j}b^{\xi_k}\otimes\bar{\iota}_{\xi_i}(\theta^{\xi'_j})\bar{\iota}_{\chi_i}(\theta^{\xi'_k})\iota_{\xi_t}(\bar{\theta}^{\eta'_i}).$$

Let

$$\alpha_1=-\sum_{i,j,k,l}\beta^{\xi_j}b^{\xi_k}\otimes\bar{\iota}_{\xi_i}(\theta^{\xi'_j})\bar{\iota}_{\chi_i}(\theta^{\xi'_k})\iota_{\xi_l}(\bar{\theta}^{\eta'_i})\theta^{\xi'_l}.$$

An OPE calculation shows that for $n\geq 0$

$$L^{tot}_{\xi_t}\circ_n\alpha_1=0,\qquad \iota^{tot}_{\xi_t}\circ_n\alpha_1=-\delta_{n,0}\sum_{i,j,k}\beta^{\xi_j}b^{\xi_k}\otimes\bar{\iota}_{\xi_i}(\theta^{\xi'_j})\bar{\iota}_{\chi_i}(\theta^{\xi'_k})\iota_{\xi_t}(\bar{\theta}^{\eta'_i}). \tag{5.7}$$

Let $\alpha=\alpha_0+\alpha_1$. It follows from (5.7) that α is G-invariant, G-chiral horizontal, and satisfies $\mathcal{L}(\alpha)=\xi_k\otimes\beta^{\xi_k}\otimes 1$.

We need to correct α so that it lies in $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)_{H-bas}$, without destroying the above properties. First, we claim that α_0 is already H-chiral horizontal. This is clear since α is a sum of terms of the form $\beta^{\xi_j}b^{\xi_k}c^{\xi'_l}\otimes f_{jkl}$ where $f_{jkl}\in\mathcal{C}^\infty(G)$, and $\bar{\iota}_\xi\circ_n$ lowers degree and only acts on the second factor of $\mathcal{W}(\mathfrak{g})\otimes\mathcal{Q}'(G)$ for $\xi\in\mathfrak{h}$.

Second, we claim that α_1 is H-chiral horizontal as well. First note that for $\xi\in\mathfrak{h}$, $\bar{\iota}_\xi\circ_n$ acts by derivations on $\mathcal{Q}'(G)$, so it suffices to show that it acts by zero on each term of the form $\bar{\iota}_{\xi_i}(\theta^{\xi'_j})$, $\bar{\iota}_{\chi_i}(\theta^{\xi'_k})$ and $\iota_{\xi_l}(\bar{\theta}^{\eta'_i})\theta^{\xi'_l}$. Clearly $\bar{\iota}_\xi\circ_n$ acts by zero on $\bar{\iota}_{\xi_i}(\theta^{\xi'_j})$ and

$\bar{\iota}_{\chi_i}(\theta^{\xi'_k})$ since these terms have degree 0 and $\bar{\iota}_\xi \circ_n$ has degree -1. Next, note that for each $\eta'_i \in (\mathfrak{g}/\mathfrak{h})^*$ we have

$$\iota_{\xi_l}(\bar{\theta}^{\eta'_i})\theta^{\xi'_l} = \bar{\theta}^{\eta'_i}.$$

which can be checked by applying ι_η, $\eta \in \mathfrak{g}/\mathfrak{h}$ to both sides. Since $\bar{\iota}_\xi \circ_n \bar{\theta}^{\eta'_i} = 0$ for all $\xi \in \mathfrak{h}$ and $\eta'_i \in (\mathfrak{g}/\mathfrak{h})^*$, the claim follows.

Finally, if α is not H-invariant, we can make it H-invariant by averaging it over H, that is, we take $\alpha' = \frac{1}{|H|}\int_H h\alpha \; d\mu$, where $d\mu$ is the Haar measure on H. Since the G and H actions commute, α' is G-invariant and G-chiral horizontal, and $\mathcal{L}(\alpha') = \xi_k \otimes \beta^{\xi_k} \otimes 1$. Moreover, since α is H-chiral horizontal, it follows that $h\alpha$ is still H-chiral horizontal for all $h \in H$, so α' is H-chiral horizontal as well. Since α' is H-invariant and lives in $\mathcal{W}(\mathfrak{g})[2] \otimes \mathcal{Q}'(G)[0]$, α' is in fact H-chiral invariant, so that $\alpha' \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G)_{H-bas}$, as desired.

This completes the proof of Theorem 5.2 in the case where G is semisimple. We now consider the general case in which G has a positive-dimensional center. As in Lemma 1.10, let $\tilde{G}$ be a finite cover of G of the form $K \times T$, where K is semisimple and T is a torus. Then

$$\mathbf{H}^*_G(\mathcal{Q}'(G/H)) = \mathbf{H}^*_{\tilde{G}}(\mathcal{Q}'(G/H)) = \mathbf{H}^*_{\tilde{G}}(\mathcal{Q}'(\tilde{G}/\tilde{H})).$$

Here $\tilde{H}$ is the inverse image of H under the projection $\tilde{G} \to G$, which is a finite cover of H. Hence without loss of generality we may assume that G is already of the form $K \times T$.

Since G acts almost effectively on G/H, $H \cap T$ is finite, and since T and H commute, H is a normal subgroup of HT. The group HT/H is a torus, which we denote by T'. The natural map $T \to T'$ is a finite cover, and we have a principal T'-bundle $G/H \to G/HT$.

Since K is semisimple and K acts almost effectively on

$$G/HT = (K \times T)/HT = K/(HT \cap K),$$

we can apply Theorem 5.2 in the semisimple case; there exists a K chiral horizontal, K invariant element $\alpha \in \mathcal{W}(\mathfrak{k}) \otimes \mathcal{Q}'(G/HT)$ such that $L^{tot}_\xi \circ_1 \alpha = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{k}$.

Since G/H is a T'-bundle over G/HT, the projection $G/H \to G/HT$ induces an isomorphism $\mathcal{Q}'(G/HT) \to \mathcal{Q}'(G/H)_{T'-bas}$. Hence we have an injection $\mathcal{W}(\mathfrak{k}) \otimes \mathcal{Q}'(G/HT) \hookrightarrow \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$. Let α_K denote the image of α under this map. Clearly

α_K is invariant under $L_\eta^{G/H}(k)$ and $\iota_\eta^{G/H}$ for $\eta \in \mathfrak{t}$ and $k \geq 0$ because α_K is a sum of terms of the form $\alpha_i \otimes \omega_i$ with $\omega_i \in \mathcal{Q}'(G/H)_{T'-bas}$. Similarly, $L_\eta^{\mathcal{W}}(k)$ and $\iota_\eta^{\mathcal{W}}(k)$ act trivially on α_K for $\eta \in \mathfrak{t}$ and $k \geq 0$ since each α_i does not depend on $b^\eta, c^{\eta'}, \beta^\eta, \gamma^{\eta'}$ for $\eta \in \mathfrak{t}', \eta' \in (\mathfrak{t}')^*$. Hence α_K is T'-chiral basic. If α_K is not T-invariant, we can make it T-invariant by averaging it over the finite group $\Gamma = Ker(T \to T')$.

Clearly α_K is G-chiral horizontal and G invariant and satisfies $L_\xi^{tot} \circ_1 \alpha_K = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{k}$ and $L_\eta^{tot} \circ_1 \alpha_K = 0$ for $\eta \in \mathfrak{t}$. We will construct another element $\alpha_T \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ which is G-invariant, G-chiral horizontal, and satisfies $L_\xi^{tot} \circ_1 \alpha_T = 0$ for $\xi \in \mathfrak{k}$ and $L_\eta^{tot} \circ_1 \alpha_T = \beta^\eta \otimes 1$ for $\eta \in \mathfrak{t}$. Then $\alpha_K + \alpha_T$ satisfies all the conditions of Lemma 4.1.

Since $H \cap T$ is finite, the action of T on G/H is locally free. Then there are connection forms $\theta^{\eta'} \in \Omega^1(G/H)$ satisfying $\iota_\eta \theta^{\eta'} = \langle \eta', \eta \rangle$. Let $\Gamma^{\eta'} = g \circ_0 \theta^{\eta'}$, which has degree zero and weight one. Let η_i be an orthonormal basis of $\mathfrak{t}$ (relative to the Killing form), and define

$$\alpha = \beta^{\eta_i} \otimes \Gamma^{\eta_i'}.$$

Clearly α is T-invariant, T-chiral horizontal, and satisfies $L_\eta^{tot} \circ_1 \alpha = \beta^\eta$ for $\eta \in \mathfrak{t}$. Since K and T commute, we can average α over K without destroying the above properties to make it G-invariant.

We claim that α is in fact G-chiral horizontal. It is enough to check this in local coordinates γ^i on G/H. For $\xi' \in \mathfrak{k}^*$, $\Gamma^{\xi'}$ is of the form $f^k \partial \gamma^k$, where the f^k are smooth functions. Similarly, for each $\xi \in \mathfrak{k}$, the vertex operator $\iota_\xi^{G/H}$ is locally of the form $f^k b^k$. Hence $\Gamma^{\xi'}$ commutes with $\iota_\xi^{G/H}$. Since α does not depend on $c^{\xi'}$ for $\xi' \in \mathfrak{k}^*$, it follows that α commutes with b^ξ for $\xi \in \mathfrak{k}$. Hence $\iota_\xi^{tot} = b^\xi + \iota_\xi^{G/H}$ commutes with α for $\xi \in \mathfrak{k}$, as claimed.

However, α need not satisfy $L_\xi^{tot} \circ_1 \alpha = 0$ for $\xi \in \mathfrak{k}$. The last step is to correct α so that this property holds. Let

$$\alpha_T = \alpha + \theta_{\mathcal{S}}^{\xi_i} L_{\xi_i}^{tot} \circ_1 \alpha,$$

where i runs over a basis of $\mathfrak{k}$. An OPE calculation using the fact that $L_{\xi_k}^{\mathcal{W}} \circ_1 \theta_{\mathcal{S}}^{\xi_i} = -\delta_{i,k}$ shows that $L_\xi^{tot} \circ_1 \alpha_T = 0$ for $\xi \in \mathfrak{k}$. Finally, since $L_\xi^{tot} \circ_1$ and $L_\eta^{tot} \circ_1$ commute for $\xi \in \mathfrak{k}$ and $\eta \in \mathfrak{t}$, it follows that for all $\eta \in \mathfrak{t}$ we have

$$L_\eta^{tot} \circ_1 (\theta_{\mathcal{S}}^{\xi_i} L_{\xi_i}^{tot} \circ_1 \alpha,) = \theta_{\mathcal{S}}^{\xi_i} L_{\xi_i}^{tot} \circ_1 (L_\eta^{tot} \circ_1 \alpha) = \theta_{\mathcal{S}}^{\xi_i} L_{\xi_i}^{tot} \circ_1 (\beta^\eta \otimes 1) = 0.$$

Thus $L_\eta^{tot} \circ_1 \alpha_T = L_\eta^{tot} \circ_1 \alpha = \beta^\eta \otimes 1$, as desired. $\square$

5.1. Finite-dimensionality of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ *for compact* M

The subspace $\mathcal{W}(\mathfrak{g})^p[n] \subset \mathcal{W}(\mathfrak{g})$ of degree p and weight n is finite-dimensional, so $\mathbf{H}^p_G(\mathbf{C})[n]$ is finite-dimensional for all $p \in \mathbf{Z}$ and $n \geq 0$. Similarly, since G/H is compact for any closed subgroup $H \subset G$, Theorem 5.2 implies that $\mathbf{H}^p_G(\mathcal{Q}'(G/H))[n]$ is finite-dimensional for all p, n. In this section, we show that if we replace G/H with an arbitrary compact M, $\mathbf{H}^p_G(\mathcal{Q}'(M))[n]$ is finite-dimensional for all p, n as well. This generalizes a well-known classical result in the case $n = 0$. Hence the generating function

$$\chi(G, M) = \sum_{p,n} dim\ \mathbf{H}^d_G(\mathcal{Q}'(M))[n]\ z^p q^n$$

is a well-defined invariant of the G-manifold M.

Lemma 5.5. *If* M *has a finite cover* $\{U_1, .., U_m\}$ *of* G*-invariant open sets, such that*

$$dim\ \mathbf{H}^p_G(\mathcal{Q}'(U_{i_1} \cap \cdots \cap U_{i_k}))[n] < \infty,$$

for each p, n *and for fixed indices* $i_1, .., i_k$*, then* $\mathbf{H}^p(M)[n]$ *is finite-dimensional.*

Proof: This is the standard generalized Mayer-Vietoris argument by induction on m. If $m = 1$, there is nothing to prove. For $m = 2$, this is the usual Mayer-Vietoris argument. Put $V = U_1 \cup \cdots \cup U_{m-1}$. Then we have

$$\cdots \to \mathbf{H}^{p-1}_G(V \cap U_m) \to \mathbf{H}^p_G(M) \to \mathbf{H}^p_G(V) \oplus \mathbf{H}^p_G(U_m) \to \cdots,$$

which we restrict to a given weight n. By inductive hypothesis, $\mathbf{H}^p_G(V)[n]$ and $\mathbf{H}^p_G(U_m)[n]$ are finite-dimensional, so the third term is finite dimensional. Note that $V \cap U_m$ is covered by the open sets $U_i \cap U_m$, $i = 1, .., m-1$, and their multiple intersections also have finite-dimensional cohomology at fixed p, n. Since there are $m - 1$ such open sets, the inductive hypothesis can be applied again. Thus the first term is also finite-dimensional, implying that the second term is finite-dimensional as well. □

Lemma 5.6. *Suppose the* G*-manifold* M *is a fiber bundle whose general fiber is* G/H*. If* M *is compact, then* $\mathbf{H}^p_G(M)[n]$ *is finite-dimensional.*

Proof: Choose a local trivializing cover of the bundle, which we can further refine to a good cover on the base M/G, i.e. each multiple intersection of open sets is a ball. The preimage under $M \to M/G$ of each multiple intersection of the open sets is equivariantly diffeomorphic to $G/H \times B$. So we can cover M by finitely many open sets whose multiple intersections are equivariantly contractible to G/H, which has finite-dimensional chiral cohomology for each p, n. The claim then follows by the preceding lemma. $\square$

Given a closed subgroup $H \subset G$, let $M_{(H)}$ denote the subset of M consisting of points with isotropy group conjugate to H. $M_{(H)}$ is a closed submanifold of M, which may be regarded as a G/H-fiber bundle over the manifold $M_{(H)}/G$. By the preceding lemma, $\mathbf{H}^p_G(\mathcal{Q}'(M_{(H)}))[n]$ is finite-dimensional for each p, n.

Theorem 5.7. *Suppose M is compact. Then $\mathbf{H}^p_G(\mathcal{Q}'(M))[n]$ is finite-dimensional.*

Proof: Since M is compact, there are only finitely many conjugacy classes (H) for which $M_{(H)}$ is nonempty. For *dim* $H > 0$, each such $M_{(H)}$ has a G-invariant tubular neighborhood $U_{(H)}$ which is equivariantly contractible to $M_{(H)}$. M has a finite cover consisting of the $U_{(H)}$ together with the open set U of points with finite isotropy group. Without loss of generality we can shrink each $U_{(H)}$ so it contains only two orbit types: (H) and (e).

By homotopy invariance, $\mathbf{H}^*_G(\mathcal{Q}'(U_{(H)})) = \mathbf{H}^*_G(\mathcal{Q}'(M_{(H)}))$, which is finite-dimemsional at each p, n. The action of G on U is locally free so $\mathbf{H}^*_G(U))_+ = 0$. Since the multiple intersections of the $U_{(H)}$ all lie in U, they also have no higher-weight cohomology. It follows from Lemma 5.5 that for $n > 0$, $H^p_G(\mathcal{Q}'(M))[n]$ is finite-dimensional for all p. For $n = 0$, the finite-dimensionality of $\mathbf{H}^*_G(\mathcal{Q}'(M))[0] = H^p_G(M)$ is classical. $\square$

6. The Structure of $\mathbf{H}^*_G(\mathcal{Q}(M))$

For any G-manifold M, we give a complete description of $\mathbf{H}^*_G(\mathcal{Q}(M))$ relative to the family of vertex algebras $\mathbf{H}^*_K(\mathbf{C})$ for connected normal subgroups $K \subset G$.

Theorem 6.1. *Let G be a compact group and let M be a G-manifold. Then*

$$\mathbf{H}^*_G(\mathcal{Q}(M)) \cong \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes H^*_{G'}(M),$$

where K_0 is the identity component of $Ker(G \to Diff(M))$ and $G' = G/K_0$. In particular, if $Ker(G \to Diff(M))$ is finite, $\mathbf{H}^*_G(\mathcal{Q}(M))_+ = 0$.

Theorem 6.1 shows that $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ is only sensitive to K_0, and in contrast to $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$, it carries no other geometric information about M.

Proof: We may assume that G acts almost effectively on M. As usual, we need to show that the operator $L^{tot}\circ_1$ coming from the quasi-conformal structure on $\mathbf{H}^*_G(\mathcal{Q}(M))$ acts by zero. By Lemma 4.1, it is enough to construct a chiral horizontal, G-invariant element $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ for which $L^{tot}_\xi \circ_1 \alpha = \beta^\xi \otimes 1$ for all $\xi \in \mathfrak{g}$. Since M can be covered by G-invariant open sets of the form $G \times_H V$ for some closed subgroup $H \subset G$ and some H-module V, it is enough to construct $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(G \times_H V)$ for any H and V, by Lemma 4.2 and Remark 4.3. Unlike the functor $\mathbf{H}^*_G(\mathcal{Q}'(-))$, $\mathbf{H}^*_G(\mathcal{Q}(-))$ is not a homotopy invariant, and in general $\mathbf{H}^*_G(\mathcal{Q}(G \times_H V)) \neq \mathbf{H}^*_G(\mathcal{Q}(G/H))$.

Fix $M = G \times_H V$, and assume that G acts almost effectively on M. Note that the action of G on the zero-section $G/H \subset M$ need not be almost effective. Let $K = Ker(G \to Diff(G/H))$, and suppose first that K is finite. Then by Theorem 5.2, there exists $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/H)$ which we can pull back to $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(M) \subset \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(M)$ with the same properties.

So assume that K has positive dimension, and let K_0 denote the identity component. By Lemma 1.10, we may assume that G is of the form $G_1 \times \cdots \times G_k \times T$, where the G_i are simple. Clearly any connected normal subgroup of G splits, so K_0 is a product of a subset of the G_i and a subtorus $T' \subset T$. Hence we may write $G = K_0 \times N$ for some N. Likewise, since K_0 is also a normal subgroup of H, we may write $H = K_0 \times L$. Then

$$M = (K_0 \times N) \times_{K_0 \times L} V = N \times_L V,$$

where N acts on the left and K_0 acts only on the factor V, and the actions of K_0 and L on V commute. Clearly $Ker(N \to Diff(N/L))$ is finite, so by Theorem 5.2 there exists an N-invariant, N-chiral horizontal element $\alpha_N \in \mathcal{W}(\mathfrak{n}) \otimes \mathcal{Q}'(N/L)$ such that

$$L^{tot}_\xi \circ_1 \alpha_N = \beta^\xi \otimes 1$$

for $\xi \in \mathfrak{n}$. Since K_0 acts trivially on $\mathcal{W}(\mathfrak{n}) \otimes \mathcal{Q}'(N/L)$, we can lift α_N to a G-chiral horizontal G-invariant element, also denoted by α_N, in $\mathcal{W}(\mathfrak{n}) \otimes \mathcal{Q}'(N \times_L V) \subset \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(N \times_L V)$

via the map induced by the projection map $N \times_L V \to N/L$ such that $L_\xi^{tot} \circ_1 \alpha_N = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{n}$. We can view this element as lying in $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(N \times_L V)$.

On the other hand, the linear action of K_0 on V is faithful because the action of G on M is effective. Thus there exists a K_0-chiral horizontal K_0-invariant element $\alpha_K \in \mathcal{W}(\mathfrak{k}) \otimes \mathcal{Q}(V)$ such that $L_\xi^{tot} \circ_1 \alpha_K = \beta^\xi \otimes 1$ for $\xi \in \mathfrak{k}$. Now L acts only on the $\mathcal{Q}(V)$ factor and it commutes with K_0 action. Thus by averaging over L, we may assume that α_K is L-invariant, and still satisfies the same equation.

Recall from (4.3) that $\alpha_K = \beta^{\xi_i} \otimes \Gamma^{\xi_i} - \beta^{\xi_i} b^{\xi_j} \otimes \iota_{\xi_j} \circ_0 \Gamma^{\xi_i}$, where $\Gamma^{\xi_i} = \beta^{\rho(\xi_i)(x_k)} \gamma^{x'_k}$. Clearly Γ^{ξ_i} is itself L-invariant (because the action of L on x_i and dual action on x'_i amounts to only a change of basis in V.) Now observe that any L-invariant element of $\mathcal{Q}(V)$ can be regarded as a global section in $\mathcal{Q}(N \times_L V)$. (This is clear by covering the bundle by open sets $U \times V$ where $U \subset N/L$, and the observing that the transition functions are elements of L.) The element α_K is now N-chiral horizontal and N-invariant in $\mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}(N \times_L V)$ and satisfies $L_\xi^{tot} \circ_1 \alpha_K = 0$ for $\xi \in \mathfrak{n}$.

Finally, $\alpha = \alpha_N + \alpha_K$ is G chiral horizontal G invariant and satisfies

$$L_\xi^{tot} \circ_1 \alpha_N = \beta^\xi \otimes 1$$

for all $\xi \in \mathfrak{g} = \mathfrak{k} + \mathfrak{n}$. This completes the proof that $\mathbf{H}_G(\mathcal{Q}(M))_+ = 0$. $\square$

6.1. *The ideal property of* $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ *and* $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$

A consequence of Theorem 6.1 is that $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ and $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ are *vertex algebra ideals*, that is, they are closed under $\alpha\circ_n$ and $\circ_n\alpha$ for all $n \in \mathbf{Z}$ and α in $\mathbf{H}^*_G(\mathcal{Q}(M))$, $\mathbf{H}^*_G(\mathcal{Q}'(M))$, respectively. It suffices to exhibit these spaces as kernels of vertex algebra homomorphisms. We begin with the case $M = pt$.

Lemma 6.2. *For any G, $\mathbf{H}^*_G(\mathbf{C})_+$ is an ideal of $\mathbf{H}^*_G(\mathbf{C})$.*

Let V be faithful representation of G, and consider the Chern-Weil map $\kappa_G : \mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_G(\mathcal{Q}(V))$. At weight zero, this is the identity map $S(\mathfrak{g}^*)^G \to S(\mathfrak{g}^*)^G$ and it vanishes beyond weight zero because $\mathbf{H}^*_G(\mathcal{Q}(V))_+ = 0$. Hence $\mathbf{H}^*_G(\mathbf{C})_+ = Ker(\kappa_G)$. $\square$

In view of Theorem 6.1 and the preceding lemma, it is immediate that $\mathbf{H}^*_G(\mathcal{Q}(M))_+$ is an ideal for any M.

Theorem 6.3. *For any G and M, $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ is an ideal of $\mathbf{H}^*_G(\mathcal{Q}'(M))$.*

Proof: Consider the map

$$\phi : \mathbf{H}^*_G(\mathcal{Q}'(M)) \to \mathbf{H}^*_G(\mathcal{Q}(M)) \tag{6.1}$$

induced by the inclusion $\mathcal{Q}'(M) \hookrightarrow \mathcal{Q}(M)$. If G acts almost effectively on M, $\mathbf{H}^*_G(\mathcal{Q}(M))_+ = 0$ by Theorem 6.1, so $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = Ker(\phi)$, as desired. If $K = Ker(G \to Diff(M))$ has positive dimension, $\mathbf{H}^*_G(\mathcal{Q}'(M)) = \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes \mathbf{H}^*_{G'}(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M)) = \mathbf{H}^*_{K_0} \otimes \mathbf{H}^*_{G'}(\mathcal{Q}(M))$, and (6.1) becomes

$$\mathbf{H}^*_{K_0}(\mathbf{C}) \otimes \mathbf{H}^*_{G'}(\mathcal{Q}'(M)) \to \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes \mathbf{H}^*_{G'}(\mathcal{Q}(M)),$$

where the map on the first factor is the identity. Since G' acts almost effectively on M, $\mathbf{H}^*_{G'}(\mathcal{Q}'(M))_+$ is an ideal. Finally,

$$\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = \mathbf{H}^*_{K_0}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G'}(\mathcal{Q}'(M)) \oplus \mathbf{H}^*_{K_0}(\mathbf{C}) \otimes \mathbf{H}^*_{G'}(\mathcal{Q}'(M))_+.$$

Since both $\mathbf{H}^*_{K_0}(\mathbf{C})_+$ and $\mathbf{H}^*_{G'}(\mathcal{Q}'(M))_+$ are ideals, the claim follows. $\square$.

Corollary 6.4. *$\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ is a module over $H^*_G(M)$ under $a \circ_n (-)$, for all $n \in \mathbf{Z}$. Here $a \in H^*_G(M)$ is regarded as a vertex operator in $\mathbf{H}^*_G(\mathcal{Q}'(M))[0]$.*

7. The Structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$

In contrast to $\mathbf{H}^*_G(\mathcal{Q}(M))$, $\mathbf{H}^*_G(\mathcal{Q}'(M))$ typically contains nontrivial geometric information about M beyond weight zero, and is a strictly stronger invariant of G manifolds than the classical equivariant cohomology. Our goal in this section is to give a *relative* description of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ in terms of the vertex algebras $\mathbf{H}^*_K(\mathbf{C})$ for connected normal subgroups $K \subset G$, together with certain geometric data about M. We will focus on three special cases: G simple, $G = G_1 \times G_2$ where G_1, G_2 are simple, and G abelian. We first describe $\mathbf{H}^*_G(\mathcal{Q}'(M))$ as a linear space using Mayer-Vietoris sequences together with Theorem 5.2, and then describe the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$.

Let $\mathcal{A} = \sum_{n\geq 0}\mathcal{A}[n]$ and $\mathcal{B} = \sum_{n\geq 0}\mathcal{B}[n]$ be weight graded vertex algebras, and let $f : \mathcal{A} \to \mathcal{B}$ be a weight-preserving vertex algebra homomorphism. Assume that $\mathcal{A}_+ = \sum_{n>0}\mathcal{A}[n]$ is an ideal of $\mathcal{A}$.

Lemma 7.1. *Suppose that the restriction of f to $\mathcal{A}_+$ is injective. Then the vertex algebra structure of $\mathcal{A}$ is uniquely determined by the ring structure of $(\mathcal{A}[0], \circ_{-1})$, the homomorphism f, and the vertex algebra structure of $\mathcal{B}$.*

Proof: We need to describe the map

$$\circ_n : \mathcal{A}[i] \otimes \mathcal{A}[j] \to \mathcal{A}[i + j - n - 1] \tag{7.1}$$

for all $i, j \geq 0$ and $n \leq i+j-1$ in terms of the above data. First, suppose that $n = i+j-1$. Since $\mathcal{A}_+$ is an ideal, it follows that (7.1) is zero unless $i = j = 0$ and $n = -1$, which is known by hypothesis. So we may assume that $n < i + j - 1$.

Let $a \in \mathcal{A}[i]$ and $b \in \mathcal{A}[j]$, and suppose that either $i > 0$ or $j > 0$. Since $\mathcal{A}_+$ is an ideal, $a \circ_n b \in \mathcal{A}_+$. Since $f : \mathcal{A}_+ \to \mathcal{B}_+$ is injective, f is an isomorphism of $\mathcal{A}_+$ onto its image $\mathcal{B}'_+ \subset \mathcal{B}_+$, and the inverse map $f^{-1} : \mathcal{B}'_+ \to \mathcal{A}_+$ is well-defined. It follows that

$$a \circ_n b = f^{-1}(f(a) \circ_n f(b)),$$

which is determined by f and the vertex algebra structure of $\mathcal{B}$.

Finally, suppose that $i = j = 0$ and $n \leq -2$. Then $a \circ_n b \in \mathcal{A}_+$, so as above, we have $a \circ_n b = f^{-1}(f(a) \circ_n f(b))$. $\square$

We will apply Lemma 7.1 in the case where $\mathcal{A} = \mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathcal{B}$ is another vertex algebra whose structure is known, to determine the vertex algebra structure on $\mathbf{H}^*_G(\mathcal{Q}'(M))$.

7.1. The case where G is simple

For simple G, we describe $\mathbf{H}^*_G(\mathcal{Q}'(M))$ in terms of $\mathbf{H}^*_G(\mathbf{C})$ together with certain classical geometric data. Let $\mathbf{i}^* : \mathbf{H}^*_G(\mathcal{Q}'(M)) \to \mathbf{H}^*_G(\mathcal{Q}'(M^G))$ be the map induced from $i : M^G \hookrightarrow M$. The restriction of $\mathbf{i}^*$ to $\mathbf{H}^*_G(\mathcal{Q}'(M))[0]$ coincides with the classical map $i^* : H^*_G(M) \to H^*_G(M^G)$.

Theorem 7.2. *(Positive-weight localization for simple group actions) Let G be simple. For any G-manifold M, $\mathbf{i}^* : \mathbf{H}^*_G(\mathcal{Q}'(M))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(M^G))_+$ is a linear isomorphism in positive weight. Hence*

$$\mathbf{H}^*_G(\mathcal{Q}'(M))_+ \cong \mathbf{H}^*_G(\mathbf{C})_+ \otimes H^*(M^G). \tag{7.2}$$

Moreover, the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ is uniquely determined by (7.2).

Proof: If M^G is non-empty, let U_0 be a G-invariant tubular neighborhood of M^G and let $U_1 = M \setminus M^G$. It suffices to show that $\mathbf{H}^*_G(\mathcal{Q}'(U_1))_+ = 0$ and $\mathbf{H}^*_G(\mathcal{Q}'(U_0 \cap U_1))_+ = 0$. In this case, we have $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = \mathbf{H}^*_G(\mathcal{Q}'(U_0 \cup U_1))_+$, and since $\mathbf{H}^*_G(\mathcal{Q}'(U_0))_+ = \mathbf{H}^*_G(\mathcal{Q}'(M^G))_+$ by homotopy invariance, the claim follows from a Mayer-Vietoris argument.

For each point $x \in U_1$, the isotropy group G_x has positive codimension in G since G is connected. Let U_x be a G-invariant neighborhood of the orbit Gx, which we may take to be a vector bundle of the form $G \times_{G_x} V$ whose zero-section is Gx.

By the proof of Theorem 5.2, there exists a G-invariant, G-chiral horizontal element $\alpha_x \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(G/G_x)$ satisfying (4.2). Via the projection $U_x \to Gx$, this pulls back to an element $\alpha_{U_x} \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(U_x)$ satisfying the same conditions. Using a G-invariant partition of unity as in Lemma 4.2, we can glue the α_{U_x} together to obtain $\alpha \in \mathcal{W}(\mathfrak{g}) \otimes \mathcal{Q}'(U_1)$ safisfying these conditions as well. It follows that $\mathbf{H}^*_G(\mathcal{Q}'(U_1))_+ = 0$. Finally, the same argument shows that $\mathbf{H}^*_G(\mathcal{Q}'(U_0 \cap U_1))_+ = 0$.

Next, let $\mathcal{A} = \mathbf{H}^*_G(\mathcal{Q}'(M))$, $\mathcal{B} = \mathbf{H}^*_G(\mathcal{Q}'(M^G))$, and let $f : \mathcal{A} \to \mathcal{B}$ be the map $\mathbf{i}^*$. Clearly the hypothesis of Lemma 7.1 is satisfied. Moreover, the ring structure of $(\mathbf{H}^*_G(\mathcal{Q}'(M))[0], \circ_{-1})$ coincides with the ring structure of $(H^*_G(M), \cup)$, which is classical. As a vertex algebra, $\mathbf{H}^*_G(\mathcal{Q}'(M^G)) = \mathbf{H}^*_G(\mathbf{C}) \otimes H^*(M^G)$ where $H^*(M^G)$ is regarded as a vertex algebra in which all products except $\circ_{-1}$ are trivial. By Lemma 7.1, this determines the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ uniquely. $\Box$

Note that $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = \mathbf{H}^*_G(\mathbf{C})_+ \otimes H^*(M^G)$ may alternatively be described as

$$\mathbf{H}^*_G(\mathbf{C})_+ \otimes_{S(\mathfrak{g}^*)^G} H^*_G(M^G). \tag{7.3}$$

Given $\alpha \in \mathbf{H}^*_G(\mathbf{C})_+$ and $\omega \in H^*_G(M)$, it follows from Theorem 7.2 that

$$\kappa_G(\alpha) \circ_{-1} \omega = \alpha \otimes i^*(\omega) \in \mathbf{H}^*_G(\mathbf{C})_+ \otimes_{S(\mathfrak{g})^*G} H^*_G(M^G).$$

Similarly, given $\omega, \nu \in H^*(M^G)$, $\alpha \otimes \omega$ and $\alpha \otimes \nu$ lie in $\mathbf{H}^*_G(\mathbf{C}) \otimes H^*(M^G)$, and

$$(\alpha \otimes \omega) \circ_{-1} (\alpha \otimes \nu) = (\alpha \circ_{-1} \alpha) \otimes (\omega \cup \nu).$$

Hence both the classical restriction map i^* and the ring structure of $H^*(M^G)$ are encoded in the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$.

7.2. $\mathbf{H}^*_G(\mathcal{Q}'(-))$ *is a stronger invariant of G-manifolds than* $H^*_G(-)$

For any simple G, we construct compact G-manifolds M and N together with a smooth, G-equivariant map $f : M \to N$ which induces a ring isomorphism $H^*_G(N) \to H^*_G(M)$ (with $\mathbf{Z}$-coefficients), such that $\mathbf{H}^*_G(\mathcal{Q}'(M)) \neq \mathbf{H}^*_G(\mathcal{Q}'(N))$. *Hence* $\mathbf{H}^*_G(\mathcal{Q}'(-))$ *is a strictly stronger invariant than* $H^*_G(-)$ *on the category of compact G-manifolds.*

We need a theorem of R. Oliver which describes the fixed-point submanifolds of group actions on disks [14].

Theorem 7.3. *(Oliver) Let F be a finite CW-complex. If G is semisimple, there exists a smooth action of G on a closed disk D with fixed point set D^G having the homotopy type of F. If G is a torus, there exists such an action if and only if F is $\mathbf{Z}$-acyclic.*

Suppose that G acts smoothly on an n-dimensional disk D such that D^G has the homotopy type of F. Note that for any $m \geq n$, there is an m-dimensional disk $\tilde{D}$ with a smooth G action such that $\tilde{D}^G$ also has the homotopy type of F. To see this, let G act trivially on $[0,1]$. Then G acts on $D\times[0,1]$ and the fixed-point set $(D\times[0,1])^G = D^G\times[0,1]$ also has the homotopy type of F. Note that $\partial D \times \{0\}$ and $\partial D \times \{1\}$ have G-invariant neighborhoods $\partial D \otimes [0,1)$ and $\partial D \otimes (0,1]$, respectively, so the corners of $D \times [0,1]$ can be smoothed equivariantly. It follows that $D \times [0,1]$ is equivariantly diffeomorphic to a disk $\tilde{D}$ of dimension $n+1$.

For any action of G on D, the projection $\pi : D \to pt$ is a homotopy equivalence, so the induced map $\pi^* : H^*_G(D) \to H^*_G(pt) = S(\mathfrak{g}^*)^G$ is a ring isomorphism (over $\mathbf{Z}$). Let G be simple, and fix a positive integer k. For $i = 0,\dots,k$, let F_i be a CW-complex consisting of i zero cells and no higher-dimensional cells. For large n, we may choose n-dimensional disks $D_0,\dots,D_k$ equipped with smooth G-actions such that D_i^G has the homotopy type of F_i.

Theorem 7.4. *For $i \neq j$, we have $\mathbf{H}^*_G(\mathcal{Q}'(D_i)) \neq \mathbf{H}^*_G(\mathcal{Q}'(D_j))$.*

Proof: This is immediate from Theorem 7.2, since $dim\ H^0(D_i^G) = i$, for $i = 0,\dots,k$. $\square$

This result shows that there exist G-manifolds with the same classical equivariant cohomology (over $\mathbf{Z}$), which can be distinguished by $\mathbf{H}^*_G(\mathcal{Q}'(-))$. Note that if we choose two disks D_1 and D_2 such that D_1^G and D_2^G have the same Betti numbers, but $H^*(D_1^G) \neq H^*(D_2^G)$ as rings, then $\mathbf{H}^*_G(\mathcal{Q}'(D_1))$ and $\mathbf{H}^*_G(\mathcal{Q}'(D_2))$ will be isomorphic as linear spaces but not as vertex algebras.

If G acts smoothly on an n-dimensional disk D, we may glue together two copies of D along their boundaries to obtain a smooth action of G on the sphere S^n. This allows us to construct *compact* G-manifolds M and N which have the same classical equivariant cohomology rings (over $\mathbf{Z}$), such that $\mathbf{H}^*_G(\mathcal{Q}'(M)) \neq \mathbf{H}^*_G(\mathcal{Q}'(N))$.

Let G be simple, and fix a positive integer k. For $i = 1,\dots k$, let F_i be a CW-complex with 3^i zero cells and no higher-dimensional cells. For large n, we may choose $2n$-dimensional disks D_i with smooth G-actions such that D_i^G has the homotopy type of F_i. Let S_i be the copy of S^{2n} obtained by gluing together two copies of D_i along their boundaries.

Lemma 7.5. *For $i = 1, \dots, k$, the classical equivariant cohomology ring $H^*_G(S_i)$ is isomorphic to $S(\mathfrak{g}^*)^G[a]/(a^2)$ with $\mathbf{Z}$-coefficients.*

Proof: Let T be a maximal torus of G, and let W be the Weyl group, so that $H^*_G(S) = H^*_T(S)^W$. By Theorem 7.3, D_i^T is $\mathbf{Z}$-acyclic, and since D_t^T contains more than two points, D_i^T is connected and $D_i^T \cap \partial D_i \neq \varnothing$. It follows that S_i^T is also connected. Since $H^0(S_i) = \mathbf{Z}$, $H^{2n}(S_i) = \mathbf{Z}$ and all other cohomology groups are zero, S_i is equivariantly formal, and

$$i^* : H^*_T(S_i) \to H^*_T(S_i^T) = H^*(S_i^T) \otimes S(\mathfrak{t}^*)$$

is injective. It follows that $H^*_T(S_i)$ is a two-dimensional free $S(\mathfrak{t}^*)$-module with generators 1 and a of degrees 0 and $2n$, respectively. Since $H^{n-1}(S_i) = 0$, it follows that $\iota_\xi(\omega) = 0$ for all $\xi \in \mathfrak{t}$. Hence $\omega^2 = 0$. Finally, since W fixes a, it follows that $\mathbf{H}^*_G(S_i) \cong S(\mathfrak{t}^*)^W[a]/(a^2) = S(\mathfrak{g}^*)^G[a]/(a^2)$ for each $i = 1, \dots, k$, as claimed.

Theorem 7.6. *For $i = 1, \dots, k$, the vertex algebras $\mathbf{H}^*_G(\mathcal{Q}'(S_i))$ are all distinct.*

Proof: Note that each connected component $\mathcal{C}$ of D_i^G gives rise to one component of S_i^G (if $\mathcal{C} \cap \partial D_i \neq \varnothing$), or two components of S_i^G (if $\mathcal{C} \cap \partial D_i = \varnothing$). Since $dim\ H^0(D_i^G) = 3^i$, we have $3^i \leq dim\ H^0(S_i^G) \leq 2 \cdot 3^i$. Thus the integers $dim_{\mathbf{R}}\ H^0(S_i^G)$ are all distinct, so the claim follows from Theorem 7.2. $\square$

Theorem 7.6 shows that we can find compact G-manifolds with the same classical equivariant cohomology, which can be distinguished by $\mathbf{H}^*_G(\mathcal{Q}'(-))$. Next, we give examples of *morphisms* $f : M \to N$ in the category of compact G-manifolds (that is, smooth G-equivariant maps) for which $f^* : H^*_G(N) \to H^*_G(M)$ is an isomorphism (over $\mathbf{Z}$), but $\mathbf{H}^*_G(\mathcal{Q}'(M)) \neq \mathbf{H}^*_G(\mathcal{Q}'(N))$.

Let D_0 be an n-dimensional disk with a smooth G-action for which $\varnothing \neq D^G \neq D$, and let S_0 be the corresponding sphere, as above. Fix a point $p \in S_0^G$, and let U be a G-invariant neighborhood of p equipped with a smooth G-equivariant map $\phi : U \to \mathbf{R}^n$, where G acts linearly on $\mathbf{R}^n$, $\phi(p) = 0$, and ϕ maps the closure $\bar{U}$ diffeomorphically onto the disk $D_1 = \{x \in \mathbf{R}^n : |x| \leq 1\}$. We identify $D_1/\partial D_1$ with another copy of S^n, which we denote by S_1. Note that S_1 is a smooth G-manifold and S_1^G is a sphere S^k for some $0 \leq k < n$. Let $q \in S_1^G$ be the point which corresponds to ∂D_1 under the projection

$\pi : D_1 \to S_1$, and let $g = \pi \circ \phi : U \to S_1$, which is clearly smooth and G-equivariant. Note that g does *not* extend to a smooth map $S_0 \to S_1$. However, we can construct a new function f which agrees with g in a neighborhood of $p \in U$, which extends smoothly to all of S_0.

Lemma 7.7. *There exists a smooth, G-equivariant map $f : S_0 \to S_1$ such that $f = g$ in a neighborhood of $p \in U$, and $f(S_0 \setminus U) = q$. Moreover, f is smoothly (but not equivariantly) homotopic to the identity map $S^n \to S^n$, so f induces isomorphisms in both singular and equivariant cohomology with $\mathbf{Z}$-coefficients.*

Proof: As above, we identify U with the open disk $D_1^\circ = \{(x : |x| < 1\}$. Let $U' \subset U$ be a G-invariant neighborhood of ∂U of the form $\{x \in U : \ 1 - \epsilon < |x| \leq 1\}$. Clearly $g(U')$ is a G-invariant neighborhood of q, which we identify with another open disk $D_2^\circ = \{y : |y| < 1\}$ equipped with a linear action of G. Note that the radius $|y|$ is a G-invariant function on D_2.

Choose a smooth function $h : [0,1] \to [0,1]$ such that $h(t) = 0$ for $0 \leq t < \frac{1}{3}$, and $h(t) = 1$ for $2/3 < t \leq 1$. Define $f : U \to S_1$ by $f(x) = g(x)h(|g(x)|)$ for $x \in U'$, and $f(x) = g(x)$ for $x \in U \setminus U'$. Since $h(|y|)$ is G-invariant, f is G-equivariant and since $f = g$ on a neighborhood of $U \setminus U'$, f is smooth. Moreover, since f maps a neighborhood of ∂U to q (which we have identified with $0 \in D_2$), f extends to a smooth, G-equivariant map $S_0 \to S_1$ sending $S_0 \setminus U \to q$, as desired. Finally, the fact that f is homotopic to $id : S^n \to S^n$ is clear because $S_0 \setminus U$ is smoothly contractible to a point in S_0. $\square$.

Theorem 7.8. *Suppose that F is a CW-complex consisting of k zero cells and no higher-dimensional cells. Let D be a disk with a smooth G-action such that D^G has the homotopy type of F, and let S_0 and S_1 be as above. If $k \geq 3$, $\mathbf{H}^*_G(\mathcal{Q}'(S_0)) \neq \mathbf{H}^*_G(\mathcal{Q}'(S_1)$.*

Proof: Since S_1^G is a k-dimensional sphere for $0 \leq k < n$, we have $1 \leq dim\ H^0(S_1^G) \leq 2$. Since $k \leq dim\ H^0(S_0^G) \leq 2k$, we have $dim\ H^0(S_0^G) > dim\ H^0(S_1^G)$ for $k \geq 3$. The claim then follows from Theorem 7.2. $\square$.

Even though $f : S_0 \to S_1$ is homotopic to $id : S^n \to S^n$, this result does not contradict Theorem 3.2 because there is no *equivariant* homotopy between f and id. Thus unlike the classical equivariant cohomology, the functor $\mathbf{H}^*_G(\mathcal{Q}'(-))$ can distinguish G-manifolds M and N which admit a G-equivariant map which is a homotopy equivalence, as long as M and N are not equivariantly homotopic.

7.3. The case $G = G_1 \times G_2$, where G_1, G_2 are simple

For simple G, Theorem 7.2 describes $\mathbf{H}^*_G(\mathcal{Q}'(M))$ in terms of $\mathbf{H}^*_G(\mathbf{C})$ together with classical geometric data. In this section, we give a similar description of $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ in the case $G = G_1 \times G_2$, where the G_1 and G_2 are simple groups. As in the case G simple, we first describe $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ as a linear space, and then describe the vertex algebra structure.

Theorem 7.9. *Let G_1, G_2 be simple and let $G = G_1 \times G_2$. Then for any G-manifold M, $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ is linearly isomorphic to*

$$\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \bigoplus \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}) \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G).$$

Let U_1, U_2 be G-invariant tubular neighborhoods of M^{G_1}, M^{G_2}, respectively. If $x \notin U_1 \cup U_2$, its stabilizer G_x contains neither G_1 nor G_2, so $\mathbf{H}^*_G(\mathcal{Q}'(G/G_x))_+ = 0$. A Mayer-Vietoris argument then shows that $\mathbf{H}^*_G(\mathcal{Q}'(M))_+ = \mathbf{H}^*_G(\mathcal{Q}'(U_1 \cup U_2))_+$. Since U_1, U_2, and $U_1 \cap U_2$ are equivariantly contractible to M^{G_1}, M^{G_2}, and M^G, respectively, we can replace $\mathbf{H}^*_G(\mathcal{Q}'(U_1))_+$, $\mathbf{H}^*_G(\mathcal{Q}'(U_1))_+$, and $\mathbf{H}^*_G(\mathcal{Q}'(U_1 \cap U_2))_+$ with $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$, $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$, and $\mathbf{H}^*_G(\mathcal{Q}'(M^G))_+$ in the Mayer-Vietoris sequence

$$\cdots \to \mathbf{H}^*_G(\mathcal{Q}'(U_1 \cup U_2))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(U_1))_+ \oplus \mathbf{H}^*_G(\mathcal{Q}(U_2))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(U_2 \cap U_2))_+ \to \cdots,$$

obtaining

$$\cdots \to \mathbf{H}^*_G(\mathcal{Q}'(M))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}^*_G(\mathcal{Q}(M^{G_2}))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(M^G))_+ \to \cdots. \tag{7.4}$$

Lemma 7.10. *The map* $\phi : \mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}^*_G(\mathcal{Q}(M^{G_2}))_+ \to \mathbf{H}^*_G(\mathcal{Q}'(M^G))_+$ *appearing in (7.4) is surjective. Hence* $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ *is canonically isomorphic to* $Ker(\phi)$.

Proof: Let $i_1 : M^G \to M^{G_1}$, $i_2 : M^G \to M^{G_2}$, $j_1 : M^{G_1} \to M$, $j_1 : M^{G_2} \to M$ denote the obvious inclusion maps. First we need to describe each of the spaces $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$, $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$, and $\mathbf{H}^*_G(\mathcal{Q}'(M^G))_+$. Since G acts trivially on M^G, $\mathbf{H}^*_G(\mathcal{Q}'(M^G))_+ = \mathbf{H}^*_G(\mathbf{C})_+ \otimes H^*(M^G)$, which is isomorphic to

$$\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(pt) \otimes H^*(M^G) \bigoplus H^*_{G_1}(pt) \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G$$

Similarly, since G_1 acts trivially on M^{G_1}, $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1})) = \mathbf{H}^*_{G_1}(\mathbf{C}) \otimes \mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1}))$. Hence $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$ is isomorphic to

$$\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \bigoplus H^*_{G_1}(pt) \otimes \mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1}))_+ \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1}))_+.$$

Since G_2 is simple, $\mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1}))_+ = \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G)$ by Theorem 7.2. Hence $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$ is isomorphic to

$$\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \bigoplus H^*_{G_1}(pt) \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G).$$

Interchanging the roles of G_1 and G_2, we see that $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))_+$ is isomorphic to

$$\mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}) \bigoplus H^*_{G_2}(pt) \otimes \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*(M^G).$$

Next, we need to describe the restriction of ϕ to the various summands of $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))_+$. Clearly ϕ maps the summand $\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \subset \mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$ to

$$\mathbf{H}_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(pt) \otimes H^*(M^G) \subset \mathbf{H}^*_G(\mathcal{Q}'(M^G))_+,$$

acting by $id \otimes i_1^*$. (For this to make sense, we need to identify $\mathbf{H}_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(pt) \otimes H^*(M^G)$ with $\mathbf{H}_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(pt) \otimes_{H^*_{G_2}(pt)} H^*_{G_2}(M^G)$, as in (7.3)). Also, ϕ acts by id on the remaining summands of $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+$.

Similarly, ϕ maps the summand $\mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}) \subset \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))_+$ to $\mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(pt) \otimes H^*(M^G)$, acting by $-id \otimes i_2^*$. Finally, ϕ acts by $-id$ the identity on the remaining summands of $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))_+$. The surjectivity of ϕ is now apparent. $\square$.

Proof of Theorem 7.9 : The following notation will be convenient. Since $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))_+$ decomposes as the direct sum of six subspaces

$$\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \bigoplus H^*_{G_1}(pt) \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G)$$

$$\bigoplus \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}) \bigoplus H^*_{G_2}(pt) \otimes \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*(M^G) \bigoplus \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*(M^G),$$

an element $\omega \in \mathbf{H}^*_G(\mathcal{Q}'(M^{G_1}))_+ \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))_+$ can be written uniquely as a 6-tuple $(\omega_1, \dots, \omega_6)$.

Let $\alpha = \sum_i \alpha_i \otimes \omega_i$ be an arbitrary element of $\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1})$. Via $i_1^* : H^*_{G_2}(M^{G_1}) \to H^*_{G_2}(M^G)$, $\omega_i \mapsto \sum_j p_{ij} \otimes \nu_{ij} \in H^*_{G_2}(pt) \otimes H^*(M^G) = H^*_{G_2}(M^G)$. Let

$$\tilde{\alpha} = \sum_{i,j} \alpha_i \otimes p_{ij} \otimes \nu_{ij} \in \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(pt) \otimes H^*(M^G),$$

and note that $(\alpha, 0, 0, 0, \tilde{\alpha}, 0)$ lies in $Ker(\phi)$ by construction. Since α was arbitrary, the assignment $\alpha \mapsto (\alpha, 0, 0, 0, \tilde{\alpha}, 0)$ identifies $\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1})$ with a linear subspace of $Ker(\phi)$.

Interchanging the roles of G_1 and G_2, for any $\alpha \in \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2})$ we can find $\tilde{\alpha} \in$ such that $(0, \tilde{\alpha}, 0, \alpha, 0, 0) \in Ker(\phi)$. Hence the assignment $\alpha \mapsto (0, \tilde{\alpha}, 0, \alpha, 0, 0)$ identifies $H^*_{G_1}(M^{G_2}) \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+$ with another subspace of $Ker(\phi)$.

Finally, given $\alpha \in \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*(M^G)$, $(0, 0, \alpha, 0, 0, \alpha) \in Ker(\phi)$, so the map $\alpha \mapsto (0, 0, \alpha, 0, 0, \alpha)$ identifies $\mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*(M^G)$ with another subspace of $Ker(\phi)$. Clearly these three subspaces of $Ker(\phi)$ intersect trivially, and account for all of $Ker(\phi)$. $\square$

Next, we use Lemma 7.1 to describe the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$. Taking $\mathcal{A} = \mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathcal{B} = \mathbf{H}^*_G(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))$, and $f : \mathcal{A} \to \mathcal{B}$ the map

$$\mathbf{H}^*_G(\mathcal{Q}'(M)) \to \mathbf{H}^*_G(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))$$

appearing in (7.4) it is clear that the hypothesis of Lemma 7.1 holds. The ring structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))[0] = H^*_G(M)$ is classical, and the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2}))$ may be described completely in terms of $\mathbf{H}^*_{G_1}(\mathbf{C})$, $\mathbf{H}^*_{G_2}(\mathbf{C})$ and classical data because of the identity

$$\mathbf{H}^*_G(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}^*_G(\mathcal{Q}'(M^{G_2})) = \mathbf{H}^*_{G_1}(\mathbf{C}) \otimes \mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1})) \oplus \mathbf{H}^*_{G_2}(\mathbf{C}) \otimes \mathbf{H}^*_{G_1}(\mathcal{Q}'(M^{G_2})).$$

Since G_1, G_2 are simple, the vertex algebra structures of both $\mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1}))$ and $\mathbf{H}^*_{G_2}(\mathcal{Q}'(M^{G_1}))$ are given by Theorem 7.2. By Lemma 7.1, this uniquely determines the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$.

Finally, via the identification of $\mathbf{H}^*_G(\mathcal{Q}'(M))_+$ with

$$\mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \bigoplus \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}) \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G)$$

given by Theorem 7.9, we can now describe all circle products in

$$H^*_G(M) \oplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}) \bigoplus \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}) \bigoplus \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G).$$

For example,

- Given $\alpha \otimes \omega \in \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1})$ and $\eta \otimes \nu \in \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2})$,

$$(\alpha \otimes \omega) \circ_{-1} (\eta \otimes \nu) = (\alpha \circ_{-1} \eta) \otimes i_1^*(\alpha) \cup i_2^*(\nu) \in \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*(M^G).$$

- Given $a \in H^*_G(M)$ and $\alpha \otimes \omega \in \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1})$,

$$a \circ_{-1} (\alpha \otimes \omega) = \alpha \otimes j_1^*(a) \cup \omega \in \mathbf{H}^*_{G_1}(\mathbf{C})_+ \otimes H^*_{G_2}(M^{G_1}).$$

- Given $a \in H^*_G(M)$ and $\alpha \otimes \omega \in \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2})$,

$$a \circ_{-1} (\alpha \otimes \omega) = \alpha \otimes j_2^*(a) \cup \omega \in \mathbf{H}^*_{G_2}(\mathbf{C})_+ \otimes H^*_{G_1}(M^{G_2}).$$

Note that the maps i_1^*, i_2^*, j_1^* and j_2^*, as well as the ring structures of $H^*_{G_1}(M^{G_2})$, $H^*_{G_2}(M^{G_1})$), and $H_G(M^G)$ are encoded in the vertex algebra structure of $\mathbf{H}^*_G(\mathcal{Q}'(M))$. For general G, we expect $\mathbf{H}^*_G(\mathcal{Q}'(M))$ to depend on the family of vertex algebras $\mathbf{H}^*_K(\mathbf{C})$ for connected normal subgroups $K \subset G$, together with the rings $H^*_{G/K}(M^K)$ and all maps $H^*_{G/K}(M^K) \to H^*_{G/K'}(M^{K'})$ when $K \subset K'$.

7.4. The case where G is a torus T

In this section, we study $\mathbf{H}^*_G(\mathcal{Q}'(M))$ in the case where G is a torus T. Recall from [11] that $\mathbf{H}^*_T(\mathcal{Q}'(M))$ can be computed using the small chiral Weil complex $\mathcal{C} = \langle \gamma, c\rangle \otimes \mathcal{Q}'(M)$, with differential $K(0)\otimes 1 + 1\otimes d_{\mathcal{Q}}$. Here $\langle\gamma, c\rangle$ is the subalgebra of $\mathcal{W}(\mathfrak{t})$ generated by the $\gamma^{\xi'_i}, c^{\xi'_i}$, $\xi' \in \mathfrak{t}^*$, and $K(0)$ is the chiral Koszul differential. $\mathbf{H}^*_T(\mathcal{Q}'(M))$ is an abelian vertex algebra, ie, a supercommutative algebra equipped with a derivation ∂ of degree 0 and weight 1. First we consider the case $T = S^1$. The next result is analogous to Theorem 7.2 for simple group actions.

Theorem 7.11. *(Positive-weight localization for circle actions) For any S^1-manifold M, $\mathbf{i}^* : \mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ \to \mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1}))_+$ is a linear isomorphism in positive weight. Hence*

$$\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ \cong \mathbf{H}^*_{S^1}(\mathbf{C})_+ \otimes H^*(M^{S^1}). \tag{7.5}$$

Moreover, the vertex algebra structure of $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ is determined completely by (7.5).

Proof: Every point in M is either an S^1-fixed point or has a finite isotropy group. If M^{S^1} is nonempty, fix an S^1-invariant tubular neighborhood U of M^{S^1}, and let $V = M\backslash M^{S^1}$. Since S^1 acts locally freely on V, $\mathbf{H}^*_{S^1}(\mathcal{Q}'(V))_+ = 0 = \mathbf{H}^*_{S^1}(\mathcal{Q}'(U\cap V))_+$. By a Mayer-Vietoris argument, $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ = \mathbf{H}^*_{S^1}(\mathcal{Q}'(U))_+$, and by homotopy invariance $\mathbf{H}^*_{S^1}(\mathcal{Q}'(U))_+ = \mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1}))_+$. Finally, since S^1 acts trivially on M^{S^1}, we have $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1}))_+ = \mathbf{H}^*_{S^1}(\mathbf{C})_+ \otimes H^*(M^{S^1})$.

As for the vertex algebra structure, taking $\mathcal{A} = \mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$, $\mathbf{B} = \mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1}))$ and $f = \mathbf{i}^*$, the hypothesis of Lemma 7.1 is clearly satisfied. The ring structure of $\mathbf{H}^*_{S^1}(\mathcal{A}'(M))[0] = H^*_{S^1}(M)$ is classical, and $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1})) = \mathbf{H}^*_{S^1}(\mathbf{C}) \otimes H^*(M^{S^1}))$ as a vertex algebra. This determines the vertex algebra structure of $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$. As in the case G simple, both the classical restriction map i^* and the ring structure of $H^*(M^{S^1})$ are encoded in the vertex algebra structure of $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$. $\Box$

Recall from Theorem 6.1 of [10] that $\mathbf{H}^*_{S^1}(\mathbf{C})$ is just the polynomial algebra $\mathbf{C}[\gamma, \partial\gamma, \partial^2\gamma, \dots]$. Hence $\mathbf{H}^*_{S^1}(\mathbf{C})_+$ is the ideal $\langle \partial\gamma, \partial^2\gamma, \cdots\rangle \subset \mathbf{C}[\gamma, \partial\gamma, \partial^2\gamma, \dots]$. Equivalently, $\mathbf{H}^*_{S^1}(\mathbf{C})_+$ may be described as the *vertex algebra ideal* generated by $\partial\gamma$. Thus Theorem 7.11 gives a complete description of $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ in terms of classical data.

Example 7.12. *$M = \mathbf{CP}^1$, where T has isotropy weights $-1, 1$ at the fixed points p_0, p_1.*

Since $M^{S^1} = \{p_0, p_1\}$, it follows from Theorem 7.11 that

$$\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ = \mathbf{H}^*_{S^1}(\mathbf{C})_+ \otimes H^*(\{p_0, p_1\}) = \mathbf{H}^*_{S^1}(\mathbf{C})_+ \oplus \mathbf{H}^*_{S^1}(\mathbf{C})_+. \tag{7.6}$$

It follows that $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ is the free abelian vertex algebra generated by $H^*_{S^1}(M)$.

Example 7.13. *$M = \mathbf{CP}^2$, where T acts with isotropy weights $-1, 0, 1$ at the fixed points.*

We claim that $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ is *not* the vertex algebra $\langle H^*_{S^1}(M)\rangle$ generated by the weight zero component. Classically, $H^*_{S^1}(M) = \mathbf{C}[t, \omega]/\langle \omega(\omega - t)(\omega + t)\rangle$, where ω is the equivariant symplectic form and t is the image of the generator of $H^*_{S^1}(pt)$ under the Chern-Weil map. From this description, it is clear that *dim* $H^2_{S^1}(M) = 2$. Hence in the vertex subalgebra $\langle H^*_{S^1}(M)\rangle \subset \mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$, the subspace of degree 2 and weight 1 can have dimension at most 2. On the other hand, M^{S^1} consists of three isolated fixed points, so $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+ = \mathbf{H}^*_{S^1}(\mathbf{C})_+ \oplus \mathbf{H}^*_{S^1}(\mathbf{C})_+ \oplus \mathbf{H}^*_{S^1}(\mathbf{C})_+$ by Theorem 7.11. In particular, *dim* $\mathbf{H}^2_{S^1}(\mathcal{Q}'(M))[1] = 3$, so it must contain elements that do not lie in $\langle H^*_{S^1}(M)\rangle$.

Theorem 7.14. *For any S^1-manifold M, $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ is generated as a vertex algebra by $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))[0] \oplus \mathbf{H}^*_{S^1}(\mathcal{Q}'(M))[1]$.*

Proof: Fix a basis $\{\alpha_i|\ i \in I\}$ of $H^*(M^{S^1})$. Consider the collection

$$\mathcal{C} = \{\partial\gamma \otimes \alpha_i|\ i \in I\} \subset \mathbf{H}^*_{S^1}(\mathbf{C})[1] \otimes H^*(M^{S^1}),$$

and let $\langle \mathcal{C}\rangle$ denote the vertex subalgebra of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ generated by $\mathcal{C}$. Since

$$\mathbf{i}^* : \mathbf{H}^*_{S^1}(\mathcal{Q}'(M)) \to \mathbf{H}^*_{S^1}(\mathcal{Q}'(M^{S^1})) \cong \mathbf{H}^*_{S^1}(\mathbf{C}) \otimes H^*(M^{S^1})$$

preserves ∂, it follows that $(\partial^{k+1}\gamma) \otimes \alpha_i = \partial^k(\partial\gamma \otimes \alpha_i)$ lies in $\langle \mathcal{C}\rangle$, for all $k \geq 0$. We claim that $\mathcal{C}$ together with $H^*_{S^1}(M)$ generates $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ as a vertex algebra. At weight zero the claim is obvious, so let $\omega \in \mathbf{H}^*_{S^1}(\mathcal{Q}'(M))_+$ be a nonzero element. In particular, this implies that M^{S^1} is nonempty.

By Theorem 7.11, we can write $\omega = \sum_{i\in I} p_i \otimes \alpha_i$, where $p_i \in \mathbf{H}^*_{S^1}(\mathbf{C})_+$. Since p_i has positive weight, it is divisible by $\partial^{k_i}\gamma$ for some $k_i > 0$, and we may write $p_i = q_i\partial^{k_i}\gamma$ where $q_i \in \mathbf{H}^*_{S^1}(\mathbf{C})$. Since M^{S^1} is nonempty, the chiral Chern-Weil map is injective, and since $\mathbf{H}^*_{S^1}(\mathbf{C})$ is generated by $H^*_{S^1}(pt)$ as a vertex algebra, $q_i \otimes 1$ lies in $\langle H^*_{S^1}(M)\rangle$. Since $p_i \otimes \alpha_i = \,:(q_i \otimes 1)(\partial^{k_i}\gamma \otimes \alpha_i):$ for each $i \in I$, the claim follows. $\square$

Corollary 7.15. *If M is a compact S^1-manifold, $\mathbf{H}^*_{S^1}(\mathcal{Q}'(M))$ is finitely generated as a vertex algebra.*

Proof: This follows immediately from the preceding lemma and the fact that $H^*(M^{S^1})$ is finitely generated for compact M. $\square$.

7.5. The case $T = S^1 \times S^1$ and $M = \mathbf{CP}^2$

Next, we consider the case where $T = S^1 \times S^1$. This is analogous to the case $G = G_1 \times G_2$ with G_1, G_2 simple, but can be more subtle because many different copies of S^1 inside T that can arise as stabilizers of points in M. As an example, we compute $\mathbf{H}^*_T(\mathcal{Q}'(M))$ in the case $M = \mathbf{CP}^2$ with the usual linear action of T. Note that T contains three copies of S^1 which arise as stabilizer subgroups, which we denote by T_i, $i = 1, 2, 3$. Each M^{T_i} is a copy of $\mathbf{CP}^1$ which we denote by M_i, and $M_i \cap M_j$ consists of a single point p_{ij}. Let U_i be a T-invariant tubular neighborhood of M_i. Clearly $\mathbf{H}^*_T(\mathcal{Q}'(U_i)) = \mathbf{H}^*_T(\mathcal{Q}'(M_i)) = \mathbf{H}^*_{T_i}(\mathbf{C}) \otimes \mathbf{H}^*_{T/T_i}(\mathcal{Q}'(M_i))$, and the action of T/T_i on M_i is the standard action of S^1 on $\mathbf{CP}^1$.

Lemma 7.16. *$\mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+$ is linearly isomorphic to*

$$\mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes H^*_{T/T_1}(M^{T_1}) \bigoplus \mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes H^*_{T/T_2}(M^{T_2}) \tag{7.7}$$

$$\bigoplus \mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_1}(\mathcal{Q}'(p_{13}))_+ \bigoplus \mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_2}(\mathcal{Q}'(p_{23}))_+ \bigoplus \mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_1}(\mathcal{Q}'(p_{12}))_+.$$

Proof: Consider the Mayer-Vietoris sequence

$$\cdots \to \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+ \to \mathbf{H}^*_T\mathcal{Q}'(U_1))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(U_2))_+ \to \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cap U_2))_+ \to \cdots,$$

which we can replace with

$$\cdots \to \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+ \to \mathbf{H}^*_T(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(M_2))_+ \to \mathbf{H}^*_T(\mathcal{Q}'(p_{12}))_+ \to \cdots. \tag{7.8}$$

By (7.6), $\mathbf{H}^*_T(\mathcal{Q}'(M_1))_+$ is linearly isomorphic to

$$\mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes H^*_{T/T_1}(M_1) \bigoplus H^*_{T_1}(pt) \otimes \mathbf{H}^*_{T/T_1}(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{13}\})$$

$$\bigoplus \mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_1}(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{13}\}).$$

Likewise, $\mathbf{H}^*_T(\mathcal{Q}'(M_2))_+$ is linearly isomorphic to

$$\mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes H^*_{T/T_2}(M_2) \bigoplus H^*_{T_2}(pt) \otimes \mathbf{H}^*_{T/T_2}(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{23}\})$$

$$\bigoplus \mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_2}(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{23}\}).$$

Thus the middle term $\mathbf{H}^*_T(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(M_2))_+$ in (7.8) can be identified with the direct sum of the above six subspaces, so we may write $\omega \in \mathbf{H}^*_T(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(M_2))_+$ as a 6-tuple $(\omega_1, \dots, \omega_6)$ as in the proof of Theorem 7.9. The map

$$\mathbf{H}^*_T(\mathcal{Q}'(M_1))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(M_2))_+ \to \mathbf{H}^*_T(\mathcal{Q}'(p_{12}))_+$$

in (7.8), which we denote by ϕ, is surjective, so we may identify $\mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+$ with $Ker(\phi)$.

Given $\alpha \in \mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes H^*_{T/T_1}(M^{T_1})$, we can find

$$\tilde{\alpha} \in H^*_{T_2}(pt) \otimes \mathbf{H}^*_{T/T_2}(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{23}\})$$

such that $(\alpha, 0, 0, 0, \tilde{\alpha}, 0) \in Ker(\phi)$. The assignment $\alpha \mapsto (\alpha, 0, 0, 0, \tilde{\alpha}, 0)$ identifies $\mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes H^*_{T/T_1}(M^{T_1})$ with a linear subspace of $Ker(\phi)$. Likewise, given $\alpha \in \mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes H^*_{T/T_2}(M_2)$, there exists

$$\tilde{\alpha} \in H^*_{T_1}(pt) \otimes \mathbf{H}^*_{T/T_1}(\mathbf{C})_+ \otimes H^*(\{p_{12}, p_{13}\})$$

such that $(0, \tilde{\alpha}, 0, \alpha, 0, 0) \in Ker(\phi)$, so $\mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes H^*_{T/T_2}(M_2)$ may be identified with another subspace of $Ker(\phi)$.

Next, note that $H^*(\{p_{12},p_{13}\}) = H^*(p_{12})\oplus H^*(p_{13})$, so that $\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(p_{13})$ may be regarded as a subspace of $\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(\{p_{12},p_{13}\})$. Clearly $\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(p_{13})$ lies in $Ker(\phi)$ since $p_{13}\notin U_1\cap U_2$. Similarly, $\mathbf{H}^*_{T_2}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_2}(\mathbf{C})_+\otimes H^*(p_{23})$ is a subspace of $\mathbf{H}^*_{T_2}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_2}(\mathbf{C})_+\otimes H^*(\{p_{12},p_{23}\})$ which lies in the $Ker(\phi)$.

Finally, note that

$$\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+ = \mathbf{H}^*_{T_2}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_2}(\mathbf{C})_+.$$

It follows that $\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(\{p_{12},p_{13}\})$ and $\mathbf{H}^*_{T_2}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_2}(\mathbf{C})_+\otimes H^*(\{p_{12},p_{23}\})$ each contain a copy of $\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(p_{12})$. Thus given $\alpha\in\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(p_{12})$, $(0,0,\alpha,0,0,\alpha)$ will lie in $Ker(\phi)$. This identifies $\mathbf{H}^*_{T_1}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_1}(\mathbf{C})_+\otimes H^*(p_{12})$ with another subspace of $Ker(\phi)$. Finally, it is straightforward to check that these five subspaces of $Ker(\phi)$ intersect pairwise trivially and account for all of $Ker(\phi)$. This completes the proof of Lemma 7.16. □

Next, since T acts locally freely on the complement of $U_1\cup U_2\cup U_3$,

$$\mathbf{H}^*_T(\mathcal{Q}'(M))_+ = \mathbf{H}^*_T(\mathcal{Q}'(U_1\cup U_2\cup U_3),$$

so we have a Mayer-Vietoris sequence

$$\cdots\to\mathbf{H}^*_T(\mathcal{Q}'(M))_+\to\mathbf{H}^*_T(\mathcal{Q}'(U_1\cup U_2))_+\oplus\mathbf{H}^*_T(\mathcal{Q}'(U_3))_+\to\mathbf{H}^*_T(\mathcal{Q}'((U_1\cup U_2)\cap U_3))_+. \tag{7.9}$$

Note that $(U_1\cup U_2)\cap U_3 = \{p_{13},p_{23}\}$, so that

$$\mathbf{H}^*_T(\mathcal{Q}'((U_1\cup U_2)\cap U_3))_+ = \mathbf{H}^*_T(\mathcal{Q}'(p_{13}))_+\oplus\mathbf{H}^*_T(\mathcal{Q}'(p_{23}))_+ = \mathbf{H}^*_T(\mathbf{C})_+\oplus\mathbf{H}^*_T(\mathbf{C})_+.$$

As for the other terms in (7.9), $\mathbf{H}^*_T(\mathcal{Q}'(U_1\cup U_2))_+$ is given by (7.7), and $\mathbf{H}^*_T(\mathcal{Q}'(U_3))_+$ is isomorphic to

$$\mathbf{H}^*_{T_3}(\mathbf{C})_+\otimes H^*_{T/T_3}(M_3)\bigoplus H^*_{T_3}(pt)\otimes\mathbf{H}^*_{T/T_3}(\mathcal{Q}'(M_3))_+\bigoplus\mathbf{H}^*_{T_3}(\mathbf{C})_+\otimes\mathbf{H}^*_{T/T_3}(\mathcal{Q}'(M_3))_+,$$

where

$$\mathbf{H}^*_{T/T_3}(\mathcal{Q}'(M_3))_+ = \mathbf{H}^*_{T/T_3}(\mathcal{Q}'(p_{13}))_+\oplus\mathbf{H}^*_{T/T_3}(\mathcal{Q}'(p_{23}))_+ = \mathbf{H}^*_{T/T_3}(\mathbf{C})\oplus\mathbf{H}^*_{T/T_3}(\mathbf{C}),$$

by (7.6). It is easy to check that the map

$$\mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(U_3))_+ \to \mathbf{H}^*_T(\mathcal{Q}'((U_1 \cup U_2) \cap U_3))_+$$

in (7.9), which we denote by ψ, is surjective, so that $\mathbf{H}^*_T(\mathcal{Q}'(M))_+ = Ker(\psi)$.

Suppose that $\alpha \in \mathbf{H}^*_{T_3}(\mathbf{C})_+ \otimes H^*_{T/T_3}(M_3) \subset \mathbf{H}^*_T(\mathcal{Q}'(M_3))_+$. Then there are unique elements

$$\alpha_1 \in \mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes H^*_{T/T_1}(M_1) \subset \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\alpha_2 \in \mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes H^*_{T/T_2}(M_2) \subset \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+,$$

such that $(\alpha_1+\alpha_2, \alpha) \in \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+ \oplus \mathbf{H}^*_T(\mathcal{Q}'(M_3))_+$ lies in $Ker(\psi)$. The assignment $\alpha \mapsto (\alpha_1 + \alpha_2, \alpha)$ identifies $\mathbf{H}^*_{T_3}(\mathbf{C})_+ \otimes H^*_{T/T_3}(M_3)$ with a linear subspace of $Ker(\psi)$. Similarly, we may identify each of the spaces

$$\mathbf{H}^*_{T_1}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_1}(\mathbf{C})_+ \otimes H^*(p_{12}) \subset \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\mathbf{H}^*_{T_2}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_2}(\mathbf{C})_+ \otimes H^*(p_{23}) \subset \mathbf{H}^*_T(\mathcal{Q}'(U_1 \cup U_2))_+,$$

$$\mathbf{H}^*_{T_3}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_3}(\mathbf{C})_+ \otimes H^*(p_{12}) \in \mathbf{H}^*_T(\mathcal{Q}'(M_3))_+$$

with a subspace of $Ker(\psi)$. It is easy to check that these subspaces intersect pairwise trivially and account for all of $Ker(\psi)$. To summarize, we have proved

Theorem 7.17. *For $M = \mathbf{CP}^2$ and $T = S^1 \times S^1$ as above, $\mathbf{H}^*_T(\mathcal{Q}'(M))_+$ is linearly isomorphic to*

$$\Big(\bigoplus_{i=1}^{3} \mathbf{H}^*_{T_i}(\mathbf{C})_+ \otimes H^*_{T/T_i}(M_i)\Big) \bigoplus \Big(\bigoplus_{i=1}^{3} \mathbf{H}^*_{T_i}(\mathbf{C})_+ \otimes \mathbf{H}^*_{T/T_i}(\mathbf{C})_+\Big). \tag{7.10}$$

As in the case of the case $G = G_1 \times G_2$ for G_1, G_2 simple, the vertex algebra structure of $\mathbf{H}^*_T(\mathcal{Q}'(M))$ may be deduced from Theorem 7.17. Let $\mathcal{A} = \mathbf{H}^*_T(\mathcal{Q}'(M))$, and $\mathcal{B} = \mathbf{H}^*_T(\mathcal{Q}'(U_1 \oplus U_2)) \oplus \mathbf{H}^*_T(\mathcal{Q}'(U_3))$, and let $f : \mathcal{A} \to \mathcal{B}$ be the map

$$\mathbf{H}^*_T(\mathcal{Q}'(M)) \to \mathbf{H}^*_T(\mathcal{Q}'(U_1 \oplus U_2)) \oplus \mathbf{H}^*_T(\mathcal{Q}'(U_3))$$

appearing in (7.9). Clearly the conditions of Lemma 7.1 are satisfied. As usual, the product $\circ_{-1}$ on $\mathbf{H}^*_T(\mathcal{Q}'(M))[0] = H^*_T(M)$ is classical, and the vertex algebra structure of $\mathcal{B}$ is determined by Lemma 7.16 and the structure of $\mathbf{H}^*_{S^1}(\mathcal{Q}'(\mathbf{CP}^1))$ given by (7.6). By Lemma 7.1, the vertex algebra structure of $\mathbf{H}^*_T(\mathcal{Q}'(M))$ is uniquely determined from this data.

Corollary 7.18. $\mathbf{H}^*_T(\mathcal{Q}'(M))$ *is generated as a vertex algebra by* $\bigoplus_{n=0}^2 \mathbf{H}^*_T(\mathcal{Q}'(M))[n]$.

Proof: This is immediate from the vertex algebra structure of $\mathbf{H}^*_T(\mathcal{Q}'(M))$ and the structure of $\mathbf{H}^*_T(\mathbf{C})$. $\square$

For a general torus T, if M is a T-manifold of finite orbit type (ie, only a finite number of subtori $T' \subset T$ can occur as isotropy groups for points in M), we expect that $\mathbf{H}^*_T(\mathcal{Q}'(M))$ will be generated as a vertex algebra by $\bigoplus_{n=0}^N \mathbf{H}^*_T(\mathcal{Q}'(M))[n]$ for some N. Note that a similar statement for $\mathbf{H}^*_G(\mathcal{Q}'(M))$ when G is nonabelian is out of reach because it is not known if $\mathbf{H}^*_G(\mathbf{C})$ is a finitely generated vertex algebra.

8. Concluding Remarks and Open Questions

Our descriptions of $\mathbf{H}^*_G(\mathcal{Q}'(M))$ and $\mathbf{H}^*_G(\mathcal{Q}(M))$ are given relative to the family of vertex algebras $\mathbf{H}^*_K(\mathbf{C})$ for various connected normal subgroups $K \subset G$. An important open question in this theory is to describe $\mathbf{H}^*_G(\mathbf{C})$ for any G. Note that for $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, we have $\mathbf{H}^*_G(\mathbf{C}) = \mathbf{H}^*_{G_1}(\mathbf{C}) \otimes \cdots \otimes \mathbf{H}^*_{G_n}(\mathbf{C})$, where $G_1, \dots, G_n$ are compact, connected Lie groups with Lie algebras $\mathfrak{g}_1, \dots, \mathfrak{g}_n$, respectively. We already know how to describe $\mathbf{H}^*_G(\mathbf{C})$ when G is abelian, so it suffices to assume that G is simple.

Question 8.1. *Recall from [10] that for simple* G*, the weight-one subspace* $\mathbf{H}^*_G(\mathbf{C})[1]$ *is isomorphic to* $Hom_G(\mathfrak{g}, S(\mathfrak{g}^*))$*, which is finitely generated as a module over* $\mathbf{H}^*_G(\mathbf{C})[0] =$

$S(\mathfrak{g}^*)^G$ by a theorem of Kostant. Is there is a similar representation-theoretic description of $\mathbf{H}^*_G(\mathbf{C})[n]$ for any n, and is $\mathbf{H}^*_G(\mathbf{C})[n]$ finitely generated as an $S(\mathfrak{g}^*)^G$-module?

Question 8.2. *Is $\mathbf{H}^*_G(\mathbf{C})$ finitely generated as a vertex algebra? Can we find a set of generators?*

Question 8.3. *Can we compute the character $\chi(G) = \sum_{p,n} dim\ \mathbf{H}^p_G(\mathbf{C})[n]\ z^p q^n$? Does $\chi(G)$ have any nice properties (modularity, relations to other objects from classical Lie theory, etc.)?*

Question 8.4. *For compact M, can we compute $\chi(G, M) = \sum_{p,n} dim\ \mathbf{H}^p_G(\mathcal{Q}'(M))[n]\ z^p q^n$? How is $\chi(G, M)$ related to other known invariants of M?*

Recall that for a topological G-space M, $H^*_G(M)$ is defined by the Borel construction $H^*_G(M) = H^*((M \times E)/G)$, where E is a contractible space on which G acts freely. When M is a smooth G-manifold and our coefficient ring is $\mathbf{R}$, this definition agrees with the de Rham-theoretic definition $H^*_G(M) = H^*_{bas}(W(\mathfrak{g}) \otimes \Omega(M))$?

Question 8.5. *Is there a vertex algebra valued equivariant cohomology $\mathbf{H}^*_G(M)$ for any topological G-space M which coincides with $\mathbf{H}^*_G(\mathcal{Q}'(M))$ when M is a smooth G-manifold? In particular, we must have $\mathbf{H}^*_G(pt) = \mathbf{H}^*_G(\mathbf{C})$; what is the topological interpretation of $\mathbf{H}^*_G(\mathbf{C})$?*

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